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DECOMPOSABLE AND STRONGLY DECOMPOSABLE ALMOST DISTRIBUTIVE LATTICES

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Abstract

The concepts of Decomposable and Strongly decomposable almost distributive lattices are introduced. Various properties of prime, minimal prime and annihilator ideals of a decomposable ADL are furnished. Some characterizations for an ideal in a strongly decomposable ADL to be totally ordered are provided.

Keywords: Almost Distributive Lattice (ADL), prime ideal, minimal prime ideal, maximal ideal, annihilator ideal, decomposable ADL, strongly decomposable ADL.

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1. Introduction

With an idea of bringing common abstraction to most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao in [13]. An Almost Distributive Lattice (ADL) is an algebra $(R, \wedge, \vee)$ of type (2,2) which

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satisfies almost all the properties of a distributive lattice except possibly the commutativity of $\wedge$ and $\vee$ and the right distributivity of $\vee$ over $\wedge$. It is interesting to note that many results which are valid for distributive lattices are also valid for ADLs, even though the techniques of the proofs in case of ADLs are slightly different, for the reason that the operations $\wedge$ and $\vee$ are not commutative. The concept of an ideal was introduced in an ADL analogues to that in a distributive lattice. A study of some properties of prime ideals, minimal prime ideals and annihilator ideals in an ADL is carried out in [6, 9, 10, 11] and [14]. Recently different types of ideals in an ADL are introduced and studied in [7] and [8].

Motivated by the characterizations of Stone lattices obtained by Gratzer and Schmidt [2], Cornish [1] and Pawar [4] characterized distributive lattices with the least element 0 in which every prime ideal contains a unique minimal prime ideal and called such lattices normal lattices. This work inspired Xinmin Lu *et al.* [3] to introduce the concept of decomposable lattices by replacing the word normality by decomposability. A distributive lattice L with the least element 0 is said to be decomposable if for any two incomparable elements $a, b \in L$, there exists $x, y \in L$ such that $a = x \vee (a \wedge b)$ and $b = y \vee (a \wedge b)$ and $x \wedge y = 0$. Further prime, minimal prime and special ideals in decomposable lattices are studied explicitly in [3]. Analogues to normal distributive lattices, normal ADLs are defined and studied by Rao and Ravikumar [11]. Hence it worth to introduce decomposability in an ADL. The work of Xinmin Lu *et al.* [3] motivates us to study some more properties of prime ideals, minimal prime ideals and annihilator ideals in a decomposable ADL.

In this paper we introduce decomposable ADL. Note that our definition is slightly different from that for lattices. Examples of decomposable ADL and non decomposable ADL are furnished. Various properties of prime, minimal prime and annihilator ideals in a decomposable ADL are proved. The concluding section deals with strongly decomposable ADL. Various characterizations for an ideal in a strongly decomposable ADL to be totally ordered are obtained.

2. Preliminaries

In this section we recall some definitions and results from references that we need for the text of this paper.

Recall from [13], an almost distributive lattice (ADL) with 0 is an algebra $\langle R, \wedge, \vee, 0\rangle$ of type (2, 2, 0) satisfying the following conditions for all $x, y, z \in R$.

(L1) $x \vee 0 = x$

(L2) $0 \wedge x = 0$

(L3) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(L4) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(L5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(L6) $(x \vee y) \wedge y = y$.

Throughout this paper, R stands for an ADL $\langle R, \wedge, \vee, 0\rangle$ with zero unless otherwise mentioned. For any $a, b \in R$, define $a \leq b$ if $a = a \wedge b$ (or equivalently $a \vee b = b$), then $\leq$ is a partial ordering on R. For any $m \in L$, m is maximal with respect to partial ordering $\leq$ if and only if $m \vee x = m$ for all $x \in L$. A non-empty subset I of R is said to be an ideal if it satisfies the conditions:

(i) $a, b \in I \Rightarrow a \vee b \in I$ and

(ii) $a \in I, x \in R \Rightarrow a \wedge x \in I$.

