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NONLOCAL HOLONOMY REPRESENTATIONS FOR LIE ALGEBRA-VALUED ASHTEKAR-BARBERO CONNECTION

Jakub Bilski

Institute of Mathematics, University of Zielona Góra
Szafrana 4a, 65–516 Zielona Góra, Poland
e-mail: j.bilski@im.uz.zgora.pl

Abstract

Holonomies of the Ashtekar-Barbero connection can be considered as abstract elements of a Lie group, exponentially mapped from their algebra representation. This idea allows for the definition of the states in loop quantum gravity, which preserve the group symmetry that is equivalent to the Ashtekar-Barbero connection symmetry of Lie algebra. The equivalence of the symmetries requires either the quadratic- or linear-order precision in the expansion of group elements either around a finite value of the expansion parameter or by taking the limit as this parameter approaches zero, respectively. These conditions put different constraints on the holonomy regularization method in loop quantum gravity, where holonomies are expanded around finite values of the related paths' lengths. This article investigates the possibility of increasing the linear-order precision, postulated in canonical loop quantum gravity, into the quadratic order. It demonstrates that the regularization method can be defined more accurately.

Keywords: gauge theory, holonomy, Lie algebra, Lie group, loop quantum gravity, nonlocality.

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1. Motivation

The Ashtekar variables [3] are the representation of the gravitational degrees of freedom that allow the expression of a particular action [21], leading to the Einstein field equations with time gauge [17] in the form of a gauge theory, analogous to the Yang-Mills model [31]. In the case of the real Ashtekar variables, they lead to the Hamiltonian [8, 21], which may be considered a candidate for the classical limit of the background-independent formulation of quantum gravity, called loop quantum gravity (LQG) [4, 24].

It is known that the symmetry transformations of operators, which preserve probabilities at the quantum level, are related to a Lie algebra — cf. [27]. Equal algebra of operators has to describe the symmetries of the classical candidates for field theory operators before quantization. This article analyzes the algebra of the modified candidates forming the canonical pair for a time gauge-fixed simplification of general relativity (GR) [2]. These variables are the real Ashtekar-Barbero connection and the densitized dreibein [4, 8, 25]. Their canonical quantization leads to the model analogous to the DeWitt equation [14], which is not background-independent. The background-independent formalism of LQG is constructed by introducing particular regularization procedures [24, 25], in which the point-located Ashtekar-Barbero connection and its curvature are replaced by different functionals of path-related holonomies. This paper indicates the difference in these procedures that lead to the pair of quantities preserving Lie algebra symmetry with unequal precision. One quantity is an order below the standard symmetry preservation at the quantum level [27]. This article provides the solution to this different precision problem. It also proposes a symmetrization method for point-located Ashtekar variables into path-related equivalents of these variables. These smeared equivalents undergo identical Lie algebra and can be used to regularize path-related canonical variables in LQG into path-related holonomies invariant under Lie group transformations.

2. Gravitational Hamiltonian in the real Ashtekar variables

The total Hamiltonian of the time gauge-fixed simplification of GR expressed in terms of the real Ashtekar variables consists of three terms. Two types of these terms are first-class constraints corresponding to the spatial diffeomorphism invariance and Lie algebra symmetry. The third term, called the Hamiltonian constraint [3, 8, 21, 24], corresponds to the abelian energy gauge and contains two propagating degrees of freedom. This term is Barbero's Hamiltonian for the Ashtekar connection field derived from the Holst action with the momentum degrees of freedom removed from denominators by Thiemann's method. It is defined by the following expression:

$$H := -\frac{4\sigma_E}{\gamma\kappa^2}\int_{\Sigma_t} d^3x N(x)\,\epsilon_{+}^{abc}\Big(F_{ab}^i(x) - (\gamma^2+1)\epsilon_{ijk}K_a^j(x)K_b^k(x)\Big)\Big\{A_c^i(x), \mathbb{V}(\Sigma_t)\Big\}\,, \tag{1}$$

where

$$K_a^j(x) := -\frac{8\,\sigma_E}{\gamma^2\kappa^3}\left\{A_a^j(x), \left\{\int_{\Sigma_t} d^3y\,\epsilon_{+}^{bcd}F_{bc}^k\Big\{A_d^k(y), \mathbb{V}(\Sigma_t)\Big\}, \mathbb{V}(\Sigma_t)\right\}\right\}, \tag{2}$$

