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ON THE PURITIES OF α-IDEALS IN ORDERED SEMIGROUPS

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Abstract

One of the important principles for characterizing ordered semigroups into classes is the purities of ideals. It was Changphas, and Sanborisoot studied this concept in ordered semigroups. They described left (resp., right) weakly regular ordered semigroups using left (resp., right) pure of two-sided ideals. Our work weakens their study by introducing new kinds of purities for α-ideals. We characterize our new version of purities; moreover, a class of ordered semigroups is characterized by this concept.

Keywords: α-ideal, β-pure, pure ideal, ordered semigroup, regularities.

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1. Introduction

When analyzing the structure of various algebraic systems, the idea of ideals is crucial. The purities are one of the essential characteristics of ideals. Numerous algebraic systems employed the purities of ideals as a subject of research. The

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purities and the purely primness of ideals in semigroups were firstly begun in 1989 by Ahsan and Takahashi (see [1]). In 2009, Bashir and Shabir [14] expanded these ideas to ternary semigroups. They discussed weakly pure ideals and examined their properties. The set of all proper purely prime ideals in ternary semigroups is topologized was also demonstrated. In 2012, Shabir and Anjum [15] started the concept of purities for k-ideals in hemirings. They explained left pure and right pure of k-ideals. Additionally, the authors used left (resp., right) pure k-ideals to characterize left (resp., right) k-weakly regular hemirings. In Γ-semirings, Jagatap [5] introduced the purities of k-ideals. The author began by studying the properties of pure k-ideals.

When Changphas and Sanborisoot established the ideas of left (resp., right) pure, weakly pure, and purely prime ideals in ordered semigroups, the research of Ahsan and Takahashi was generalized to a higher level. Changphas and Sanborisoot [4] characterized left (resp., right) pure ideals and used them to describe left (resp., right) weakly regular ordered semigroups. In the same year, the same authors extended the investigations of Bashir and Shabir (see [13]). The new purities versions were initially introduced in ordered hemirings by Pibaljommee and Palakawong na Ayutthaya (see [12]). They defined the properties of quasi-pure and bi-pure of k-ideals. The authors used two new types of pure k-ideals to characterize ordered hemirings: those that are both left and right weakly ordered k-regular and those that are k-regular. It seems that the trend of quasi-purity and bi-purity is significant. Palakawong na Ayutthaya and Pibaljommee started these notions for ideals and k-ideals in ordered semirings in [10] and [11], respectively. They characterized left (resp., right quasi-, bi-) pure for both ideals and k-ideals. The characterizations of left (resp., right) weakly ordered (k-)regular semirings were provided by using left (resp., right quasi-, bi-) purity of ideals and k-ideals.

In addition to being crucial for characterizing algebraic systems, the purities of ideals are also essential for understanding hyperalgebraic systems. In ordered Krasner hyperrings, left (resp., right) purity of left (resp., right) hyperideals was attempted to be defined by Omidi *et al.* in 2007 (see [9]). They investigated the properties of left (resp., right) pure hyperideals. The authors also developed the idea of purely maximum hyperideals and examined their associated features. Changphas and Davvaz [3] investigated the properties of left (resp., right) pure hyperideals in ordered semihypergroups in a similar approach. Their characterizations were present. The authors also started the concept of left (resp., right) weakly pure hyperideals. They showed that the set of all purely prime hyperideals forms topology on an appropriate set. Recently, Shao *et al.* [16] studied the concept of left (resp., right) pure and purely prime hyperideals in ordered semihyperrings.

As mentioned earlier, we can see from the motivation that the purities concept is typically defined for two-sided ideals. Additionally, the variety of ideal purities seems to be limited. In the current study, we introduce the concept of β-purity of α-ideals in ordered semigroups utilizing the concept of α-ideals. By a suitable ideal, we characterize β-pure α-ideals. In addition, we describe γ-regular ordered semigroups by β-pure α-ideals.

2. Preliminaries

The basic notions used in this paper are recalled in this section. The notion of words is present in the first subsection. The second subsection provides the concept of ordered semigroups and their auxiliary results. The readers can be found some of these basic notations in [17].

2.1. Word

Throughout this paper, let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of all natural numbers, and $\mathbb{N}_0$ to be the set of all nonnegative integers, that is, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let X be a nonempty set.