The set of all ideals of R is denoted by $\mathfrak{I}(R)$. Note that for $a, b \in R$ and $I \in \mathfrak{I}(R)$ we have $a \wedge b \in I$ if and only if $b \wedge a \in I$. The set $S = \{a \wedge x : x \in R\}$ is the smallest ideal of R containing a. We denote it by $S = (a]$ and it is called the principal ideal generated by a. An ideal I of R is said to be proper if $I \neq R$. A proper ideal P of R is said to be prime if for any $x, y \in R$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. A prime ideal P of R is called minimal if there exists no prime ideal Q of R such that $Q \subset P$. Every prime ideal of R contains a minimal prime ideal. A proper ideal M of R is said to be maximal if it is not properly contained in any proper ideal of R. R is called a pm-ADL if every prime ideal is contained in a unique maximal of R (see [5]). For any $x \in R$, an x-maximal ideal is an ideal of R which is maximal with respect to not containing x. For any non-empty subset A of an ADL R, define $A^* = \{x \in R : a \wedge x = 0, \text{ for all } a \in A\}$. Then A^* is called the annihilator of A. For any non-empty subset A of R, A^* is an ideal of R and $A^* \cap A \subseteq \{0\}$. An ideal I of R is said to be aniihilator ideal if $I = I^{**}$. For $x \in R$, the annulet $\{x\}^*$ of x is defined as $\{x\}^* = \{y \in R : x \wedge y = 0\}$. An element $x \in R$ is said to be dense in R if, $\{x\}^* = \{0\}$. Two distinct ideals I and J in R are said to be co-maximal if $I \vee J = R$. Dually we can define filter, prime filter, minimal prime filter and maximal filter. Let R be an ADL with maximal elements and A be the set of all maximal elements in R. An element $x \in R$ is said to be dual dense if $\{x\}^+ = A$, where $\{x\}^+ = \{y \in R : x \vee y = m \text{ for some } m \in A\}$. Any two distinct filters F, G in R are said to be weakly co-maximal, if $F \vee G$ contains a dual dense element in R. An ADL, R with maximal elements is called a dually semi-complemented if, for each non zero element $x \in R$ there exists a non-maximal element $y \in R$ such that $x \vee y$ is maximal. For an ADL R, let $\mathfrak{M}(R)$ denotes the set of all minimal prime ideals of R.

Proposition 1 [13]**.** *For $a, b \in R$ we have, $(a \wedge b] = (a] \cap (b]$ and $(a \vee b] = (a] \vee (b]$.*

Proposition 2 [13]**.** *The set $\mathfrak{I}(R)$ of all ideals of R is a complete distributive lattice with the least elements $\{0\}$ and the greatest element R under set inclusion, in which for any $I, J \in \mathfrak{I}(R)$, $I \cap J$ is the infimum of I and J and the supremum is given by $I \vee J = \{i \vee j : i \in I,\ j \in J\}$.*

Proposition 3 [6]**.** *Let X be a non-empty subset of R such that $0 \notin X$. Then $\bigcup\{\{a\}^* : a \in X\} = \bigcap\{M \in \mathfrak{M}(R) \ : \ M \cap X = \phi\}$.*

Proposition 4 [9]**.** *For any nonempty subset of A of R, A^* is an ideal of R.*

Proposition 5 [9]**.** *For any nonempty subset I of R we have $I^* \cap I^{**} = (0]$.*

Proposition 6 [9]**.** *The set of all annihilator ideal of R forms a complete Boolean algebra.*

Proposition 7 [12]**.** *P is a minimal prime ideal of R if and only if $R \setminus P$ is a maximal prime filter of R.*

Proposition 8 [12]**.** *Every prime ideal of R contains a minimal prime ideal.*

Proposition 9 [6]**.** *For $a \in R$, any a-maximal ideal in R is prime.*

Proposition 10 [6]**.** *For any subset A of R, $A^* = \bigcap\{M \in \mathfrak{M}(R) \ : \ A \nsubseteq M\}$.*

3. Decomposable ADL

For $I, J \in \mathfrak{I}(R)$ we write $I \parallel J$ when the ideals I and J are incomparable in the poset $(\mathfrak{I}(R), \subseteq)$. At the outset we define a decomposable ADL.

Definition. An ADL R is said to be decomposable if for any $I \parallel J$, where $I, J \in \mathfrak{I}(R)$, there exists $x \in I \setminus J$ and $y \in J \setminus I$ such that $x \wedge y = 0$.

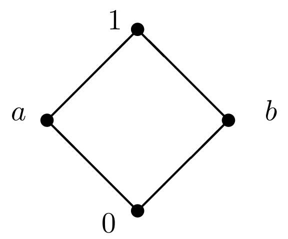

Figure 1. Decomposable ADL.

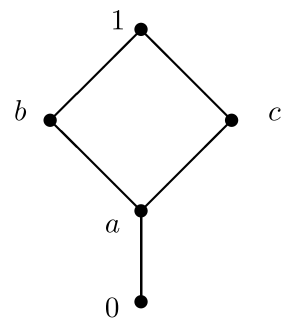

Figure 2. Non Decomposable ADL.

Example 11. An ADL represented by the Hasse Digramme as in Figure 1 is decomposable while an ADL represented in by the Hasse Digramme as in Figure 2 is not decomposable.

Example 12. Let X be any non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define $\wedge$ and $\vee$ on X by $x \wedge y = y$, $x \vee y = x$ if $x \neq x_0$ and $x_0 \wedge x = x_0$, $x_0 \vee x = x$. Then $(X, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element (see [13]). It can be verified that this ADL is decomposable.

A necessary and sufficient condition for a proper ideal to be prime in a decomposable *ADL* is proved in the following theorem.

