NONLOCAL HOLONOMY REPRESENTATIONS FOR LIE ALGEBRA-VALUED ASHTEKAR-BARBERO CONNECTION

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Abstract

Holonomies of the Ashtekar-Barbero connection can be considered as abstract elements of a Lie group, exponentially mapped from their algebra representation. This idea allows for the definition of the states in loop quantum gravity, which preserve the group symmetry that is equivalent to the Ashtekar-Barbero connection symmetry of Lie algebra. The equivalence of the symmetries requires either the quadratic- or linear-order precision in the expansion of group elements either around a finite value of the expansion parameter or by taking the limit as this parameter approaches zero, respectively. These conditions put different constraints on the holonomy regularization method in loop quantum gravity, where holonomies are expanded around finite values of the related paths' lengths. This article investigates the possibility of increasing the linear-order precision, postulated in canonical loop quantum gravity, into the quadratic order. It demonstrates that the regularization method can be defined more accurately.

Keywords: gauge theory, holonomy, Lie algebra, Lie group, loop quantum gravity, nonlocality.

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1. MOTIVATION

The Ashtekar variables [3] are the representation of the gravitational degrees of freedom that allow the expression of a particular action [21], leading to the Einstein field equations with time gauge [17] in the form of a gauge theory, analogous to the Yang-Mills model [31]. In the case of the real Ashtekar variables, they lead to the Hamiltonian [8, 21], which may be considered a candidate for the classical limit of the background-independent formulation of quantum gravity, called loop quantum gravity (LQG) [4, 24].

It is known that the symmetry transformations of operators, which preserve probabilities at the quantum level, are related to a Lie algebra — cf. [27]. Equal algebra of operators has to describe the symmetries of the classical candidates for field theory operators before quantization. This article analyzes the algebra of the modified candidates forming the canonical pair for a time gauge-fixed simplification of general relativity (GR) [2]. These variables are the real Ashtekar-Barbero connection and the densitized dreibein [4, 8, 25]. Their canonical quantization leads to the model analogous to the DeWitt equation [14], which is not background-independent. The background-independent formalism of LQG is constructed by introducing particular regularization procedures [24, 25], in which the point-located Ashtekar-Barbero connection and its curvature are replaced by different functionals of path-related holonomies. This paper indicates the difference in these procedures that lead to the pair of quantities preserving Lie algebra symmetry with unequal precision. One quantity is an order below the standard symmetry preservation at the quantum level [27]. This article provides the solution to this different precision problem. It also proposes a symmetrization method for point-located Ashtekar variables into path-related equivalents of these variables. These smeared equivalents undergo identical Lie algebra and can be used to regularize path-related canonical variables in LQG into path-related holonomies invariant under Lie group transformations.

2. Gravitational Hamiltonian in the real Ashtekar variables

The total Hamiltonian of the time gauge-fixed simplification of GR expressed in terms of the real Ashtekar variables consists of three terms. Two types of these terms are first-class constraints corresponding to the spatial diffeomorphism invariance and Lie algebra symmetry. The third term, called the Hamiltonian constraint [3, 8, 21, 24], corresponds to the abelian energy gauge and contains two propagating degrees of freedom. This term is Barbero's Hamiltonian for the Ashtekar connection field derived from the Holst action with the momentum degrees of freedom removed from denominators by Thiemann's method. It is defined by the following expression:

$$(1) \quad H := -\frac{4\sigma_E}{\gamma\kappa^2}\int_{\Sigma_t}d^3x\,N(x)\,\epsilon_+^{abc}\Big(F_{ab}^i(x) - (\gamma^2+1)\epsilon_{ijk}K_a^j(x)K_b^k(x)\Big)\Big\{A_c^i(x),\mathbb{V}(\Sigma_t)\Big\}\,,$$

where

$$(2) \qquad K_a^j(x) := -\frac{8\sigma_E}{\gamma^2\kappa^3}\Bigg\{A_a^j(x),\bigg\{\int_{\Sigma_t}d^3y\,\epsilon_+^{bcd}F_{bc}^k\Big\{A_d^k(y),\mathbb{V}(\Sigma_t)\Big\},\mathbb{V}(\Sigma_t)\bigg\}\Bigg\}\,,$$

$$(3) \qquad \mathbb{V}(\Sigma_t) := \int_{\Sigma_t}d^3x\,\sqrt{|E(x)|}\,, \qquad E := \frac{1}{3!}\epsilon_{abc}^-\epsilon^{ijk}E_i^aE_j^bE_k^c\,,$$