We denote the free monoid over the set X by X^*. The identity of X^* is called the empty word, denoted by ε. The notation $X^+ := X^* \setminus \{\varepsilon\}$ denotes the free semigroup over the set X. Let $\alpha \in X^*$. We define $\mathrm{var}(\alpha)$ and $|\alpha|$ to be the set of variables occurring in the word α and the length of α, respectively. We denote by X^*_{f} the set $\{\alpha \in X^* : \mathrm{var}(\alpha) = X\}$. For any $n \in \mathbb{N}$, we let

$$X^*_n := \{\alpha \in X^* : |\alpha| = n\}$$

and

$$X^*_{\mathrm{f},n} := \{\alpha \in X^*_{\mathrm{f}} : |\alpha| = n\}.$$

Any $\alpha \in X^*$ and $n \in \mathbb{N}$, we denote $\underbrace{\alpha \cdots \alpha}_{n \text{ times}}$ by α^n. Let $\alpha \in X^+$, and $R := \{i \to j : i, j \in X \text{ and } i \neq j\}$. We define α_R to be the word obtained by replacing i in the word α by j for all $i \to j \in R$. In particular, if $R = \{i_1 \to j_1, \ldots, i_k \to j_k\}$ is finite, then we may write α_R by $\alpha_{i_1 \to j_1, \ldots, i_k \to j_k}$.

Let $m, n \in \mathbb{N}$ be such that $m \leq n$. We define $[m, n] := \{i \in \mathbb{N} : m \leq i \leq n\}$. For any $k \in \mathbb{N}$, we denote the set $\{[m, n] : 1 \leq m \leq n \leq k\}$ by $\mathsf{Int}(k)$.

For any $n \in \mathbb{N}$, we say that an $I \in \mathsf{I}_n$ if I is either an m-element subset of $\mathsf{Int}(n)$, where $1 \leq m \leq n$, such that

$$I = \big\{[b_1, c_1], \ldots, [b_m, c_m] : [b_j, c_j] \cap [b_k, c_k] = \emptyset \text{ for all } 1 \leq j < k \leq m\big\},$$

or $I = \{\emptyset\}$. We denote $\mathsf{I} := \bigcup_{n \in \mathbb{N}} \mathsf{I}_n$. Note that $\mathsf{I}_k \subset \mathsf{I}_{k+1}$ for all $k \in \mathbb{N}$.

Let $\alpha = a_1 \cdots a_n \in \{0,1\}^*_n$ and $I = \{[b_1, c_1], \ldots, [b_m, c_m]\} \in \mathsf{I}_n$, where $n \in \mathbb{N}$. We define a new word α_I obtained by α and I by

$$\alpha_I := a_1 \cdots a_{b_1-1} 0 a_{c_1+1} \cdots a_{b_m-1} 0 a_{c_m+1} \cdots a_n.$$

If $I = \{\emptyset\}$, then we define $\alpha_I := \alpha$. We set $a_1 \cdots a_{b_1-1} := \varepsilon$ if $b_1 = 1$ and $a_{c_{m+1}} \cdots a_n := \varepsilon$ if $c_m = n$.

For any $\alpha \in \{0,1\}^+$, a mapping $1_\alpha \colon \{0,1\}^+ \to \{0,1\}^+$ assigns for any $\beta = b_1 \cdots b_n \in \{0,1\}^*_n$, where $n \in \mathbb{N}$, by $1_\alpha(\beta) := \alpha_1 \cdots \alpha_n$ such that

$$\alpha_i = \begin{cases} 0 & \text{if } b_i = 0, \\ \alpha & \text{if } b_i = 1, \end{cases}$$

where $1 \leq i \leq n$. In particular, for any $n \in \mathbb{N}$, we denote $1^n_\alpha(\alpha)$ by $\alpha^{(n)}$. We note that $\alpha^{(0)} = \alpha$.

Let $\alpha \in \{0,1\}^+$. We define

$$\mathsf{minor}(\alpha) := \left\{\beta \in \{0,1\}^+ : (\beta^{(m)})_I = \alpha \text{ for some } m \in \mathbb{N}_0 \text{ and } I \in \mathsf{I}\right\}.$$

Let $\alpha, \beta \in \{0,1,2\}^+$ such that $|\alpha| = m$ and $|\beta| = n$, where $m, n \in \mathbb{N}$. Suppose that $\beta = b_1 \cdots b_n$. We define the word $r_\alpha(\beta)$ obtained by β with respect to α by $r_\alpha(\beta) := \beta$ if there is no $i \in \{1, \ldots, n-m+1\}$ such that $b_i \cdots b_{i+m-1} = \alpha_{1\to 2}$, otherwise,

$$r_\alpha(\beta) := b_1 \cdots b_{i-1} 2 b_{i+m} \cdots b_n.$$

In particular, $r_\alpha(\beta) := 2b_{1-m} \cdots b_n$ and $r_\alpha(\beta) := b_1 \cdots b_{i+m-2} 2$ if $i = 1$ and $i = b_{i+m-1}$, respectively. For any $k \in \mathbb{N}_0$, we define $r^k_\alpha(\beta) = r_\alpha(r^{k-1}_\alpha(\beta))$, where $r^k_\alpha(\beta) := \beta$ if $k = 0$.