Theorem 13. *In a decomposable ADL R, a proper ideal P is prime if and only if the set $\{I \in \mathfrak{I}(R) : I \supseteq P\}$ is totally ordered.*

Proof. Let a proper ideal P of R be such that the set $\{I \in \mathfrak{I}(R) | I \supseteq P\}$ is a totally ordered subset of $\mathfrak{I}(R)$. Let P be not prime. Then there exist $a, b \in R$ such that $a \wedge b \in P$ with $a \notin P$ and $b \notin P$. As $P \vee (a] \supset P$ and $P \vee (b] \supset P$, by assumption $P \vee (a] \subseteq P \vee (b]$ or $P \vee (b] \subseteq P \vee (a]$. Assume $P \vee (a] \subseteq P \vee (b]$. As $a \wedge b \in P$, we get

$$\begin{aligned} P \vee (a \wedge b] &= P \vee [(a] \cap (b]], && \text{(by Result 1)} \\ &= [P \vee (a]] \cap [P \vee (b]], && \text{(by Result 2)} \\ &= P \vee (a]. && \text{(since } P \vee (a] \subseteq P \vee (b]). \end{aligned}$$

This shows that $a \in P$; a contradiction. Hence P must be a prime ideal.

For converse, let $\{I \in \mathfrak{I}(R) : I \supseteq P\}$ be not totally ordered. Then there exists $I, J \in \mathfrak{I}(R)$ containing P and $I \parallel J$. As R is a decomposable ADL there exist $x \in I \setminus J$ and $y \in J \setminus I$ with $x \wedge y = 0$. As $x \wedge y = 0 \in P$ and P is a prime ideal, we get $x \in P$ or $y \in P$. But then $x \in J$ or $y \in I$ leading to the contradiction. Hence $\{I \in \mathfrak{I}(R) : I \supseteq P\}$ must be a totally ordered set. ∎

Note that $\{0\}$ need not be a prime ideal in R (e.g. an ADL R represented in Figure 1). But if R is a decomposable ADL then following is a direct consequence of Theorem 13.

Corollary 14. *In a decomposable ADL R, $\{0\}$ is a prime ideal in R if and only if R is totally ordered.*

From Theorem 13 and from the fact that no two maximal ideals in R are comparable we have the following consequence.

Corollary 15. *Any decomposable ADL is a pm-ADL.*

For a prime ideal P of R define $S_P = \bigcap\{M \in \mathfrak{M}(R) : M \subseteq P\}$. Necessary and sufficient condition for any two prime ideals to be comparable in a decomposable ADL is proved in the following theorem.

Theorem 16. *In a decomposable ADL R, two prime ideals P and Q are comparable if and only if $S_P \subseteq Q$ or $S_Q \subseteq P$.*

Proof. The proof of only if part being obvious, we prove if part only. Let $S_P \subseteq Q$ and $P \parallel Q$. As R is decomposable, there exists $x \in P \setminus Q$ and $y \in Q \setminus P$ such that $x \wedge y = 0$. As for $M \in \mathfrak{M}(R)$ contained in P, $0 = x \wedge y \in M$ implies

$x \in M$ as $y \notin P$. But then $x \in \cap\{M \in \mathfrak{M}(R) | M \subseteq P\} = S_P$ implies $x \in Q$; a contradiction. Hence P and Q must be comparable. Similarly we prove if $S_Q \subseteq P$, then prime ideals P and Q are comparable. ∎

In the following theorem we prove a property of finite number of mutually incomparable prime ideals of a decomposable ADL.

Theorem 17. *Let R be a decomposable ADL and $0 < a \in R$. If $P_1, P_2, P_3, \ldots, P_n$ are mutually incomparable prime ideals of R and $a \notin \bigcup_{i=1}^{n} P_i$. Then there exist $a_i \in \bigcap_{i \neq j}^{n} P_j \setminus P_i$ such that $0 < a_i < a$ and $a_i \wedge a_j = 0$ for $i \neq j, 1 \leq i, j \leq n$.*

Proof. We prove this theorem by induction on $n \geq 2$.

Step 1. $n = 2$. Let $P_1 \parallel P_2$ and $a \notin P_1 \cup P_2$. As R is decomposable, there exists $x_1 \in P_2 \setminus P_1$ and $x_2 \in P_1 \setminus P_2$ such that $x_1 \wedge x_2 = 0$. Define $a_1 = a \wedge x_1$ and $a_2 = a \wedge x_2$. Then $a \notin P_1$ and $x_1 \notin P_1$ imply $a \wedge x_1 = a_1 \notin P_1$. Again $a \notin P_2$ and $x_2 \notin P_2$ give $a \wedge x_2 = a_2 \notin P_2$. Clearly $0 < a_1$ and $0 < a_2$. If $a_1 = a$, then $a \wedge x_1 = a$ which implies $a \leq x_1 \in P_2$ which means $a \in P_2$; a contradiction. Therefore $a_1 < a$. Similarly, we get $a_2 < a$. Thus $0 < a_1 < a$ and $0 < a_2 < a$. Further $a_1 \wedge a_2 = (a \wedge x_1) \wedge (a \wedge x_2) = a \wedge (x_1 \wedge a \wedge x_2) = a \wedge (a \wedge x_1 \wedge x_2) = a \wedge (a \wedge 0) = a \wedge 0 = 0$. This shows that the result is true for $n = 2$.