71 and $\sigma_E := \text{sgn}(E)$. The Poisson brackets for canonical fields are specified regard-
72 ing the metric tensor variables with a time gauge [2] providing the factor $-\gamma\kappa/2$
73 in front of the definition

74 (4) $$\{X,Y\} := -\frac{\gamma\kappa}{2}\int d^3x \left(\frac{\delta X}{\delta A_a^i(x)}\frac{\delta Y}{\delta E_i^a(x)} - \frac{\delta X}{\delta E_i^a(x)}\frac{\delta Y}{\delta A_a^i(x)} \right).$$

75 The symbols ϵ_+^{abc} and ϵ_{abc}^- represent the Levi-Civita tensor density and inverse
76 density, respectively. The scalar γ is the real Barbero-Immirzi parameter [8, 22],
77 and $\kappa := 16\pi G$, where the speed of light is normalized to unity. The quantity N
78 is a Lagrange multiplier, F_{ab}^i is the curvature of the real Ashtekar-Barbero con-
79 nection field A_a^i, and E_i^a is the densitized dreibein. It represents the momentum
80 canonically conjugated to A_a^i. The Hamiltonian constraint in (1) is the start-
81 ing point for graph regularization — the procedure directly preceding canonical
82 quantization.

83 It is also worth bringing up that the choice of the complex Ashtekar variables
84 would lead to rather nonphysical candidates for observables — see the expansion
85 in (24) in which all parameters have to be real. In this case, auxiliary effective
86 methods [9, 26] of reality conditions implementation on the classical phase space
87 would be needed.

88 Apart from the reality of the Lie algebra-valued variables, it is required that
89 the related generators are either unitary and linear or antiunitary and antilinear
90 [28]. As in the case of the standard operators in quantum mechanics and quantum
91 field theory, they are unitary and linear in general [27]. All of the aforementioned
92 facts encouraged the investigation of the holonomy representation algebra. This
93 article verifies that this regularized representation can be used to form a gauge-
94 invariant equivalent of the Ashtekar-Barbero connection's degrees of freedom. It
95 is then demonstrated that the standard equivalent of the connection's degrees
96 of freedom in LQG, invariant under Lie algebra representation [24, 25], becomes
97 a Lie group representation. However, in the case of the connection field, the
98 invariance of the standard algebra-to-group transformation in LQG is one order
99 less accurate than in the case of the connection's curvature field.

100 An alternative algebra-to-group map for the Ashtekar connection can solve
101 this problem. This map, unifying the accuracy of the connection and its curvature
102 regularization in LQG to higher precision, is proposed in this article. Finally,
103 Wigner's construction of operators is recalled to demonstrate the relevance of the
104 accuracy of the algebra-to-group maps in quantum theories.

105   ## 3.   HOLONOMY OF THE ASHTEKAR-BARBERO CONNECTION

106 **Definition.** The parallel transport of a vector bundle element over the manifold
107 M to another bundle along a smooth oriented path $\ell(s) : [0, 1] \to M$ is determined

108 by the expression

109 (5)
$$h^{-1}_{[0,1]}[\boldsymbol{A}] := \mathcal{P} \exp\left(-\int_0^1 ds\, \dot{\ell}^a(s)\, \boldsymbol{A}_a\big(\ell(s)\big)\right).$$

110 This object is called the holonomy of the vector potential $\boldsymbol{A} := A^I \boldsymbol{t}_I$ along the
111 path ℓ, where $\boldsymbol{t}_I$ are the generators of a Lie algebra.

112 **Remark 1.** The 'inverse notation', $\boldsymbol{h}^{-1}$, follows the convention established in the
113 LQG-related literature [4, 25]. Consequently, $\boldsymbol{h}$ denotes the inverse holonomy.

114 **Remark 2.** The Ashtekar-Barbero connection coefficients $A^I \in \mathbb{R}$ are specified
115 by the following normalization of the Lie bracket,

116 (6)
$$[\boldsymbol{t}_J, \boldsymbol{t}_K] = \frac{1}{2} C^I{}_{JK} \boldsymbol{t}_I \,,$$

117 where $C^I{}_{JK} = -C^I{}_{KJ} \in \mathbb{R}$ are the structure constants. By definition, this
118 normalization resolves the issue of reality conditions implementation.