$$\mathbb{V}(\Sigma_t) := \int_{\Sigma_t} d^3x\sqrt{|E(x)|}\,, \qquad E := \frac{1}{3!}\epsilon_{abc}^{-}\epsilon^{ijk}E_i^aE_j^bE_k^c\,, \tag{3}$$

and $\sigma_E := \mathrm{sgn}(E)$. The Poisson brackets for canonical fields are specified regarding the metric tensor variables with a time gauge [2] providing the factor $-\gamma\kappa/2$ in front of the definition

$$\{X,Y\} := -\frac{\gamma\kappa}{2}\int d^3x \left(\frac{\delta X}{\delta A_a^i(x)}\frac{\delta Y}{\delta E_i^a(x)} - \frac{\delta X}{\delta E_i^a(x)}\frac{\delta Y}{\delta A_a^i(x)}\right). \tag{4}$$

The symbols ϵ_+^{abc} and ϵ_{abc}^- represent the Levi-Civita tensor density and inverse density, respectively. The scalar γ is the real Barbero-Immirzi parameter [8, 22], and $\kappa := 16\pi G$, where the speed of light is normalized to unity. The quantity N is a Lagrange multiplier, F_{ab}^i is the curvature of the real Ashtekar-Barbero connection field A_a^i, and E_i^a is the densitized dreibein. It represents the momentum canonically conjugated to A_a^i. The Hamiltonian constraint in (1) is the starting point for graph regularization — the procedure directly preceding canonical quantization.

It is also worth bringing up that the choice of the complex Ashtekar variables would lead to rather nonphysical candidates for observables — see the expansion in (24) in which all parameters have to be real. In this case, auxiliary effective methods [9, 26] of reality conditions implementation on the classical phase space would be needed.

Apart from the reality of the Lie algebra-valued variables, it is required that the related generators are either unitary and linear or antiunitary and antilinear [28]. As in the case of the standard operators in quantum mechanics and quantum field theory, they are unitary and linear in general [27]. All of the aforementioned facts encouraged the investigation of the holonomy representation algebra. This article verifies that this regularized representation can be used to form a gauge-invariant equivalent of the Ashtekar-Barbero connection's degrees of freedom. It is then demonstrated that the standard equivalent of the connection's degrees of freedom in LQG, invariant under Lie algebra representation [24, 25], becomes a Lie group representation. However, in the case of the connection field, the invariance of the standard algebra-to-group transformation in LQG is one order less accurate than in the case of the connection's curvature field.

An alternative algebra-to-group map for the Ashtekar connection can solve this problem. This map, unifying the accuracy of the connection and its curvature regularization in LQG to higher precision, is proposed in this article. Finally, Wigner's construction of operators is recalled to demonstrate the relevance of the accuracy of the algebra-to-group maps in quantum theories.

3. HOLONOMY OF THE ASHTEKAR-BARBERO CONNECTION

Definition. The parallel transport of a vector bundle element over the manifold M to another bundle along a smooth oriented path $\ell(s) : [0,1] \to M$ is determined by the expression

$$
\boldsymbol{h}^{-1}_{[0,1]}[\boldsymbol{A}] := \mathcal{P}\exp\Big(-\int_0^1 ds\, \dot{\ell}^a(s)\, \boldsymbol{A}_a\big(\ell(s)\big)\Big). \tag{5}
$$

This object is called the holonomy of the vector potential $\boldsymbol{A} := A^I \boldsymbol{t}_I$ along the path ℓ, where $\boldsymbol{t}_I$ are the generators of a Lie algebra.

Remark 1. The 'inverse notation', $\boldsymbol{h}^{-1}$, follows the convention established in the LQG-related literature [4, 25]. Consequently, $\boldsymbol{h}$ denotes the inverse holonomy.

Remark 2. The Ashtekar-Barbero connection coefficients $A^I \in \mathbb{R}$ are specified by the following normalization of the Lie bracket,

$$
[\boldsymbol{t}_J, \boldsymbol{t}_K] = \frac{1}{2} C^I{}_{JK} \boldsymbol{t}_I \,, \tag{6}
$$

where $C^I{}_{JK} = -C^I{}_{KJ} \in \mathbb{R}$ are the structure constants. By definition, this normalization resolves the issue of reality conditions implementation.