Let $\alpha \in \{0,1\}^+$. We define $\langle \alpha \rangle := \{\alpha\}$ if $|\alpha| = 1$, otherwise, $\langle \alpha \rangle := \{\alpha\} \cup \{1^j : 1 \leq j \leq |\alpha| - 1\}$. For any $\alpha_1, \ldots, \alpha_k \in \{0,1\}^+$, where $k \in \mathbb{N}$, we denote $\langle \alpha_1 \rangle \times \cdots \times \langle \alpha_k \rangle$ by $\Pi^k_{i=1} \langle \alpha_i \rangle$.

Let $\alpha \in \{1,2\}^*_n$, where $n \in \mathbb{N}$. Suppose that $\alpha = a_1 \cdots a_n$. For any $1 \leq i \leq n$, we define

$$\pi_i(\alpha) = \begin{cases} \langle \alpha_{2\to 0} \rangle & \text{if } a_i = 1, \\ \{2\} & \text{if } a_i = 2. \end{cases}$$

We denote $\pi_1(\alpha) \times \cdots \times \pi_n(\alpha)$ by $\pi(\alpha)$.

Any k-ary tuple $\mathbf{x} = (x_1, \ldots, x_k)$ of $\{0,1,2\}^+$ induces a word assigned by $\underline{\mathbf{x}} := x_1 \cdots x_k$.

For any $k \in \mathbb{N}$ and $\alpha \in \{0,1\}^*_{\mathrm{f}}$, we define the set $\mathsf{comp}_k(\alpha)$ to be the set

$$\left\{(\alpha_1, \ldots, \alpha_k) \in (\{0,1\}^* \setminus \{1\}^*)^k : \underline{\mathbf{x}} \in \mathsf{minor}(\alpha) \text{ for all } \mathbf{x} \in \Pi^k_{i=1} \langle \alpha_i \rangle\right\}.$$

Let $\alpha \in \{0,1\}^+$ and $k \in \mathbb{N}$. We define the set

$$\mathsf{K}_\alpha^{(k)} := \{(\alpha_1, \dots, \alpha_k) \in \mathsf{comp}_k(\alpha) : \alpha \in \mathsf{minor}(\alpha_1 \cdots \alpha_k)\}.$$

The set $\mathsf{K}_\alpha^{(k)}$ is a tool for characterizing all classes of ordered semigroups which we will present in the following subsection.

2.2. Ordered semigroup

An *ordered semigroup* is an algebraic structure $\langle S, \cdot, \leq \rangle$ of type $(2;2)$ such that $\langle S, \cdot \rangle$ is a semigroup, and $\langle S, \leq \rangle$ is a partially ordered set such that $\leq$ is compatible with the binary operation $\cdot$. That is, for any $x, y \in S$ such that $x \leq y$, we have that $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for all $z \in S$.

From now on, we denote an ordered semigroup $\langle S, \cdot, \leq \rangle$ with the bold letter $\mathbf{S}$ of its universe set. Moreover, we denote the product $x \cdot y$ by xy.

Let $\mathbf{S}$ be an ordered semigroup, and A, B and C subsets of S. Define the sets AB and $(C]$ as follows: $AB := \emptyset$ if A or B is empty, otherwise,

$$AB := \{ab : a \in A \text{ and } b \in B\}$$

and

$$(C] := \{x \in S : x \leq c \text{ for some } c \in C\}.$$

The following lemma is important in investigating ordered semigroups.

Lemma 1 [6]**.** *Let $\mathbf{S}$ be an ordered semigroup, and A and B nonempty subsets of S. Then the following statements valid:*

1. $A \subseteq (A]$ *and* $((A]] = (A]$;
2. $A \subseteq B$ *implies* $(A] \subseteq (B]$;
3. $A(B], (A]B, (A]\,(B] \subseteq (AB]$;
4. $(A \cup B] = (A] \cup (B]$.

In 2013, Towwun and Changphas [18] introduced the notion of α-ideals in ordered semigroups as a generalization of one-sided ideals. Miccoli and Pondělíček, who first proposed the idea of α-ideals in semigroups, were the inspiration for Towwun and Changphas' research (see [8]). It is, therefore, preferable to recollect a few concepts related to the idea of α-ideals in ordered semigroups.