119   The formalism of LQG (cf. [4, 25]) introduces a graph structure, which al-
120 lows defining the associated scalar product as a relation between the positions of
121 variables on this graph, hence without using a metric. This technique is based on
122 the framework constructed in [5, 6, 7]. If the real Ashtekar-Barbero connection
123 fields A^i_a, theirs curvature F^i_{ab}, and densitized dreibeins E^a_i in the Hamiltonian
124 constraint in (1) can be rigorously replaced by holonomies and the fluxes of den-
125 sitized dreibeins[i], then the graph is naturally introduced. Its edges are the paths
126 ℓ in the holonomy definition (5). Hence, the graph's introduction technique is re-
127 duced to the inversion of the holonomy-to-connection map in the just mentioned
128 definition.

129 **Theorem 3.** *Let the quantity* $\mathbb{L}_0$ *denote a fiducial length scale. Let also the*
130 *indices* $p, q, r, \ldots$ *do not indicate any spatial directions but enumerate edges, so*
131 *the Einstein summation convention is not applied to these edge indices. The*
132 *holonomy* $\boldsymbol{h}^{-1}_p = \mathcal{P} \exp\big(-\int_{l_p} \boldsymbol{A}\big)$, *adjusted to a particular graph's edge (of length*
133 $l_p := \mathbb{L}_0 \varepsilon_p$), *can be expanded around the infinitesimal value of the dimensionless*
134 *regularization parameter* $\varepsilon_p := \varepsilon_{l_p} \in (0, 1)$,

135 (7)
$$\boldsymbol{h}^{\mp 1}_p[\boldsymbol{A}] = \mathbb{1} \mp \mathbb{L}_0 \varepsilon_p \boldsymbol{A}_a(0) \dot{\ell}^a_p\big|_{\varepsilon_p=0} + \frac{1}{2}\big(\mathbb{L}_0 \varepsilon_p\big)^2 \boldsymbol{A}_a(0) \boldsymbol{A}_b(0) \dot{\ell}^a_p\big|_{\varepsilon_p=0} \dot{\ell}^b_p\big|_{\varepsilon_p=0}$$
$$\mp \frac{1}{2}\big(\mathbb{L}_0 \varepsilon_p\big)^2 \partial_a \boldsymbol{A}_b(0) \dot{\ell}^a_p\big|_{\varepsilon_p=0} \dot{\ell}^b_p\big|_{\varepsilon_p=0} + \mathcal{O}\big(\varepsilon^3\big) \,.$$

136 ***Proof.*** See [1, 16, 18].   ∎

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[i]Their construction (see e.g. [4, 25]) is not relevant to the analysis presented in this article.

137 **Remark 4.** It is worth emphasizing that all the elements in (7) are oriented in
138 the same direction, $\dot{\ell}_p\big|_{\varepsilon_p=0}$, tangent to $\ell_p(s)$ at the point where the edge of a zeroth
139 length is localized. This tangent direction determines a one-dimensional system
140 and defines the spatial basis for the expansion in (7). For instance, the term
141 $\partial_a \boldsymbol{A}_b(0)$ in the lower line of this expression is the coefficient of the directional
142 derivative of a one-form, where both quantities are oriented toward the direction
143 of $\dot{\ell}_p\big|_{\varepsilon_p=0}$. This tangent can be easily completed to the orthonormal Euclidean
144 basis defined by the Frenet-Serret frame [19, 23].

145 **Remark 5.** The left-hand side of (7) is a non-local, path-related (edge-related)
146 object, while the right-hand side is a local point-related (vertex-related) expres-
147 sion.

148 **Remark 6.** The quadratic terms in the expansion on the right-hand side of (7)
149 contain derivatives of connections, neither being fields nor holonomies. These
150 terms have to be removed to find a connection-to-holonomy map.

151 **Theorem 7.** *The equivalent of the expansion on the right-hand side of (7), which*
152 *is as nonlocal as the holonomy and does not contain derivatives of connections,*
153 *equals*

154 $$(8) \quad \boldsymbol{h}_p^{\mp 1}[\boldsymbol{A}] = \mathbb{1} \mp \frac{1}{2}\mathbb{L}_0\varepsilon_p\big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A}_{p_{\text{fin}}}(\ell_{\text{fin}})\big) + \frac{1}{2}\big(\mathbb{L}_0\varepsilon_p\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}})\big)^2 + \mathcal{O}\big(\varepsilon^3\big)\,.$$

155 ***Proof.*** In order to remove the problematic derivatives in (7), the following ap-
156 proximation of the directional derivative along the curve ℓ_p has been applied:

157 $$(9) \qquad \mathbb{L}_0\varepsilon_p\partial_a\boldsymbol{A}_b(0)\dot{\ell}_p^a\big|_{\varepsilon_p=0}\dot{\ell}_p^b\big|_{\varepsilon_p=0} = \boldsymbol{A}_a(1)\dot{\ell}_p^a\big|_{\varepsilon_p=1} - \boldsymbol{A}_a(0)\dot{\ell}_p^a\big|_{\varepsilon_p=0} + \mathcal{O}\big(\varepsilon^2\big)\,.$$