The formalism of LQG (cf. [4, 25]) introduces a graph structure, which allows defining the associated scalar product as a relation between the positions of variables on this graph, hence without using a metric. This technique is based on the framework constructed in [5, 6, 7]. If the real Ashtekar-Barbero connection fields A^i_a, theirs curvature F^i_{ab}, and densitized dreibeins E^a_i in the Hamiltonian constraint in (1) can be rigorously replaced by holonomies and the fluxes of densitized dreibeins[i], then the graph is naturally introduced. Its edges are the paths ℓ in the holonomy definition (5). Hence, the graph's introduction technique is reduced to the inversion of the holonomy-to-connection map in the just mentioned definition.

Theorem 3. *Let the quantity $\mathbb{L}_0$ denote a fiducial length scale. Let also the indices $p, q, r, \dots$ do not indicate any spatial directions but enumerate edges, so the Einstein summation convention is not applied to these edge indices. The holonomy $\boldsymbol{h}^{-1}_p = \mathcal{P}\exp\big(-\int_{l_p} \boldsymbol{A}\big)$, adjusted to a particular graph's edge (of length $l_p := \mathbb{L}_0 \varepsilon_p$), can be expanded around the infinitesimal value of the dimensionless regularization parameter $\varepsilon_p := \varepsilon_{l_p} \in (0,1)$,*

$$
\begin{aligned}
\boldsymbol{h}^{\mp 1}_p[\boldsymbol{A}] = \mathbb{1} &\mp \mathbb{L}_0 \varepsilon_p \boldsymbol{A}_a(0)\, \dot{\ell}^a_p\big|_{\varepsilon_p=0} + \frac{1}{2}\big(\mathbb{L}_0 \varepsilon_p\big)^2 \boldsymbol{A}_a(0) \boldsymbol{A}_b(0)\, \dot{\ell}^a_p\big|_{\varepsilon_p=0} \dot{\ell}^b_p\big|_{\varepsilon_p=0} \\
&\mp \frac{1}{2}\big(\mathbb{L}_0 \varepsilon_p\big)^2 \partial_a \boldsymbol{A}_b(0)\, \dot{\ell}^a_p\big|_{\varepsilon_p=0} \dot{\ell}^b_p\big|_{\varepsilon_p=0} + \mathcal{O}\big(\varepsilon^3\big)\,.
\end{aligned} \tag{7}
$$

Proof. See [1, 16, 18]. ∎

[i] Their construction (see e.g. [4, 25]) is not relevant to the analysis presented in this article.

Remark 4. It is worth emphasizing that all the elements in (7) are oriented in the same direction, $\dot{\ell}_p\big|_{\varepsilon_p=0}$, tangent to $\ell_p(s)$ at the point where the edge of a zeroth length is localized. This tangent direction determines a one-dimensional system and defines the spatial basis for the expansion in (7). For instance, the term $\partial_a \boldsymbol{A}_b(0)$ in the lower line of this expression is the coefficient of the directional derivative of a one-form, where both quantities are oriented toward the direction of $\dot{\ell}_p\big|_{\varepsilon_p=0}$. This tangent can be easily completed to the orthonormal Euclidean basis defined by the Frenet-Serret frame [19, 23].

Remark 5. The left-hand side of (7) is a non-local, path-related (edge-related) object, while the right-hand side is a local point-related (vertex-related) expression.

Remark 6. The quadratic terms in the expansion on the right-hand side of (7) contain derivatives of connections, neither being fields nor holonomies. These terms have to be removed to find a connection-to-holonomy map.

Theorem 7. *The equivalent of the expansion on the right-hand side of* (7)*, which is as nonlocal as the holonomy and does not contain derivatives of connections, equals*

$$\text{(8)} \quad \boldsymbol{h}_p^{\mp 1}[\boldsymbol{A}] = \mathbb{1} \mp \frac{1}{2}\mathbb{L}_0\,\varepsilon_p\big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A}_{p_{\text{fin}}}(\ell_{\text{fin}})\big) + \frac{1}{2}\big(\mathbb{L}_0\,\varepsilon_p\,\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}})\big)^2 + \mathcal{O}\big(\varepsilon^3\big)\,.$$