Step 2. Now assume that the result is true for any $(n-1)$ prime ideals.

Step 3. We prove that the result holds for n. Let $P_1 \parallel P_2 \parallel P_3 \cdots \parallel P_n$ and $a \notin \bigcup_{i=1}^{n} P_i$. Then $P_1 \parallel P_2 \parallel P_3 \cdots \parallel P_{n-1}$ and $a \notin \bigcup_{i=1}^{n-1} P_i$. By induction hypothesis, there exist $b_i \notin P_i$ such that $0 < b_i < a$ and $b_i \wedge b_j = 0$ for $i \neq j$ where $i, j \in \{1, 2, 3, \ldots, n-1\}$. Also we have $P_2 \parallel P_3 \parallel P_4 \cdots \parallel P_n$ and $a \notin \bigcup_{i=2}^{n} P_i$. Therefore again by induction hypothesis, there exist $c_i \notin P_i$ such that $0 < c_i < a$ and $c_i \wedge c_j = 0$ for $i \neq j$, where $i, j \in \{2, 3, \ldots, n\}$. Further $P_1 \parallel P_n$ and $a \notin P_1 \cup P_n$. Therefore by step 1, there exist $d_1 \notin P_1, d_n \notin P_n$ such that $0 < d_1 < a$ and $0 < d_n < a$ and $d_1 \wedge d_n = 0$. Since $b_i \notin P_i$ and $c_i \notin P_i$, we get $f_i = b_i \wedge c_i \notin P_i$ for $i \in \{2, 3, \ldots, n-1\}$. It is easy to verify that $f_i < a$ and $f_i \wedge f_j = 0$ for $i \neq j$. If $f_i = b_i \wedge c_i = 0$, then $b_i \in P_i$ or $c_i \in P_i$, which is not true. Therefore we must have $f_i > 0$. Let $f_1 = b_1 \wedge d_1$ and $f_n = c_n \wedge d_n$. Since $b_1 \notin P_1$ and $d_1 \notin P_1$ we get $f_1 = b_1 \wedge d_1 \notin P_1$. Also $0 < f_1 < a$. Similarly $c_n \notin P_n$ and $d_n \notin P_n \Rightarrow f_n = c_n \wedge d_n \notin P_n$. Further $0 < f_n < a$ and $f_1 \wedge f_n = b_1 \wedge d_1 \wedge c_n \wedge d_n = b_1 \wedge c_n \wedge d_1 \wedge d_n = b_1 \wedge c_n \wedge 0 = 0$. This shows that there exist $f_1, f_2, \ldots, f_n$ in R such that $f_i \notin P_i$, $0 < f_i < a$ and $f_i \wedge f_j = 0$ for $i, j \in \{1, 2, \ldots, n\}$. This shows that the result is true for n.

Hence by the principal of induction, the result is true for all $n \geq 2$. ∎

Let $\tilde{N}(R)$ denote the set of all annihilator ideal in R i.e., $\tilde{N}(R) = \{I \in \mathfrak{I}(R) : I = I^{**}\}$. We know that $\tilde{N}(R)$ forms a complete Boolean algebra (by

Proposition 6). In the view of definition of decomposable ADL, Theorem 6 in [6] can be restated as follows.

Theorem 18. *In a decomposable ADL R, for any prime ideal P in R, the following statements are equivalent.*

(i) *For any $I \in \mathfrak{I}(R)$, I and P are comparable.*

(ii) *For any $N \in \tilde{N}(R) \setminus R, N \subseteq P$.*

(iii) *For any $M \in \mathfrak{M}(R), M \subseteq P$.*

(iv) *For any $a \notin P, \{a\}^* = \{0\}$. (i.e., each element of $(R \setminus P)$ is dense in R).*

Proof. By Theorem 5 in [6], we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Therefore it is sufficient to prove (iv) $\Rightarrow$ (i).

Suppose, if possible, that there is an ideal I of R such that $I \parallel P$. Since R is decomposable, there exist $x \in I \setminus P$ and $y \in P \setminus I$ such that $x \wedge y = 0$. Then $x > 0, y > 0$ and $y \in \{x\}^*$. Also by assumption, $x \notin P$ implies $\{x\}^* = \{0\}$. Thus $y \in \{x\}^* = \{0\}$ gives $y = 0$; a contradiction. Hence I and P must be comparable. This proves $(iv) \Rightarrow (i)$. ■

4. STRONGLY DECOMPOSABLE ADL

In this section we introduce strongly decomposable ADL and prove some characterizations for an ideal in a strongly decomposable ADL to be totally ordered. For $a, b \in R$, we write $a \parallel b$, when a, b are incomparable in $(R, \leq)$.