158 Its welcome side effect is the invariance of the right-hand side of (8) under the
159 operation $\boldsymbol{h}_p^{\mp 1}[\boldsymbol{A}] \to \boldsymbol{h}_{p^{-1}}^{\pm 1}[\boldsymbol{A}]$, which replaces the $\ell_{\text{init}} := \ell(0)$ point with $\ell_{\text{fin}} :=$
160 $\ell(1)$. ∎

161 Analogously, one can expand the loop holonomy $\boldsymbol{h}_{qr}^{-1}$ defined around the
162 parallelepiped-like loop composed of two pairs of identical opposite edges, ℓ_q
163 and ℓ_r.

164 **Theorem 8.** *Let the loop initiate along the outgoing edge ℓ_q, hence along the*
165 *path cooriented with ℓ_q and let it return along the outgoing edge ℓ_r, hence along*
166 *the path cooriented with ℓ_q^{-1}. As a result, the loop holonomy, describing the paral-*
167 *lel transport of a vector, factorizes into the following composition of holonomies,*
168 *$\boldsymbol{h}_{qr}^{-1} = \boldsymbol{h}_q^{-1}\boldsymbol{h}_r^{-1}\boldsymbol{h}_q\boldsymbol{h}_r$ (the 'inverse notation' holds in the whole article). The out-*
169 *come of the expansion of the parallelepiped-like loop holonomy around the in-*
170 *finitesimal value of regularization parameters ε_q and ε_r is*

(10) $$\boldsymbol{h}_{\square qr}^{\mp 1}[\boldsymbol{A}] = \mathbb{1} \mp \mathbb{L}_0^2 \varepsilon_q \varepsilon_r \boldsymbol{F}_{ab}(0) \left.\dot{\ell}_q^a\right|_{\varepsilon_q=0} \left.\dot{\ell}_r^b\right|_{\varepsilon_r=0} + \mathcal{O}(\varepsilon^3)\,,$$

where $\boldsymbol{F}_{ab} := F_{ab}^I \boldsymbol{t}_I$ are the spatial coefficients of the curvature of $\boldsymbol{A}$.

Proof. See [1, 18][i].   ∎

Instead of quadrilateral loops, the LQG model requires the analogous expansion for triangular loops [4, 25].

Theorem 9. *Let the neighboring pair of edges ℓ_q and ℓ_r be the part of a triangular loop. The expansion analogous to (8) leads to the expression similar to (10),*

(11) $$\boldsymbol{h}_{\triangle qr}^{\mp 1}[\boldsymbol{A}] = \mathbb{1} \mp \frac{1}{2} \mathbb{L}_0^2 \varepsilon_q \varepsilon_r \boldsymbol{F}_{ab}(0) \left.\dot{\ell}_q^a\right|_{\varepsilon_q=0} \left.\dot{\ell}_r^b\right|_{\varepsilon_r=0} + \mathcal{O}(\varepsilon^3)\,.$$

Proof. See [1, 18] or simply consider the triangular path formed of the neighboring pair of edges in some parallelepiped-like quadrilateral and the diagonal of this quadrilateral.   ∎

Corollary 10. *These formulas can be simplified into a single expression analogous to (8),*

(12) $$\boldsymbol{h}_{qr}^{\mp 1}[\boldsymbol{A}] = \mathbb{1} \mp \lambda \mathbb{L}_0^2 \varepsilon_q \varepsilon_r \boldsymbol{F}_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}}) + \mathcal{O}(\varepsilon^3)\,.$$

Parameter λ takes value $1/2$ in the case of a triangular loop and 1 in the case of a parallelepiped-like quadrilateral.

Remark 11. It is worth emphasizing that formulas (10) and (11) require the approximation in (9) of the directional derivative along a curve. It is also worth indicating that the closure of the path in the definition of the loop holonomy entails the identification $\boldsymbol{F}(\ell_{\text{init}}) = \boldsymbol{F}(\ell_{\text{fin}})$. Therefore, the formula in (12) is already invariant under the simultaneous inverse of both the holonomy and the loop encircling direction, analogous to the one discussed below expression (9).