Let $\mathbf{S}$ be an ordered semigroup, and $\alpha = a_1 \cdots a_n = \{0,1\}_n^*$, where $n \in \mathbb{N}_0$. For any $A \subseteq S$, we define $f_\alpha^A \colon P(S) \to P(S)$ by $f_\alpha^A(B) := \emptyset$ if $n = 0$, otherwise, $f_\alpha^A(B) := C_1 \cdots C_n$, where

$$C_i = \begin{cases} A & \text{if } a_i = 0, \\ B & \text{if } a_i = 1, \end{cases}$$

for all $B \subseteq S$ and $1 \leq i \leq n$. In particular, if $B = \{b\}$, then we write $f_\alpha^A(b)$ instead of $f_\alpha^A(\{b\})$. Here $P(S)$ is the set of all subsets of S.

Miccoli and Pondělíček gave the result below in terms of semigroups, but we can also present it in terms of ordered semigroups.

Lemma 2 [8]**.** *Let* $\mathbf{S}$ *be an ordered semigroup,* A *and* B *nonempty subsets of* S*, and* $\alpha, \beta \in \{0,1\}^*$*. Then the statements holds:*

1. $f_\alpha^S(A) \subseteq f_\alpha^S(B)$*, where* $A \subseteq B$;
2. $f_{\alpha\beta}^S(A) = f_\alpha^S(A) f_\beta^S(A)$.

The result that follows is simple to acquire.

Corollary 3. *Let* $\mathbf{S}$ *be an ordered semigroup,* A, B *nonempty subsets of* S*, and* $\alpha \in \{0,1\}^*$*. Then* $f_\alpha^A(S) \subseteq f_\alpha^B(S)$ *whenever* $A \subseteq B$.

Proof. We observe that $f^S_{\alpha_{0\to1,1\to0}}(A) = f_\alpha^A(S)$ and $f^S_{\alpha_{0\to1,1\to0}}(B) = f_\alpha^B(S)$. By applying Lemma 2(1), we have $f_\alpha^A(S) = f^S_{\alpha_{0\to1,1\to0}}(A) \subseteq f^S_{\alpha_{0\to1,1\to0}}(B) = f_\alpha^B(S)$. ■

Following is a definition of α-ideals.

Definition [18]**.** Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S such that $(A] = A$ is said to be an *α-ideal* of $\mathbf{S}$ if $AA \subseteq A$ and $f_\alpha^S(A) \subseteq A$.

It is not difficult to see that any left, right, bi-, and (m,n)-ideal is a 01-, 10-, 101-, and 1^m01^n-ideal, respectively (see [17]).

Towwun and Changphas proved the following result.

Theorem 4 [18]**.** *Let* $\mathbf{S}$ *be an ordered semigroup,* A *a nonempty subset of* S *and* $\alpha \in \{0,1\}^* \setminus \{1\}^*$ *such that* $|\alpha| \geq 2$*. Then the smallest* α*-ideal of* $\mathbf{S}$ *containing* A *is of the form*

$$\left(A \cup \cdots \cup A^{|\alpha|-1} \cup f_\alpha^S(A)\right].$$

We denote the smallest α-ideal of $\mathbf{S}$ containing A by $\langle A \rangle_\alpha$. For the convenience, let $\langle A \rangle_\alpha := S$ if $\alpha = 0$. In particular, if $A = \{a\}$, where $a \in S$, then we write $\langle a \rangle_\alpha$ instead of $\langle \{a\} \rangle_\alpha$.

An ordered semigroup $\mathbf{S}$ is said to be *α-regular* if any $a \in S$, we have $a \in \left(f_\alpha^S(a)\right]$. The concept of α-regular ordered semigroups can describe all classes of ordered semigroups determined by a linear inequation. Tiprachot *et al.* [17] characterized the concept of α-regular ordered semigroups in 2022.

Theorem 5 [17]**.** *Let* $\mathbf{S}$ *be an ordered semigroup, and* $\alpha \in \{0,1\}^*_{\mathrm{f}}$*. Suppose that* $(\alpha_1, \ldots, \alpha_k) \in \mathsf{K}_\alpha^{(k)}$*, where* $k \geq 2$*, and* $\alpha_1, \ldots, \alpha_k \in \{0,1\}^* \setminus \{1\}^*$*. Then the following statements are equivalent:*

1. $\mathbf{S}$ *is* α*-regular;*
2. $A_1 \cap \cdots \cap A_k \subseteq (A_1 \cdots A_k]$ *for any* α_i*-ideal* A_i*, where* $1 \leq i \leq k$*.*

It, then, turns out that the above generalizes, for example, Theorem 3.2 and Lemma 4.1 in [2] and Theorem 2 in [7].