Definition. An ADL R is said to be strongly decomposable if for $a \parallel b, a, b \in R$, there exist $x, y \in R$ such that $a = x \vee (a \wedge b)$ and $b = y \vee (a \wedge b)$ and $x \wedge y = 0$.

Let $< R, \wedge, \vee, 0 >$ be an ADL where $R = \{0, a, b, c\}$ and $\wedge$ and $\vee$ defined on R as shown by the following tables

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

The ADL $< R, \wedge, \vee, 0 >$ is strongly decomposable.

Naturally, we have the relation between strongly decomposable ADL and decomposable ADL as given below.

Theorem 19. *Everly strongly decomposable ADL is decomposable.*

Proof. Let R be strongly decomposable. Let $I \parallel J$ in $\mathfrak{I}(R)$. Therefore there exist $x \in I \setminus J$ and $y \in J \setminus I$ such that $x \parallel y$. Since R is strongly decomposable, there exist $a, b \in R$ such that $x = a \vee (x \wedge y)$ and $y = b \vee (x \wedge y)$, with $a \wedge b = 0$. Now $a \leq x$ gives $a = x \wedge a \in I$ (as $x \in I$). Thus $a \in I$. As $y \in J$ we have $x \wedge y \in J$. If $a \in J$ we will have $a \vee (x \wedge y) \in J$ i.e., $x \in J$; a contradiction. Therefore $a \notin J$. Thus $a \in I \setminus J$. Similarly, $b \in J \setminus I$. Thus for $I \parallel J$ in $\mathfrak{I}(R)$ there exist $a \in I \setminus J$ and $b \in J \setminus I$ such that $a \wedge b = 0$. This proves that R is decomposable. ■

A sufficient condition for I^{**} to be be totally ordered is proved in the following theorem, where $I \neq \{0\}$ is an ideal of a strongly decomposable ADL R.

Theorem 20. *Let R be a strongly decomposable ADL and $I \neq \{0\}$ be an ideal of R. Then I^{**} is a totally ordered ideal, if I^* is a prime ideal of R.*

Proof. By Proposition 4, we have I^{**} is an ideal in R. Assume, if possible, that I^{**} is not totally ordered. Then there exist $x, y \in I^{**}$ such that $x \parallel y$. As R is strongly decomposable, there exist $a, b \in R$ such that $x = a \vee (x \wedge y)$ and $y = b \vee (x \wedge y)$ with $a \wedge b = 0$. Therefore $a \leq x$ and $b \leq y$ which imply $a = a \wedge x$ and $b = b \wedge y$. Since $x \in I^{**}$ and I^{**} is an ideal of R, we get $a \wedge x \in I^{**}$ which means $a \in I^{**}$. Similarly we obtain $b \in I^{**}$. Since I^* is a prime ideal of R and $a \wedge b = 0 \in I^*$ we get $a \in I^*$ or $b \in I^*$. If $a \in I^*$, then $a \in I^* \cap I^{**} = (0]$ which means $a = 0$. But then $x = a \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y$. This gives $x \leq y$; which is not possible (as $x \parallel y$). Similarly $b \in I^*$ implies $b = 0$ and consequently $y \leq x$; which is again absurd. Hence I^{**} must be a totally ordered ideal. ■

We characterize non-zero, totally ordered ideal in a strongly decomposable ADL R in the following theorem.

Theorem 21. *In a strongly decomposable ADL R, following statements are equivalent for any ideal $I \neq \{0\}$ in R.*

(i) *I is totally ordered.*
(ii) *For any $0 < a \in I, \{a\}^* = I^*$.*
(iii) *I^* is a prime ideal of R.*
(iv) *I^* is a minimal prime ideal of R.*
(v) *I^{**} is a maximal totally ordered ideal of R.*
(vi) *I^{**} is a minimal annihilator ideal of R.*
(vii) *I^* is a maximal annihilator ideal of R.*
(viii) *For any $0 < x \in I$, x-maximal ideal is unique.*
(ix) *Each $0 < z \in I$ is contained in a unique maximal filter of R.*

Proof. (i) $\Rightarrow$ (ii) Let $0 < a \in I$. Then clearly $I^* \subseteq \{a\}^*$. Let, if possible, $I^* \subset \{a\}^*$. Select $x \in \{a\}^* \setminus I^*$. Then $x > 0, x \wedge a = 0$ and $x \wedge b \neq 0$ for some $b \in I$. As I is totally ordered, either $x \wedge b \leq a$ or $a \leq x \wedge b$. If $x \wedge b \leq a$, then $x \wedge b = x \wedge b \wedge a = x \wedge a \wedge b = (x \wedge a) \wedge b = 0 \wedge b = 0$. Thus $x \wedge b = 0$; a contradiction. If $a \leq x \wedge b$, then $a = a \wedge x \wedge b = x \wedge a \wedge b = 0 \wedge b = 0$; a contradiction. Therefore we must have $I^* = \{a\}^*$ for $0 < a \in I$.