All these results naturally lead to the relations between differences of reciprocal pairs of holonomies and vector or tensor fields located at specific points, namely

(13a) $$\boldsymbol{h}_p - \boldsymbol{h}_p^{-1} = \mathbb{L}_0 \varepsilon_p \big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A}_{p_{\text{fin}}}(\ell_{\text{fin}})\big) + \mathcal{O}(\varepsilon^3)\,,$$

(13b) $$\boldsymbol{h}_{qr} - \boldsymbol{h}_{qr}^{-1} = 2\lambda \mathbb{L}_0^2 \varepsilon_q \varepsilon_r \boldsymbol{F}_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}}) + \mathcal{O}(\varepsilon^3)\,.$$

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[i]It is worth noting that the relation between $\boldsymbol{h}_{\text{loop}}^{-1}$ and $\boldsymbol{F}$ is not calculated according to the standard formalism of differential geometry as in [10, 13]. Instead, the method in [1, 18], explicitly indicating the location and directions of the curvature regarding the loop, is applied. This transforms the problem of the $\boldsymbol{h}_{\text{loop}}^{-1}$ expansion into the multiplication of expansions of the holonomies at vertices of quadrilateral loops accordingly to the formula in (7).

198 The left-hand sides of these relations depend on specific lattice structures. The
199 equally-accurate expansions on the right-hand sides should correspond to the
200 symmetries of these structures. It is the case regarding the expression in (13a)
201 but the right-hand side of (13b) is not invariant under cyclic permutations of
202 edges because $\boldsymbol{F}(\ell_{\text{init}})$ is assigned to a particular vertex.

203 **Theorem 12.** *The invariance under the cyclic permutations of edges is restored*
204 *by replacing the point-related field $\boldsymbol{F}_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}})$ in (13b) with the arithmetical mean*
205 *$\bar{\boldsymbol{F}}_{qr}$ of the point-related fields assigned to all the vertices in a considered loop.*

206 **Proof.** Consider the consecutive expansion of the same loop holonomy, with
207 the loop each time initiated at a different vertex. For instance, regarding the
208 parallelepiped-like quadrilateral loop, compositions of holonomies: $\boldsymbol{h}_q^{-1} \boldsymbol{h}_r^{-1} \boldsymbol{h}_q \boldsymbol{h}_r$,
209 $\boldsymbol{h}_r^{-1} \boldsymbol{h}_q \boldsymbol{h}_r \boldsymbol{h}_q^{-1}$, $\boldsymbol{h}_q \boldsymbol{h}_r \boldsymbol{h}_q^{-1} \boldsymbol{h}_r^{-1}$, and $\boldsymbol{h}_r \boldsymbol{h}_q^{-1} \boldsymbol{h}_r^{-1} \boldsymbol{h}_q$ are the same object, but the vector
210 potential $\boldsymbol{F}_{ab}(0) \dot{\ell}_q^a\big|_{\varepsilon_q=0} \dot{\ell}_r^b\big|_{\varepsilon_r=0}$ in (10) is consecutively related to a different vertex
211 each time. The arithmetical mean of four holonomy compositions is the loop
212 holonomy $\boldsymbol{h}_{qr}^{-1}$. However, the quarter of the sum of different point-related fields
213 is none of these fields. Instead, it is a definition of the arithmetical mean $\bar{\boldsymbol{F}}_{qr}$.
214 The proof regarding a triangular loop is analogous. ∎

215 **Remark 13.** The sum of differently localized vertices in (13a) can be analogously
216 replaced by $2\bar{\boldsymbol{A}}_p := \boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A}_{p_{\text{fin}}}(\ell_{\text{fin}})$.

217 4. Linear and surface symmetrization of the Ashtekar variables

218 At this point, one should recognize the problem with the unsymmetrical outcomes
219 of holonomies' expansions in (13). The expansion in (13b) provides the object
220 which lacks either a triangular or quadrilateral loop symmetry. This object can
221 be easily symmetrized by the arithmetic mean $\bar{\boldsymbol{F}}_{qr}$ of the consecutive expansions
222 at different vertices. As a result, the right-hand sides of (13) become expressed
223 in terms of the nonlocal combinations $\bar{\boldsymbol{A}}_p$ and $\bar{\boldsymbol{F}}_{qr}$ of vector potentials and their
224 curvatures, respectively. However, the Hamiltonian in (1) is written in terms
225 of point-related quantities, not their sums. This difference between local and
226 nonlocal objects is the source of the gap in the regularization procedure in LQG.
227 To fill this gap and to make the regularization in LQG more similar to the
228 canonical quantization in established field theories, one should define a diffeomor-
229 phism-invariant and spatially-symmetric smearing of the connection's degrees of
230 freedom, generalizing the arithmetic means $\bar{\boldsymbol{A}}_p$ and $\bar{\boldsymbol{F}}_{qr}$.