3. Results

The idea of purities of α-ideals in ordered semigroups is introduced and described in our final section. We also characterize a class of ordered semigroups using this concept. Before we proceed, a notion from the previous section needs to be adjusted to make it more helpful in identifying the purities.

Let $\mathbf{S}$ be an ordered semigroup, and $\alpha = a_1 \cdots a_n = \{0, 1, 2\}^*_n$, where $n \in \mathbb{N}_0$. For any $A \subseteq S$, we define $f^A_\alpha \colon P(S) \times P(S) \to P(S)$ by $f^A_\alpha(B, C) := \emptyset$ if $n = 0$, otherwise, $f^A_\alpha(B, C) := D_1 \cdots D_n$, where

$$D_i = \begin{cases} A & \text{if } a_i = 0, \\ B & \text{if } a_i = 1, \\ C & \text{if } a_i = 2, \end{cases}$$

for all $B, C \subseteq S$ and $1 \leq i \leq n$.

The following consequence is obtained immediately by the above definition.

Lemma 6. *Let* $\mathbf{S}$ *be an ordered semigroup, and* $\alpha, \beta \in \{0, 1, 2\}^+$*. Then* $f^S_{\alpha\beta}(A, B) = f^S_\alpha(A, B) f^S_\beta(A, B)$ *for any* $A, B \subseteq S$*.*

Moreover, Definition 2.2 can be restated as follows.

Definition. Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S such that $(A] = A$ is said to be an *α-ideal* of $\mathbf{S}$ if $AA \subseteq A$ and $f^S_\alpha(A, S) \subseteq A$.

Now, we are ready to define the concept of purities of α-ideals.

Definition. Let $\mathbf{S}$ be an ordered semigroup, $\alpha \in \{0, 1\}^+ \setminus \{1\}^+$, and $\beta \in \{1, 2\}^+$. An α-ideal A of $\mathbf{S}$ is *β-pure* if $a \in \left(f^S_\beta(a, A)\right]$ for any $a \in A$.

The following result demonstrates the relationship between different purities of an α-ideal in ordered semigroups.

Proposition 7. *Let* $\mathbf{S}$ *be an ordered semigroup,* $\alpha \in \{0, 1\}^+ \setminus \{1\}^+$ *and* $\beta \in \{0, 1, 2, \}^+$*. Suppose that* B *is an* α*-ideal of* $\mathbf{S}$*. Then*

$$f^S_\beta(A, B) \subseteq f^S_{r^k_\alpha(\beta)}(A, B)$$

for any $A \subseteq S$ *and* $k \in \mathbb{N}_0$*.*

Proof. Let $A, B \subseteq S$. The proof is achieved if $k = 0$. We let $P(k)$ to be the statement: $f^S_\beta(A, B) \subseteq f_{r^k_\alpha(\beta)}(A, B)$. We show this statement by induction on k. Let us suppose more that $\alpha = a_1 \cdots a_m$ and $\beta = b_1 \cdots b_n$. It is clear that $f^S_\beta(A, B) \subseteq f_{r_\alpha(\beta)}(A, B)$ whenever $m > n$. Assume that $m \leq n$. If $r_\alpha(\beta) = \beta$, then $f^S_\beta(A, B) \subseteq f_{r_\alpha(\beta)}(A, B)$. Suppose that there exists $i \in \{1, \ldots, n - m + 1\}$ such that $b_i \cdots b_{i+m-1} = \alpha_{1\to 2}$. Then

$$
\begin{aligned}
f^S_\beta(A, B) &= f^S_{b_1\cdots b_n}(A, B) \\
&= f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{b_i\cdots b_{i+m-1}}(A, B) f^S_{b_{i+m}\cdots b_n}(A, B) \\
&= f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{\alpha_{1\to 2}}(A, B) f^S_{b_{i+m}\cdots b_n}(A, B) \\
&= f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{\alpha}(B, A) f^S_{b_{i+m}\cdots b_n}(A, B) \\
&= f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{\alpha}(B, S) f^S_{b_{i+m}\cdots b_n}(A, B) \\
&\subseteq f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{1}(B, S) f^S_{b_{i+m}\cdots b_n}(A, B) \qquad \text{(since } B \text{ is an } \alpha\text{-ideal)} \\
&= f^S_{b_1\cdots b_{i-1}}(A, B) f^S_{2}(A, B) f^S_{b_{i+m}\cdots b_n}(A, B) \\
&= f^S_{b_1\cdots b_{i-1}2b_{i+m}\cdots b_n}(A, B) \\
&= f^S_{r_\alpha(\beta)}(A, B).
\end{aligned}
$$