Proof. In order to remove the problematic derivatives in (7), the following approximation of the directional derivative along the curve ℓ_p has been applied:

$$\text{(9)} \qquad \mathbb{L}_0\,\varepsilon_p \partial_a \boldsymbol{A}_b(0)\,\dot{\ell}^a_p\big|_{\varepsilon_p=0}\dot{\ell}^b_p\big|_{\varepsilon_p=0} = \boldsymbol{A}_a(1)\,\dot{\ell}^a_p\big|_{\varepsilon_p=1} - \boldsymbol{A}_a(0)\,\dot{\ell}^a_p\big|_{\varepsilon_p=0} + \mathcal{O}\big(\varepsilon^2\big)\,.$$

Its welcome side effect is the invariance of the right-hand side of (8) under the operation $\boldsymbol{h}_p^{\mp 1}[\boldsymbol{A}] \to \boldsymbol{h}_{p^{-1}}^{\pm 1}[\boldsymbol{A}]$, which replaces the $\ell_{\text{init}} := \ell(0)$ point with $\ell_{\text{fin}} := \ell(1)$. ■

Analogously, one can expand the loop holonomy $\boldsymbol{h}_{qr}^{-1}$ defined around the parallelepiped-like loop composed of two pairs of identical opposite edges, ℓ_q and ℓ_r.

Theorem 8. *Let the loop initiate along the outgoing edge ℓ_q, hence along the path cooriented with ℓ_q and let it return along the outgoing edge ℓ_r, hence along the path cooriented with ℓ_q^{-1}. As a result, the loop holonomy, describing the parallel transport of a vector, factorizes into the following composition of holonomies, $\boldsymbol{h}_{qr}^{-1} = \boldsymbol{h}_q^{-1}\boldsymbol{h}_r^{-1}\boldsymbol{h}_q\boldsymbol{h}_r$ (the 'inverse notation' holds in the whole article). The outcome of the expansion of the parallelepiped-like loop holonomy around the infinitesimal value of regularization parameters ε_q and ε_r is*

$$h^{\mp 1}_{\square qr}[A] = \mathbb{1} \mp \mathbb{L}_0^2 \varepsilon_q \varepsilon_r F_{ab}(0)\, \dot{\ell}^a_q\big|_{\varepsilon_q=0}\, \dot{\ell}^b_r\big|_{\varepsilon_r=0} + \mathcal{O}\big(\varepsilon^3\big)\,, \tag{10}$$

where $F_{ab} := F^I_{ab} t_I$ are the spatial coefficients of the curvature of A.

Proof. See [1, 18][i]. ■

Instead of quadrilateral loops, the LQG model requires the analogous expansion for triangular loops [4, 25].

Theorem 9. *Let the neighboring pair of edges ℓ_q and ℓ_r be the part of a triangular loop. The expansion analogous to* (8) *leads to the expression similar to* (10),

$$h^{\mp 1}_{\triangle qr}[A] = \mathbb{1} \mp \frac{1}{2}\mathbb{L}_0^2 \varepsilon_q \varepsilon_r F_{ab}(0)\, \dot{\ell}^a_q\big|_{\varepsilon_q=0}\, \dot{\ell}^b_r\big|_{\varepsilon_r=0} + \mathcal{O}\big(\varepsilon^3\big)\,. \tag{11}$$

Proof. See [1, 18] or simply consider the triangular path formed of the neighboring pair of edges in some parallelepiped-like quadrilateral and the diagonal of this quadrilateral. ■

Corollary 10. *These formulas can be simplified into a single expression analogous to* (8),

$$h^{\mp 1}_{qr}[A] = \mathbb{1} \mp \lambda \mathbb{L}_0^2 \varepsilon_q \varepsilon_r F_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}}) + \mathcal{O}\big(\varepsilon^3\big)\,. \tag{12}$$

Parameter λ takes value $1/2$ in the case of a triangular loop and 1 in the case of a parallelepiped-like quadrilateral.

Remark 11. It is worth emphasizing that formulas (10) and (11) require the approximation in (9) of the directional derivative along a curve. It is also worth indicating that the closure of the path in the definition of the loop holonomy entails the identification $F(\ell_{\text{init}}) = F(\ell_{\text{fin}})$. Therefore, the formula in (12) is already invariant under the simultaneous inverse of both the holonomy and the loop encircling direction, analogous to the one discussed below expression (9).