231 **Theorem 14.** *The path-smeared connection's degrees of freedom* [11]:

$$\text{(14a)} \qquad \boldsymbol{\alpha}_p := \int_0^{\mathbb{L}_0 \varepsilon_p} ds \, \dot{\ell}^a(s) \, \boldsymbol{A}_a\big(\ell(s)\big)$$

$$\text{(14b)} \qquad \boldsymbol{\varphi}_{qr} := \int_0^{\mathbb{L}_0 \varepsilon_q} ds \int_0^{\mathbb{L}_0 \varepsilon_r} dt \, \dot{\ell}^a(s) \dot{\ell}^b(t) \, \boldsymbol{F}_{ab}\big(\ell(s,t)\big) \,.$$

are spatially-symmetric and diffeomorphism-invariant along the paths in their definitions.

Proof. The spatial symmetry is evident due to the uniform path-smearing. Any diffeomorphism transformation of these definitions changes only the structure of paths, along which the line integrals are defined, adjusting the connection's degrees of freedom to the transformed geometry. ∎

The symmetric path averaging method in (14a) requires to locate the path's midpoint at the position of the vector potential, and the symmetric surface averaging in (14b) should position the surface's centroid at the curvature's location. Consequently, the point-related connections and curvatures in (1) would be considered as the objects in the middle of spatially-extended structures.

Definition. The symmetric and diffeomorphism-invariant link holonomy representation of $\boldsymbol{A} = \boldsymbol{A}\big(\ell(\mathbb{L}_0 \varepsilon_p/2)\big)$ and the loop holonomy representation of $\boldsymbol{F} = \boldsymbol{F}\big(\ell(\mathbb{L}_0 \varepsilon_q/2, \mathbb{L}_0 \varepsilon_r/2)\big)$ are defined by the relations [11]:

$$\text{(15a)} \qquad \boldsymbol{\mathcal{A}}_p(l_p) := \frac{\boldsymbol{h}_p - \boldsymbol{h}_p^{-1}}{2\mathbb{L}_0 \varepsilon_p} \approx \frac{\boldsymbol{\alpha}_p}{\mathbb{L}_0 \varepsilon_p} \,,$$

$$\text{(15b)} \qquad \boldsymbol{\mathcal{F}}_{qr}(l_q, l_r) := \frac{\boldsymbol{h}_{qr} - \boldsymbol{h}_{qr}^{-1}}{2\lambda \mathbb{L}_0^2 \varepsilon_q \varepsilon_r} \approx \frac{\boldsymbol{\varphi}_{qr}}{\mathbb{L}_0^2 \varepsilon_q \varepsilon_r} \,.$$

Theorem 15. *Both definitions of the group representations on the left-hand sides approximate the algebra representations on the right-hand sides with the quadratic-order precision for a sufficiently small value of ε.*

Proof. By construction, the approximations in these definitions have quadratic-order precision in the case of the holonomies and curvatures that are constant along the paths of integration in (14). To preserve this precision in the general case, it is enough to consider ε sufficiently small that within a ball of radius $\mathbb{L}_0 \varepsilon/2$, the means of fields along paths in (14) are approximable with ε^2 precision by the arithmetical means of these fields at vertices located along considered paths, namely,

$$\text{(16a)} \qquad \boldsymbol{\alpha}_p = \mathbb{L}_0 \varepsilon_p \bar{\boldsymbol{A}}_p + \mathcal{O}\big(\varepsilon^3\big) \,,$$

$$\text{(16b)} \qquad \boldsymbol{\varphi}_{qr} = \mathbb{L}_0^2 \varepsilon_q \varepsilon_r \bar{\boldsymbol{F}}_{qr} + \mathcal{O}\big(\varepsilon^3\big) \,.$$

262 The existence of a sufficiently small ε is easily demonstrable, for instance, by
263 choosing the following value of this regulator,

264
265 (17) $\varepsilon = \sqrt{\max_s \left| \boldsymbol{A}(\ell(s)) - \bar{\boldsymbol{A}} \right|}$ (example of a possible choice) .   ∎

266   5.   Wigner's construction of operators

267 The connection coefficient $\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) = \boldsymbol{A}^I_{p_{\text{init}}}(\ell_{\text{init}})\boldsymbol{t}_I$ is the Lie algebra element,
268 specified by the relation in (6). Corollary, $\bar{\boldsymbol{A}}_p$ and $\boldsymbol{\alpha}_p$ are also the elements of
269 the same Lie algebra. The idea to replace these gauge fields with the related
270 holonomies is inspired by the Wilson loop representation [30] constructed on a
271 piecewise linear lattice. LQG postulates the construction of the combination of
272 links-located Hilbert spaces for the Ashtekar connection operators $\hat{\boldsymbol{A}}$ that are
273 the gauge algebra representations of the symmetry determined by holonomies
274 [4, 24, 25]. However, in this construction, the Hilbert spaces are located regarding
275 non-linear links.