This shows that $P(1)$ is true. Assume that $P(k)$ is true. That is, $f^S_\beta(A, B) \subseteq f^S_{r^k_\alpha(\beta)}(A, B)$. Suppose that $r^k_\alpha(\beta) = c_1 \cdots c_l$ for some $l \in \mathbb{N}$. If $l < m$, then $f^S_{r^k_\alpha(\beta)}(A, B) = f^S_{r^{k+1}_\alpha(\beta)}(A, B)$. This implies $f^S_\beta(A, B) \subseteq f^S_{r^{k+1}_\alpha(\beta)}(A, B)$. Suppose that $m \leq l$. If $r^k_\alpha(\beta) = r^{k+1}_\alpha(\beta)$, then we have that $f^S_\beta(A, B) \subseteq f^S_{r^{k+1}_\alpha(\beta)}(A, B)$. Suppose that there exists $i \in \{1, \ldots, l - m + 1\}$ such that $c_i \cdots c_{i+m-1} = \alpha_{1\to 2}$. Then

$$
\begin{aligned}
f^S_{r^k_\alpha(\beta)}(A, B) &= f^S_{c_1\cdots c_l}(A, B) \\
&= f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{c_i\cdots c_{i+m-1}}(A, B) f^S_{c_{i+m}\cdots c_l}(A, B) \\
&= f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{\alpha_{1\to 2}}(A, B) f^S_{c_{i+m}\cdots c_l}(A, B) \\
&= f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{\alpha}(B, A) f^S_{c_{i+m}\cdots c_l}(A, B) \\
&= f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{\alpha}(B, S) f^S_{c_{i+m}\cdots c_l}(A, B) \\
&\subseteq f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{1}(B, S) f^S_{c_{i+m}\cdots c_l}(A, B) \qquad \text{(since } B \text{ is an } \alpha\text{-ideal)} \\
&= f^S_{c_1\cdots c_{i-1}}(A, B) f^S_{2}(A, B) f^S_{c_{i+m}\cdots c_l}(A, B) \\
&= f^S_{c_1\cdots c_{i-1}2c_{i+m}\cdots c_l}(A, B) \\
&= f^S_{r^{k+1}_\alpha(\beta)}(A, B).
\end{aligned}
$$

This shows that $P(k+1)$ is true. By mathematical induction, we obtain that

$f^S_\beta(A,B) \subseteq f^S_{r^k_\alpha(\beta)}(A,B)$ for all $k \in \mathbb{N}$. Altogether, we have that, $f^S_\beta(A,B) \subseteq f^S_{r^k_\alpha(\beta)}(A,B)$ for any $A,B \subseteq S$ and $k \in \mathbb{N}_0$. ■

Lemma 8. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\alpha \in \{1,2\}^*$. *Then*

$$f^S_\alpha\left(\bigcup_{w\in\langle\alpha_{2\to 0}\rangle} f^S_w(A,B), B\right) = \bigcup_{\mathbf{x}\in\pi(\alpha)} f^S_{\underline{\mathbf{x}}}(A,B)$$

for any $A,B \subseteq S$.

Proof. Suppose that $|\alpha| = n$, where $n \in \mathbb{N}$, such that $\alpha = a_1 \cdots a_n$. Let $A,B \subseteq S$. Assume that $y \in f^S_\alpha\Big(\bigcup_{w\in\langle\alpha_{2\to 0}\rangle} f^S_w(A,B), B\Big)$. Then $y \in y_1 \cdots y_n$, where $y_i = B$ if $a_i = 2$ and $y_i = f^S_w(A,B)$ for some $w \in \langle\alpha_{2\to 0}\rangle$ if $a_i = 1$. This means that $y \in f^S_{w_1}(A,B)\cdots f^S_{w_n}(A,B)$ such that

$$w_i \in \begin{cases} \langle\alpha_{2\to 0}\rangle & \text{if } a_i = 1, \\ \{2\} & \text{if } a_i = 2. \end{cases}$$

We observe that $(w_1,\dots,w_n) \in \pi(\alpha)$. Thus

$$y \in f^S_{w_1\cdots w_n}(A,B) \subseteq \bigcup_{\mathbf{x}\in\pi(\alpha)} f^S_{\underline{\mathbf{x}}}(A,B).$$