All these results naturally lead to the relations between differences of reciprocal pairs of holonomies and vector or tensor fields located at specific points, namely

$$h_p - h_p^{-1} = \mathbb{L}_0 \varepsilon_p \big(A_{p_{\text{init}}}(\ell_{\text{init}}) + A_{p_{\text{fin}}}(\ell_{\text{fin}})\big) + \mathcal{O}\big(\varepsilon^3\big)\,, \tag{13a}$$

$$h_{qr} - h_{qr}^{-1} = 2\lambda \mathbb{L}_0^2 \varepsilon_q \varepsilon_r F_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}}) + \mathcal{O}\big(\varepsilon^3\big)\,. \tag{13b}$$

[i]It is worth noting that the relation between h^{-1}_{loop} and F is not calculated according to the standard formalism of differential geometry as in [10, 13]. Instead, the method in [1, 18], explicitly indicating the location and directions of the curvature regarding the loop, is applied. This transforms the problem of the h^{-1}_{loop} expansion into the multiplication of expansions of the holonomies at vertices of quadrilateral loops accordingly to the formula in (7).

The left-hand sides of these relations depend on specific lattice structures. The equally-accurate expansions on the right-hand sides should correspond to the symmetries of these structures. It is the case regarding the expression in (13a) but the right-hand side of (13b) is not invariant under cyclic permutations of edges because $\boldsymbol{F}(\ell_{\text{init}})$ is assigned to a particular vertex.

Theorem 12. *The invariance under the cyclic permutations of edges is restored by replacing the point-related field* $\boldsymbol{F}_{q_{\text{init}} r_{\text{init}}}(\ell_{\text{init}})$ *in* (13b) *with the arithmetical mean* $\bar{\boldsymbol{F}}_{qr}$ *of the point-related fields assigned to all the vertices in a considered loop.*

Proof. Consider the consecutive expansion of the same loop holonomy, with the loop each time initiated at a different vertex. For instance, regarding the parallelepiped-like quadrilateral loop, compositions of holonomies: $\boldsymbol{h}_q^{-1}\boldsymbol{h}_r^{-1}\boldsymbol{h}_q\boldsymbol{h}_r$, $\boldsymbol{h}_r^{-1}\boldsymbol{h}_q\boldsymbol{h}_r\boldsymbol{h}_q^{-1}$, $\boldsymbol{h}_q\boldsymbol{h}_r\boldsymbol{h}_q^{-1}\boldsymbol{h}_r^{-1}$, and $\boldsymbol{h}_r\boldsymbol{h}_q^{-1}\boldsymbol{h}_r^{-1}\boldsymbol{h}_q$ are the same object, but the vector potential $\boldsymbol{F}_{ab}(0)\,\dot{\ell}_q^a\big|_{\varepsilon_q=0}\dot{\ell}_r^b\big|_{\varepsilon_r=0}$ in (10) is consecutively related to a different vertex each time. The arithmetical mean of four holonomy compositions is the loop holonomy $\boldsymbol{h}_{qr}^{-1}$. However, the quarter of the sum of different point-related fields is none of these fields. Instead, it is a definition of the arithmetical mean $\bar{\boldsymbol{F}}_{qr}$. The proof regarding a triangular loop is analogous. ∎

Remark 13. The sum of differently localized vertices in (13a) can be analogously replaced by $2\bar{\boldsymbol{A}}_p := \boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A}_{p_{\text{fin}}}(\ell_{\text{fin}})$.

4. Linear and surface symmetrization of the Ashtekar variables

At this point, one should recognize the problem with the unsymmetrical outcomes of holonomies' expansions in (13). The expansion in (13b) provides the object which lacks either a triangular or quadrilateral loop symmetry. This object can be easily symmetrized by the arithmetic mean $\bar{\boldsymbol{F}}_{qr}$ of the consecutive expansions at different vertices. As a result, the right-hand sides of (13) become expressed in terms of the nonlocal combinations $\bar{\boldsymbol{A}}_p$ and $\bar{\boldsymbol{F}}_{qr}$ of vector potentials and their curvatures, respectively. However, the Hamiltonian in (1) is written in terms of point-related quantities, not their sums. This difference between local and nonlocal objects is the source of the gap in the regularization procedure in LQG.

To fill this gap and to make the regularization in LQG more similar to the canonical quantization in established field theories, one should define a diffeomorphism-invariant and spatially-symmetric smearing of the connection's degrees of freedom, generalizing the arithmetic means $\bar{\boldsymbol{A}}_p$ and $\bar{\boldsymbol{F}}_{qr}$.