(ii) $\Rightarrow$ (iii) By Proposition 4, I^* is an ideal in R. On contrary assume that I^* is not prime. Then there exist $a, b \in R$ such that $a \notin I^*$ and $b \notin I^*$ but $a \wedge b \in I^*$. Then $a \wedge x > 0$ for some $x \in I$. For, if $a \wedge x = 0$ for all $x \in I$, then $a \in I^*$ which is not possible. Similarly $b \wedge y > 0$ for some $y \in I$.

Now, $(a \wedge x) \wedge (b \wedge y) = a \wedge (x \wedge b \wedge y) = a \wedge (b \wedge x \wedge y) = (a \wedge b) \wedge (x \wedge y)$. Since $x \wedge y \in I$ and $a \wedge b \in I^*$, we have $(a \wedge b) \wedge (x \wedge y) = 0$. Therefore $(a \wedge x) \wedge (b \wedge y) = 0$ which means $b \wedge y \in \{a \wedge x\}^*$. As $0 < a \wedge x \in I$, by hypothesis we get $\{a \wedge x\}^* = I^*$. Therefore $b \wedge y \in I \cap I^* = \{0\}$ which yields $b \wedge y = 0$; a contradiction. Therefore I^* is a prime ideal in R.

(iii) $\Rightarrow$ (iv) If possible assume that the prime ideal I^* is not a minimal prime ideal of R. By Proposition 8, there exists a minimal prime ideal M of R such that $M \subset I^*$. If $I \nsubseteq M$, then $I^* \subseteq M$ (since from Proposition 10 we have $I^* = \cap\{M \in \mathfrak{M}(R) : I \nsubseteq M\}$); which contradicts to the fact that M is a minimal prime ideal of R. Therefore $I \subseteq M$. Thus $I = I \cap M \subseteq I \cap I^* = \{0\}$ which means $I = \{0\}$; a contradiction. Therefore I^* must be a minimal prime ideal of R.

(iv) $\Rightarrow$ (v) By Theorem 20, we have I^{**} is a totally ordered ideal. We need to prove that I^{**} is a maximal totally ordered ideal of R. Let if possible there exists a totally ordered proper ideal J of R containing I^{**} properly. Select $x \in J \setminus I^{**}$. Then $x > 0$ and $x \wedge y > 0$ for some $y \in I^*$. But then $x \wedge y \in I^*$. Also $x \in J$ and J is an ideal imply $x \wedge y \in J$. Pick up any $0 < a \in I$. Then $a \in I \subseteq I^{**} \subset J$ i.e., $a \in J$. Since J is totally ordered and a, $x \wedge y \in J$, we have $a \leq x \wedge y$ or $x \wedge y \leq a$. If $a \leq x \wedge y$, then $a = a \wedge (x \wedge y) = 0$ and if $x \wedge y \leq a$, then $x \wedge y = (x \wedge y) \wedge a = 0$ (since $x \wedge y \in I^*$ and $a \in I$). Thus in either case we get a contradiction. Therefore our assumption is wrong. This shows that I^{**} is a maximal totally ordered ideal of R.

(v) $\Rightarrow$ (vi) Obviously I^{**} is an annihilator ideal. Let T be a non zero annihilator ideal of R contained in I^{**}. Since I^{**} is totally ordered (by assumption), we get T is also totally ordered ideal of R. From previously established parts of this theorem we have $(i) \Rightarrow (v)$, therefore we get T^{**} is a maximal totally ordered ideal of R. As $T = T^{**}$ we get $T = I^{**}$. This shows that no non-zero annihilator ideal is contained in I^{**} properly. Hence I^{**} is a minimal annihilator ideal of R.

(vi) $\Rightarrow$ (vii) We have $\tilde{N}(R)$, the set of all annihilator ideal of R, is a Boolean algebra (from Proposition 6). Define a mapping $\theta : \tilde{N}(R) \to \tilde{N}(R)$ by $\theta(I) = I^*$. Then θ is a dual isomorphism. Therefore if I^{**} is a minimal element in $\tilde{N}(R)$, then

$\theta(I^{**}) = I^*$ is a maximal element in $\tilde{N}(R)$. Hence I^* is a maximal annihilator ideal of R.