Conversely, let $y \in \bigcup_{\mathbf{x}\in\pi(\alpha)} f^S_{\underline{\mathbf{x}}}(A,B)$. Then $y \in f^S_{\underline{\mathbf{x}}}(A,B)$ for some $\mathbf{x} \in \pi(\alpha)$. Since $\mathbf{x} \in \pi(\alpha)$, we see that $\mathbf{x} = (x_1,\dots,x_n) \in \pi_1(\alpha)\times\cdots\times\pi_n(\alpha)$. If there exists $i \in \{1,\dots,n\}$ such that $a_i = 1$, then $x_i \in \pi_i(\alpha) = \langle\alpha_{2\to 0}\rangle$. Thus,

$$\begin{aligned} y \in f^S_{\underline{\mathbf{x}}}(A,B) &= f^S_{x_1}(A,B)\cdots f^S_{x_n}(A,B) = f^S_\alpha\left(f^S_{x_i}(A,B), B\right) \\ &\subseteq f^S_\alpha\left(\bigcup_{w\in\langle\alpha_{2\to 0}\rangle} f^S_w(A,B), B\right). \end{aligned}$$

If there is no $i \in \{1,\dots,n\}$ such that $a_i = 1$, then

$$\begin{aligned} y \in f^S_{\underline{\mathbf{x}}}(A,B) &= f^S_{x_1}(A,B)\cdots f^S_{x_n}(A,B) = B^n \\ &= f^S_\alpha\left(\bigcup_{w\in\langle\alpha_{2\to 0}\rangle} f^S_w(A,B), B\right). \end{aligned}$$

This shows that $\bigcup_{\mathbf{x}\in\pi(\alpha)} f^S_{\underline{\mathbf{x}}}(A,B) \subseteq f^S_\alpha\Big(\bigcup_{w\in\langle\alpha_{2\to 0}\rangle} f^S_w(A,B), B\Big)$. Therefore, we complete the proof. ■

We define $\mathsf{D} \subseteq \{0,1\}^+ \times \{1,2\}^+$ as follows

$$\mathsf{D} := \{(\alpha, \beta) : \text{ for any } \mathbf{x} \in \pi(\beta) \text{ there exist } n \in \mathbb{N}_0 \text{ and } I \in \mathsf{I} \text{ with } r_\alpha^n(\underline{\mathbf{x}_I}) = \beta\}.$$

We can describe β-pure of α-ideals by the following theorem based on the notion of the set D.

Theorem 9. *Let* $\mathbf{S}$ *be an ordered semigroup,* $A \subseteq S$*,* $\alpha \in \{0,1\}^+ \smallsetminus \{1\}^+$*, and* $\beta \in \{1,2\}^+$*. Suppose that* A *is an* α*-ideal of* $\mathbf{S}$ *and* $(\alpha, \beta) \in \mathsf{D}$*. Then the following statement are equivalent:*

1. A *is* β*-pure;*
2. $B \cap A \subseteq \left(f_\beta^S(B, A)\right]$ *for all* $\beta_{2\to 0}$*-ideal* B *of* $\mathbf{S}$*.*

Proof. (1) $\Rightarrow$ (2) Let B be an $\beta_{2\to 0}$-ideal of $\mathbf{S}$ and $x \in B \cap A$. Since A is β-pure, $x \in \left(f_\beta^S(x, A)\right] \subseteq \left(f_\beta^S(B, A)\right]$. This shows that $B \cap A \subseteq \left(f_\beta^S(B, A)\right]$.

(2) $\Rightarrow$ (1) Let $x \in A$. Then,

$$\langle x \rangle_{\beta_{2\to 0}} = \left(\bigcup_{w \in \langle \beta_{2\to 0}\rangle} f_w^S(x, S)\right] = \left(\bigcup_{w \in \langle \beta_{2\to 0}\rangle} f_w^S(x, A)\right]$$

is a $\beta_{2\to 0}$-ideal of $\mathbf{S}$ containing x. By assumption, we have that

$$\begin{aligned}
x &\in \left(\bigcup_{w \in \langle \beta_{2\to 0}\rangle} f_w^S(x, A)\right] \cap A \\
&\subseteq \left(f_\beta^S\left(\left(\bigcup_{w \in \langle \beta_{2\to 0}\rangle} f_w^S(x, A)\right], A\right)\right] \\
&\subseteq \left(f_\beta^S\left(\bigcup_{w \in \langle \beta_{2\to 0}\rangle} f_w^S(x, A), A\right)\right] \\
&= \left(\bigcup_{\mathbf{x} \in \pi(\beta)} f_{\underline{\mathbf{x}}}^S(x, A)\right] && \text{(by Lemma 8)} \\
&= \bigcup_{\mathbf{x} \in \pi(\beta)} \left(f_{\underline{\mathbf{x}}}^S(x, A)\right].
\end{aligned}$$