276 **Theorem 16** (Wigner). *Let $U(\boldsymbol{\theta})$ denote a representation of a connected Lie*
277 *group described by a finite set of real continuous parameters θ^I and Hermitian*
278 *generators $\boldsymbol{s}_I$. Any representation of a symmetry transformation of a ray space*
279 *is either a unitary and linear or else antiunitary and antilinear transformation*
280 *of a Hilbert space.*

281 ***Proof of Wigner's theorem.*** See [28, 29].   ∎

282   By expanding $U(\boldsymbol{\theta})$ around a trivial transformation, i.e., the identity, one can
283 focus only on the unitary generators [27]. Consequently, in a finite neighborhood
284 of the identity, one obtains the expansion

285 (18) $U(\boldsymbol{\theta}) = \mathbb{1} + \mathrm{i}\theta^I \boldsymbol{s}_I - \frac{1}{2}\theta^J \theta^K \boldsymbol{s}_J \boldsymbol{s}_K - \frac{\mathrm{i}}{2}\theta^J \theta^K C^I_{JK} \boldsymbol{s}_I + \mathcal{O}(\theta^3) .$

286 Assuming the representation of the same Lie group as in Section 3, C^I_{JK} are the
287 same real structure constants, resulting from the following Lie bracket,

288 (19) $[\boldsymbol{s}_J, \boldsymbol{s}_K] = \mathrm{i}C^I_{JK}\boldsymbol{s}_I .$

289 By comparing the expressions in (6) and (19), one finds the explicit form of the
290 internal representation generators of $\boldsymbol{A}$,

291 (20) $\boldsymbol{t}_I = -\frac{\mathrm{i}}{2}\boldsymbol{s}_I ,$

292 where $\boldsymbol{s}_I$ is Hermitian and unitary.

The next natural question is whether it is possible to compare the formula in (8) with (18). The comparison requires to postulate the following identification,

$$(21) \qquad U(\boldsymbol{\theta}) = \boldsymbol{h}_p^{-1}[\boldsymbol{A}]\,.$$

Theorem 17. *The holonomy of the vector potential* $\boldsymbol{A} := A^I \boldsymbol{t}_I$ *is a Lie group element.*

Proof. Let $\boldsymbol{A}$, $\boldsymbol{A'}$, and so $(\boldsymbol{A} + \boldsymbol{A'})$ be gauge algebra representations. The Lie group element must be the exponential map from the related representation. The explicit calculation of the product

$$
\begin{aligned}
(22) \qquad \boldsymbol{h}_p^{-1}[\boldsymbol{A}]\boldsymbol{h}_p^{-1}[\boldsymbol{A'}] = {}& \mathbb{1} - \mathbb{L}_0 \varepsilon_p \big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}}) \big) \\
& - \frac{1}{2} \big(\mathbb{L}_0 \varepsilon_p \big)^2 \partial_{p_{\text{init}}} \big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}}) \big) \\
& + \frac{1}{2} \big(\mathbb{L}_0 \varepsilon_p \big)^2 \big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}}) \big)^2 + \mathcal{O}\big(\varepsilon^3 \big) \\
= {}& \boldsymbol{h}_p^{-1}[\boldsymbol{A} + \boldsymbol{A'}]\,,
\end{aligned}
$$

demonstrates that the holonomy is a group element. ∎

The application of the holonomy representation to the Hamiltonian in (1) aims to construct a quantum theory on the graph, the edges of which have finite length, linearly dependent on the value of the regularization parameter ε [6]. Therefore, the quadratic-order precision presumed in this article is not postulated only to increase the accuracy of the model assumed in LQG [4, 25]. It seems inevitable for the precise gauge symmetry transformation of the connection algebra representation into the holonomy group representation. This quadratic-order precision is required in the standard construction of states in noncommutative quantum field theories [27].