Theorem 14. *The path-smeared connection's degrees of freedom* [11]:

$$\boldsymbol{\alpha}_p := \int_0^{\mathbb{L}_0\varepsilon_p} ds\, \dot{\ell}^a(s)\, \boldsymbol{A}_a\big(\ell(s)\big) \tag{14a}$$

$$\boldsymbol{\varphi}_{qr} := \int_0^{\mathbb{L}_0\varepsilon_q} ds \int_0^{\mathbb{L}_0\varepsilon_r} dt\, \dot{\ell}^a(s)\, \dot{\ell}^b(t)\, \boldsymbol{F}_{ab}\big(\ell(s,t)\big)\,. \tag{14b}$$

are spatially-symmetric and diffeomorphism-invariant along the paths in their definitions.

Proof. The spatial symmetry is evident due to the uniform path-smearing. Any diffeomorphism transformation of these definitions changes only the structure of paths, along which the line integrals are defined, adjusting the connection's degrees of freedom to the transformed geometry. ∎

The symmetric path averaging method in (14a) requires to locate the path's midpoint at the position of the vector potential, and the symmetric surface averaging in (14b) should position the surface's centroid at the curvature's location. Consequently, the point-related connections and curvatures in (1) would be considered as the objects in the middle of spatially-extended structures.

Definition. The symmetric and diffeomorphism-invariant link holonomy representation of $\boldsymbol{A} = \boldsymbol{A}\big(\ell(\mathbb{L}_0\varepsilon_p/2)\big)$ and the loop holonomy representation of $\boldsymbol{F} = \boldsymbol{F}\big(\ell(\mathbb{L}_0\varepsilon_q/2, \mathbb{L}_0\varepsilon_r/2)\big)$ are defined by the relations [11]:

$$\boldsymbol{\mathcal{A}}_p(l_p) := \frac{\boldsymbol{h}_p - \boldsymbol{h}_p^{-1}}{2\mathbb{L}_0\varepsilon_p} \approx \frac{\boldsymbol{\alpha}_p}{\mathbb{L}_0\varepsilon_p}\,, \tag{15a}$$

$$\boldsymbol{\mathcal{F}}_{qr}\big(l_q, l_r\big) := \frac{\boldsymbol{h}_{qr} - \boldsymbol{h}_{qr}^{-1}}{2\lambda\mathbb{L}_0^2\varepsilon_q\varepsilon_r} \approx \frac{\boldsymbol{\varphi}_{qr}}{\mathbb{L}_0^2\varepsilon_q\varepsilon_r}\,. \tag{15b}$$

Theorem 15. *Both definitions of the group representations on the left-hand sides approximate the algebra representations on the right-hand sides with the quadratic-order precision for a sufficiently small value of ε.*

Proof. By construction, the approximations in these definitions have quadratic-order precision in the case of the holonomies and curvatures that are constant along the paths of integration in (14). To preserve this precision in the general case, it is enough to consider ε sufficiently small that within a ball of radius $\mathbb{L}_0\varepsilon/2$, the means of fields along paths in (14) are approximable with ε^2 precision by the arithmetical means of these fields at vertices located along considered paths, namely,

$$\boldsymbol{\alpha}_p = \mathbb{L}_0\varepsilon_p\bar{\boldsymbol{A}}_p + \mathcal{O}\big(\varepsilon^3\big)\,, \tag{16a}$$

$$\boldsymbol{\varphi}_{qr} = \mathbb{L}_0^2\varepsilon_q\varepsilon_r\bar{\boldsymbol{F}}_{qr} + \mathcal{O}\big(\varepsilon^3\big)\,. \tag{16b}$$

The existence of a sufficiently small ε is easily demonstrable, for instance, by choosing the following value of this regulator,

$$\varepsilon = \sqrt{\max_s \left|\boldsymbol{A}\big(\ell(s)\big) - \bar{\boldsymbol{A}}\right|} \qquad \text{(example of a possible choice)}\,. \tag{17}$$

■

5. WIGNER'S CONSTRUCTION OF OPERATORS

The connection coefficient $\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) = \boldsymbol{A}^I_{p_{\text{init}}}(\ell_{\text{init}})\boldsymbol{t}_I$ is the Lie algebra element, specified by the relation in (6). Corollary, $\bar{\boldsymbol{A}}_p$ and $\boldsymbol{\alpha}_p$ are also the elements of the same Lie algebra. The idea to replace these gauge fields with the related holonomies is inspired by the Wilson loop representation [30] constructed on a piecewise linear lattice. LQG postulates the construction of the combination of links-located Hilbert spaces for the Ashtekar connection operators $\hat{\boldsymbol{A}}$ that are the gauge algebra representations of the symmetry determined by holonomies [4, 24, 25]. However, in this construction, the Hilbert spaces are located regarding non-linear links.