Thus, $x \in \left(f_{\underline{\mathbf{x}}}^S(x, A)\right]$ for some $\mathbf{x} \in \pi(\beta)$. Since $(\alpha, \beta) \in \mathsf{D}$, we have that $r_\alpha^n(\underline{\mathbf{x}_\mathbf{I}}) = \beta$ for some $n \in \mathbb{N}_0$ and $I \in \mathsf{I}$. By Proposition 7, $x \in \left(f_{\underline{\mathbf{x}_\mathbf{I}}}^S(x, A)\right] \subseteq \left(f_{r_\alpha^n(\underline{\mathbf{x}_\mathbf{I}})}^S(x, A)\right] = \left(f_\beta^S(x, A)\right]$. Therefore, we have that A is β-pure. ■

In our final result, we characterize a class of ordered semigroups using the notion of purities of α-ideals. We establish a notion to describe this class of ordered semigroups.

Let $\beta \in \{1,2\}^* \setminus \{1\}^*$ such that $\beta = b_1 \cdots b_n$ for some $n \in \mathbb{N}$. For any $\alpha \in \{0,1\}^* \setminus \{1\}^*$, we define an n-tuple $\beta(\alpha)$ obtained by α and β by $(\gamma_1, \ldots, \gamma_n)$, where

$$\gamma_i = \begin{cases} \beta_{2\to 0} & \text{if } b_i = 1, \\ \alpha & \text{if } b_i = 2, \end{cases}$$

for all $1 \leq i \leq n$.

Theorem 10. *Let* $\mathbf{S}$ *an ordered semigroup,* $\alpha \in \{0,1\}^+ \setminus \{1\}^+$*,* $\beta \in \{1,2\}^+_{\mathrm{f}}$ *such that* $|\beta| = n$ *for some* $n \geq 2$ *and* $\gamma \in \{0,1\}^+_{\mathrm{f}}$*. Suppose that* $(\alpha, \beta) \in \mathsf{D}$ *and* $\beta(\alpha) = (\gamma_1, \ldots, \gamma_n) \in \mathsf{K}^{(n)}_\gamma$*. Then the following statements are equivalent:*

1. $\mathbf{S}$ *is* γ*-regular;*
2. *every* α*-ideal* A *of* $\mathbf{S}$ *is* β*-pure.*

Proof. (1) $\Rightarrow$ (2). Let A be an α-ideal and B a $\beta_{2\to 0}$-ideal of $\mathbf{S}$. Since $\mathbf{S}$ is γ-regular, $\beta(\alpha) \in \mathsf{K}^{(n)}_\gamma$ and Theorem 5, we have $C_1 \cap \cdots \cap C_n \subseteq (C_1 \cdots C_n]$ for any γ_i-ideal C_i of $\mathbf{S}$, where

$$C_i = \begin{cases} B & \text{if } \gamma_i = \beta_{2\to 0}, \\ A & \text{if } \gamma_i = \alpha, \end{cases}$$

for all $1 \leq i \leq n$. That is, $A \cap B \subseteq (C_1 \cdots C_n]$, where $\beta = b_1 \cdots b_n$ and

$$C_i = \begin{cases} B & \text{if } b_i = 1, \\ A & \text{if } b_i = 2, \end{cases}$$

for all $1 \leq i \leq n$. Therefore, $A \cap B \subseteq \left(f^S_\beta(B, A) \right]$. By Theorem 9, we have that A is β-pure.

(2) $\Rightarrow$ (1). Let A be an α-ideal and B a $\beta_{2\to 0}$-ideal of $\mathbf{S}$. Then $A \cap B \subseteq \left(f^S_\beta(B, A) \right]$. We note that $f^S_\beta(B, A) = C_1 \cdots C_n$, where

$$C_i = \begin{cases} B & \text{if } \gamma_i = \beta_{2\to 0}, \\ A & \text{if } \gamma_i = \alpha, \end{cases}$$

for all $1 \leq i \leq n$. By Theorem 5, $\mathbf{S}$ is a γ-regular ordered semigroup. ■

4. Conclusion

This study introduces the idea of β-pure α-ideals in ordered semigroups. As we can see, there are several kinds of purities of α-ideals; hence, our study may provide some insight into researching other forms of purities. In our first main consequence, we establish a relationship between various purities of an α-ideal. With the aid of an appropriate ideal, β-pure α-ideals are described. The characterization of a class of ordered semigroups is given using the idea of β-pure of α-ideals. We can ask if we can examine something similar for other algebraic systems through this work.

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