Theorem 18. *The formula in* (8) *is equivalent to the expression*

$$\boldsymbol{h}_p^{-1}[\boldsymbol{\alpha}] = \mathbb{1} - \boldsymbol{\alpha}_p + \frac{1}{2}\boldsymbol{\alpha}_p \boldsymbol{\alpha}_p + \mathcal{O}\big(\varepsilon^3\big)$$

$$(23) \qquad = \mathbb{1} - \mathbb{L}_0 \varepsilon_p \boldsymbol{A}\big(\ell(\mathbb{L}_0 \varepsilon_p/2)\big) + \frac{1}{2}\big(\mathbb{L}_0 \varepsilon_p\big)^2 \boldsymbol{A}\big(\ell(\mathbb{L}_0 \varepsilon_p/2)\big) \boldsymbol{A}\big(\ell(\mathbb{L}_0 \varepsilon_p/2)\big) + \mathcal{O}\big(\varepsilon^3\big)\,.$$

Moreover, all the terms on the right-hand side have their equivalents in (18).

Proof. Expression (23) is derived by applying the definition in (14a) to (8). Using then the identification of algebra generators in (20), the expansion of the group element $U(\boldsymbol{\theta})$ around the identity in (18), takes the form:

$$(24) \qquad U(\boldsymbol{\alpha}) = \mathbb{1} - \alpha^I \boldsymbol{t}_I + \frac{1}{2}\alpha^J \alpha^K \boldsymbol{t}_J \boldsymbol{t}_K + \frac{1}{2}\alpha^J \alpha^K [\boldsymbol{t}_J, \boldsymbol{t}_K] + \mathcal{O}\big(\alpha^3\big)\,,$$

320 where the $\alpha^I = 2\theta^I$ normalization of parameters were applied to identify the
321 linear terms in (18) and (23). The cubic corrections in (23) and (24) are both
322 of order ε^3, which is evident from (16a). The same formula can be used to
323 demonstrate that the term with Lie brackets is approximately equal to zero,

324
325 $$(25) \qquad \alpha^J \alpha^K [\boldsymbol{t}_J, \boldsymbol{t}_K] = \left(\mathbb{L}_0 \varepsilon_p\right)^2 \left[\bar{\boldsymbol{A}}, \bar{\boldsymbol{A}}\right] + \mathcal{O}(\varepsilon^3) = \mathcal{O}(\varepsilon^3). \qquad \blacksquare$$

326 This result verifies that the holonomy expansion in (8) coincides with the
327 group expansion around identity in (24) up to the terms of order ε^2. It also
328 confirms the internal consistency concerning the equal orientation of both factors
329 in the quadratic term in the holonomy expansion in (8). This orientation along
330 the same link provides the symmetry of the elements inside the Lie brackets in
331 (25), so they are vanishing.
332 Finally, the arithmetic mean $\bar{\boldsymbol{A}}_p$ corresponds to the abstract representation
333 $\boldsymbol{\alpha}_p$ by the formula (16a) under the short link's length (small ε) constraint, e.g.
334 (17). Therefore, both $\bar{\boldsymbol{A}}_p$ and $\boldsymbol{\alpha}_p / \mathbb{L}_0 \varepsilon_p$ are the properly symmetrized candidates
335 for the link-related object that is quantizable into the operator representation
336 of the point-related connection $\boldsymbol{A}\left(\ell(\mathbb{L}_0 \varepsilon_p / 2)\right)$. Its corresponding holonomy repre-
337 sentation $\boldsymbol{\mathcal{A}}_p(l_p)$, is given by formula (15a).

6. Conclusions

339 Demonstrating that the holonomy expansion around a short length of links equals
340 the expansion around identity proves that the links-located Hilbert spaces in LQG
341 satisfy Wigner's theorem [28, 29]. Thus, this quantum theory on a lattice can
342 be defined by the standard Dirac-Heisenberg-DeWitt [14, 15, 20] construction of
343 operators, known as the canonical representation (in quantum mechanics).
344 Moreover, the equality of the analyzed expansions is an alternative demon-
345 stration for the reality requirement of the Ashtekar variables. If the Ashtekar con-
346 nection was imaginary, the related parameter would also be imaginary, $\varepsilon_p \in (0, \mathrm{i})$.
347 The complex connection, composed of real and imaginary terms, is excluded. The
348 imaginary Ashtekar connection would lead to the imaginary length of the graph's
349 edges, hence to a non-physical model.
350 Considering then the real Ashtekar variables, the Ashtekar-Barbero-Holst-
351 Thiemann Hamiltonian [3, 8, 21, 24] in (1) is the correct candidate for quanti-
352 zation after the improved lattice regularization method described in this article.
353 The increased precision in the construction of the lattice-smeared analog (cf. [12])
354 of the Hamiltonian constraint in (1) guarantees that the accuracy of the model
355 after quantization will be comparable with standard quantum field theories.

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