Theorem 16 (Wigner)**.** *Let $U(\boldsymbol{\theta})$ denote a representation of a connected Lie group described by a finite set of real continuous parameters θ^I and Hermitian generators $\boldsymbol{s}_I$. Any representation of a symmetry transformation of a ray space is either a unitary and linear or else antiunitary and antilinear transformation of a Hilbert space.*

Proof of Wigner's theorem. See [28, 29]. ■

By expanding $U(\boldsymbol{\theta})$ around a trivial transformation, i.e., the identity, one can focus only on the unitary generators [27]. Consequently, in a finite neighborhood of the identity, one obtains the expansion

$$U(\boldsymbol{\theta}) = \mathbb{1} + \mathrm{i}\theta^I \boldsymbol{s}_I - \frac{1}{2}\theta^J\theta^K \boldsymbol{s}_J\boldsymbol{s}_K - \frac{\mathrm{i}}{2}\theta^J\theta^K C^I{}_{JK}\boldsymbol{s}_I + \mathcal{O}\big(\theta^3\big)\,. \tag{18}$$

Assuming the representation of the same Lie group as in Section 3, $C^I{}_{JK}$ are the same real structure constants, resulting from the following Lie bracket,

$$[\boldsymbol{s}_J, \boldsymbol{s}_K] = \mathrm{i}C^I{}_{JK}\boldsymbol{s}_I\,. \tag{19}$$

By comparing the expressions in (6) and (19), one finds the explicit form of the internal representation generators of $\boldsymbol{A}$,

$$\boldsymbol{t}_I = -\frac{\mathrm{i}}{2}\boldsymbol{s}_I\,, \tag{20}$$

where $\boldsymbol{s}_I$ is Hermitian and unitary.

The next natural question is whether it is possible to compare the formula in (8) with (18). The comparison requires to postulate the following identification,

$$U(\boldsymbol{\theta}) = \boldsymbol{h}_p^{-1}[\boldsymbol{A}]\,. \tag{21}$$

Theorem 17. *The holonomy of the vector potential $\boldsymbol{A} := A^I \boldsymbol{t}_I$ is a Lie group element.*

Proof. Let $\boldsymbol{A}$, $\boldsymbol{A'}$, and so $(\boldsymbol{A}+\boldsymbol{A'})$ be gauge algebra representations. The Lie group element must be the exponential map from the related representation. The explicit calculation of the product

$$\begin{aligned}\boldsymbol{h}_p^{-1}[\boldsymbol{A}]\boldsymbol{h}_p^{-1}[\boldsymbol{A'}] &= \mathbb{1} - \mathbb{L}_0\varepsilon_p\big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}})\big)\\ &\quad - \frac{1}{2}\big(\mathbb{L}_0\varepsilon_p\big)^2\partial_{p_{\text{init}}}\big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}})\big)\\ &\quad + \frac{1}{2}\big(\mathbb{L}_0\varepsilon_p\big)^2\big(\boldsymbol{A}_{p_{\text{init}}}(\ell_{\text{init}}) + \boldsymbol{A'}_{p_{\text{init}}}(\ell_{\text{init}})\big)^2 + \mathcal{O}\big(\varepsilon^3\big)\\ &= \boldsymbol{h}_p^{-1}[\boldsymbol{A}+\boldsymbol{A'}]\,,\end{aligned} \tag{22}$$

demonstrates that the holonomy is a group element. ■

The application of the holonomy representation to the Hamiltonian in (1) aims to construct a quantum theory on the graph, the edges of which have finite length, linearly dependent on the value of the regularization parameter ε [6]. Therefore, the quadratic-order precision presumed in this article is not postulated only to increase the accuracy of the model assumed in LQG [4, 25]. It seems inevitable for the precise gauge symmetry transformation of the connection algebra representation into the holonomy group representation. This quadratic-order precision is required in the standard construction of states in noncommutative quantum field theories [27].