(vii) $\Rightarrow$ (viii) Suppose there exists some $0 < x \in I$ which has two distinct x-maximal ideals say M_1 and M_2. $x \notin M_1$ and M_1 is prime imply $I^* \subseteq M_1$. Similarly $x \notin M_2$ and M_2 is prime gives $I^* \subseteq M_2$. Thus $I^* \subseteq M_1 \cap M_2$. As $M_1 \parallel M_2$, there exist $s \in M_1 \setminus M_2$ and $t \in M_2 \setminus M_1$ such that $s \wedge t = 0$ (by Theorem 19). Since $s \wedge t = 0$ we have $t \wedge s = 0$ which implies $a \wedge (t \wedge s) = 0$. Therefore $(t \wedge a) \wedge s = 0$ i.e., $s \wedge (t \wedge a) = 0$ which yields $s \in \{t \wedge a\}^* = \{a \wedge t\}^*$. Also as $a \in I$ we have $a \wedge t \in I$. This gives $\{a \wedge t\}^* \supseteq I^*$. Since I^* is a maximal annihilator ideal of R we get $\{a \wedge t\}^* = I^*$. Thus $s \in I^*$ and hence $s \in M_2$; a contradiction. Hence for each $0 < x \in I$, there exist only one x-maximal ideal of R.

(viii) $\Rightarrow$ (i) Assume, if possible, that I is not totally ordered. Then there exist x, y in I such that $x \parallel y$. Since R is strongly decomposable, there exist $a, b \in R$ such that $x = a \vee (x \wedge y)$ and $y = b \vee (x \wedge y)$ with $a \wedge b = 0$. As $x, y \in I$ we get $a, b \in I$ and $a \neq b$. Let M_1 denote a-maximal ideal and M_2 denote b-maximal ideal of R. Therefore $M_1 \neq M_2$. For, if $M_1 = M_2$, then Proposition 9 and $a \wedge b = 0 \in M_1$ imply $a \in M_1$ or $b \in M_1$. As M_1 is a-maximal ideal, we have $b \in M_1 = M_2$ i.e., $b \in M_2$; which contradicts to the fact that M_2 is b-maximal ideal. Now $a \notin M_1$ implies $a \vee b \notin M_1$. Therefore M_1 is a $a \vee b$-maximal ideal. Similarly $b \notin M_2$ gives $a \vee b \notin M_2$. Therefore M_2 is a $a \vee b$-maximal ideal. Thus $0 < a \vee b \in I$ has two distinct $a \vee b$-maximal ideals. This contradicts to our hypothesis (viii). Hence I must be a totally ordered ideal.

(vii) $\Rightarrow$ (ix) Let there exist some $0 < z \in I$ such that it is contained in two distinct maximal filters say F_1 and F_2 of R. Define $Q_1 = R \setminus F_1$ and $Q_2 = R \setminus F_2$. Then by Proposition 7, Q_1 and Q_2 are distict minimal prime ideals of R and hence they are incomparable. R being a decomposable ADL (by Theorem 19), there exist $x \in Q_1 \setminus Q_2$, $y \in Q_2 \setminus Q_1$ such that $x \wedge y = 0$. Now $z \notin Q_1$ implies $I^* \subseteq Q_1$ and $z \notin Q_2$ implies $I^* \subseteq Q_2$. Thus $I^* \subseteq Q_1 \cap Q_2$. Also $z \wedge y \in I$ gives $\{z \wedge y\}^* \supseteq I^*$. By hypothesis (vii) we get $\{z \wedge y\}^* = I^*$. Now $x \wedge z \wedge y = z \wedge x \wedge y = z \wedge 0 = 0$. Therefore $x \in \{z \wedge y\}^* = I^*$. This in turn gives $x \in Q_1 \cap Q_2$; which is absurd. This shows that each $0 < z \in I$ must be contained in a unique maximal filter.

(ix) $\Rightarrow$ (i) Assume, if possible, that I is not totally ordered. Therefore there exist $x, y \in I$ such that $x \parallel y$. As R is strongly decomposable, there exist $a, b \in R$ such that $x = a \vee (x \wedge y)$, $y = b \vee (x \wedge y)$ with $a \wedge b = 0$. Now, $a \leq a \vee (x \wedge y) = x$ and $b \leq b \vee (x \wedge y) = y$. Therefore $a = a \wedge x$ and $b = b \wedge y$. As $x \in I$ we have $a \wedge x \in I$ which means $a \in I$. Similarly, $y \in I$ gives $b \in I$. As $0 < a \in I, a$ must be contained in a unique maximal filter say M_1 of R. Also $0 < b \in I$ implies b must be contained in a unique maximal filter say M_2 of R. Further $a \wedge b = 0$ will give $M_1 \neq M_2$. Since $a \in M_1$ and as $b \in M_2$ we have $a \vee b \in M_1$ and $a \vee b \in M_2$.

Thus $0 < a \vee b \in I$ and $a \vee b$ is contained in two distinct maximal filter M_1 and M_2; a contradiction. Hence I must be a totally ordered ideal.

Hence all the statements are equivalent. ■

A sufficient condition for a non-zero, proper annihilator ideal of R to be a minimal prime ideal of a strongly decomposable ADL R is furnished in the following theorem.

Theorem 22. *Let N be a non-zero, proper annihilator ideal of a strongly decomposable ADL R i.e., $N \in \tilde{N}(R) \setminus \{R, \{0\}\}$. If any two annihilator ideals of R are either comparable or co-maximal then N is minimal prime ideal of R.*

Proof. Let R be strongly decomposable ADL such that any two annihilator ideals of R are either comparable or co-maximal. Let $N \in \tilde{N}(R) \setminus \{R, \{0\}\}$. Assume that N is not a minimal prime ideal of R. Let us denote $N^* = A$. Then A is an ideal of R. As N is an annihilator ideal we have $N = N^{**}$ and therefore $N = A^*$. Thus $N = A^*$ is not a minimal prime ideal in R. But then A is not a totally ordered ideal (by Theorem 21). Hence there exist $x, y \in A$ such that $x \parallel y$. As R is strongly decomposable, there exist $a, b \in R$ such that $x = a \vee (x \wedge y)$ and $y = b \vee (x \wedge y)$ with $a \wedge b = 0$. Then $a, b \in A$ and $0 < a$ and $0 < b$. As $\{a\}^*$ and $\{b\}^*$ are annihilator ideals of R, by hypothesis they are co-maximal or comparable.

Case 1. Let $\{a\}^*$ and $\{b\}^*$ be co-maximal i.e., $\{a\}^* \vee \{b\}^* = R$. Then using Proposition 2 and Proposition 5 we have
$\{a\}^{**} = \{a\}^{**} \cap R = \{a\}^{**} \cap [\{a\}^* \vee \{b\}^*] = [\{a\}^{**} \cap \{a\}^*] \vee [\{a\}^{**} \cap \{b\}^*] = \{0\} \vee [\{a\}^{**} \cap \{b\}^*] = \{a\}^{**} \cap \{b\}^*$. Therefore $\{a\}^{**} \subseteq \{b\}^*$ $\cdots$ (I)
Again by hypothesis the annihilator ideals $\{a\}^{**}$ and $\{b\}^{**}$ are co-maximal or comparable.

Subcase 1. Suppose $\{a\}^{**}$ and $\{b\}^{**}$ are co-maximal i.e., $\{a\}^{**} \vee \{b\}^{**} = R$. Then $\{a\}^* = \{a\}^* \cap R = \{a\}^* \cap [\{a\}^{**} \vee \{b\}^{**}] = [\{a\}^* \cap \{a\}^{**}] \vee [\{a\}^* \cap \{b\}^{**}] = \{0\} \vee [\{a\}^* \cap \{b\}^{**}] = \{a\}^* \cap \{b\}^{**}$.
Thus $\{a\}^* \subseteq \{b\}^{**}$ which means $\{b\}^* \subseteq \{a\}^{**}$ $\cdots$ (II).

Combining the inclusions from (I) and (II) we get $\{b\}^* = \{a\}^{**}$. As $a, b \in A$ we have $\{a\}^* \supseteq A^*$ and $\{b\}^* \supseteq A^*$. Therefore $\{a\}^* \cap \{b\}^* \supseteq A^*$ which gives $\{a\}^* \cap \{a\}^{**} \supseteq A^*$ i.e., $A^* = \{0\}$ (by Proposition 5). Therefore $N = A^* = \{0\}$; a contradiction.

Subcase 2. Suppose $\{a\}^{**}$ and $\{b\}^{**}$ are comparable. Then either $\{a\}^{**} \subseteq \{b\}^{**}$ or $\{b\}^{**} \subseteq \{a\}^{**}$. If $\{a\}^{**} \subseteq \{b\}^{**}$ then $\{a\}^{**} \subseteq \{a\}^*$ (using (I)). Therefore $\{a\}^{**} = \{a\}^{**} \cap \{a\}^* = \{0\}$ which yields $\{a\}^* = R$. This is impossible. Similarly $\{b\}^{**} \subseteq \{a\}^{**}$ leads us to $\{b\}^* = R$ which is again impossible.

Case 2. Let $\{a\}^*$ and $\{b\}^*$ be comparable. Then either $\{a\}^* \subseteq \{b\}^*$ or $\{b\}^* \subseteq \{a\}^*$. If $\{a\}^* \subseteq \{b\}^*$ then as $a \wedge b = 0$ we get $b \in \{a\}^* \subseteq \{b\}^*$. This gives $b = 0$, which is absurd. Similarly $\{b\}^* \subseteq \{a\}^*$ gives $a \in \{a\}^*$ which means $a = 0$; a contradiction. Thus from Case 1 and Case 2 it follows that our assumption is wrong. Hence any non-zero, proper annihilator ideal N is minimal prime ideal of R. ■

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