Theorem 18. *The formula in* (8) *is equivalent to the expression*

$$\begin{aligned}\boldsymbol{h}_p^{-1}[\boldsymbol{\alpha}] &= \mathbb{1} - \boldsymbol{\alpha}_p + \frac{1}{2}\boldsymbol{\alpha}_p\boldsymbol{\alpha}_p + \mathcal{O}\big(\varepsilon^3\big)\\ &= \mathbb{1} - \mathbb{L}_0\varepsilon_p\boldsymbol{A}\big(\ell(\mathbb{L}_0\varepsilon_p/2)\big) + \frac{1}{2}\big(\mathbb{L}_0\varepsilon_p\big)^2\boldsymbol{A}\big(\ell(\mathbb{L}_0\varepsilon_p/2)\big)\boldsymbol{A}\big(\ell(\mathbb{L}_0\varepsilon_p/2)\big) + \mathcal{O}\big(\varepsilon^3\big)\,.\end{aligned} \tag{23}$$

Moreover, all the terms on the right-hand side have their equivalents in (18).

Proof. Expression (23) is derived by applying the definition in (14a) to (8). Using then the identification of algebra generators in (20), the expansion of the group element $U(\boldsymbol{\theta})$ around the identity in (18), takes the form:

$$U(\boldsymbol{\alpha}) = \mathbb{1} - \alpha^I\boldsymbol{t}_I + \frac{1}{2}\alpha^J\alpha^K\boldsymbol{t}_J\boldsymbol{t}_K + \frac{1}{2}\alpha^J\alpha^K[\boldsymbol{t}_J,\boldsymbol{t}_K] + \mathcal{O}\big(\alpha^3\big)\,, \tag{24}$$

where the $\alpha^I = 2\theta^I$ normalization of parameters were applied to identify the linear terms in (18) and (23). The cubic corrections in (23) and (24) are both of order ε^3, which is evident from (16a). The same formula can be used to demonstrate that the term with Lie brackets is approximately equal to zero,

$$\alpha^J \alpha^K [\boldsymbol{t}_J, \boldsymbol{t}_K] = \big(\mathbb{L}_0 \varepsilon_p\big)^2 \big[\bar{\boldsymbol{A}}, \bar{\boldsymbol{A}}\big] + \mathcal{O}\big(\varepsilon^3\big) = \mathcal{O}\big(\varepsilon^3\big)\,. \tag{25}$$ ■

This result verifies that the holonomy expansion in (8) coincides with the group expansion around identity in (24) up to the terms of order ε^2. It also confirms the internal consistency concerning the equal orientation of both factors in the quadratic term in the holonomy expansion in (8). This orientation along the same link provides the symmetry of the elements inside the Lie brackets in (25), so they are vanishing.

Finally, the arithmetic mean $\bar{\boldsymbol{A}}_p$ corresponds to the abstract representation $\boldsymbol{\alpha}_p$ by the formula (16a) under the short link's length (small ε) constraint, e.g. (17). Therefore, both $\bar{\boldsymbol{A}}_p$ and $\boldsymbol{\alpha}_p/\mathbb{L}_0\varepsilon_p$ are the properly symmetrized candidates for the link-related object that is quantizable into the operator representation of the point-related connection $\boldsymbol{A}\big(\ell(\mathbb{L}_0\varepsilon_p/2)\big)$. Its corresponding holonomy representation $\boldsymbol{\mathcal{A}}_p(l_p)$, is given by formula (15a).

6. Conclusions

Demonstrating that the holonomy expansion around a short length of links equals the expansion around identity proves that the links-located Hilbert spaces in LQG satisfy Wigner's theorem [28, 29]. Thus, this quantum theory on a lattice can be defined by the standard Dirac-Heisenberg-DeWitt [14, 15, 20] construction of operators, known as the canonical representation (in quantum mechanics).

Moreover, the equality of the analyzed expansions is an alternative demonstration for the reality requirement of the Ashtekar variables. If the Ashtekar connection was imaginary, the related parameter would also be imaginary, $\varepsilon_p \in (0, \mathrm{i})$. The complex connection, composed of real and imaginary terms, is excluded. The imaginary Ashtekar connection would lead to the imaginary length of the graph's edges, hence to a non-physical model.

Considering then the real Ashtekar variables, the Ashtekar-Barbero-Holst-Thiemann Hamiltonian [3, 8, 21, 24] in (1) is the correct candidate for quantization after the improved lattice regularization method described in this article. The increased precision in the construction of the lattice-smeared analog (cf. [12]) of the Hamiltonian constraint in (1) guarantees that the accuracy of the model after quantization will be comparable with standard quantum field theories.

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