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ON 1-ABSORBING PRIME AND WEAKLY 1-ABSORBING PRIME IDEALS OF SEMIRINGS

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Abstract

In this study, we introduce the concept of 1-absorbing prime ideals and weakly 1-absorbing prime ideals of commutative semirings with non-zero identity. A proper ideal I of a commutative semiring S is said to be a 1-absorbing (resp. a weakly 1-absorbing) prime ideal if whenever $abc \in I$ (resp. $0 \neq abc \in I$) for some non-units $a, b, c \in S$, then either $ab \in I$ or $c \in I$. The relationships among 1-absorbing prime ideals, prime ideals, 2-prime ideals, and 2-absorbing ideals are investigated. 1-absorbing prime ideals of subtractive valuation domain are studied. Many properties, results, and characterizations of 1-absorbing prime (resp. weakly 1-absorbing prime) ideals are given.

Keywords: semiring, 1-absorbing prime ideals, weakly 1-absorbing prime ideals, 2-prime ideals.

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1. INTRODUCTION

The concept of semiring was introduced by H.S. Vandiver in 1935, as a generalization of both rings and distributive lattices. Several authors have applied this concept in various disciplines in many ways. Semirings have wide applications in different branches of mathematics and computer science such as Optimization Theory, Automata Theory, Cryptography, Graph Theory, Topology, etc. A non-empty set S together with two binary operations addition '+' and multiplication '$\cdot$' is called semiring if:

(1) $(S, +)$ is a commutative monoid with identity element 0.

(2) $(S, \cdot)$ is a monoid with identity element $1 \neq 0$.

(3) The multiplication both from left and right is distributive over addition.

(4) $0 \cdot a = a \cdot 0 = 0$ for every $a \in S$.

A semiring S is commutative if $ab = ba$ for all $a, b \in S$. In this paper, all semirings are assumed to be commutative and with non-zero identity.

A non-empty subset I of a semiring S is called an ideal of S if for a, $b \in I$ and $r \in S$, $a+b \in I$ and $ra \in I$. An ideal I is proper if $I \neq S$. An ideal I of a semiring S is called subtractive ideal (or k-ideal) if a, $a+b \in I$ and $b \in S$, implies $b \in I$. A semiring S is called subtractive if every ideal of S is a subtractive ideal. For an ideal I of S and $a \in S$ define $(I : a) = \{x \in S : ax \in I\}$. It is easy to see that if I is a subtractive ideal of S, then $(I : a)$ is a subtractive ideal of S. A semiring S is said to be a regular semiring if for every element $r \in S$ there exists an element $x \in S$ such that $rxr = r$. A proper ideal I of a semiring S is said to be a strong ideal if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ [4]. The radical of an ideal I is defined as $\sqrt{I} = \{a \in S : a^n \in \text{I for some positive integer n}\}$. An ideal I is irreducible, if and only if $A \cap B = I$ for two ideals A, B implies $A = I$ or $B = I$. An element $u \in S \setminus \{0\}$ is called a unit if there exists an element u' such that $uu' = 1$. We denote the set of all units of S by $U(S)$. A non-zero element $a \in S$ is said to be a semi-unit in S if there exists $r, s \in S$ such that $1 + ra = sa$. Let $(S_1, +, \cdot)$ and $(S_2, +, \cdot)$ be two semirings with zero elements 0_{S_1}, 0_{S_2} and identity elements 1_{S_1}, 1_{S_2} respectively, a mapping $f : S_1 \longrightarrow S_2$ is said to be a semiring homomorphism if $f(a+b) = f(a)+f(b)$, $f(ab) = f(a)f(b)$, $f(0_{S_1}) = 0_{S_2}$ and $f(1_{S_1}) = 1_{S_2}$ for all $a, b \in S_1$. A one-one homomorphism is called monomorphism. Let U be a multiplicatively closed subset of a semiring S. The relation is defined on the set $S \times U$ by $(s, a) \sim (t, b) \iff xsb = xat$ for some $x \in U$ is an equivalence relation and the equivalence class of $(s, a) \in S \times U$ denoted by s/a. The set of all equivalence classes of $S \times A$ under "$\sim$" is denoted by S_U. S_U forms a semiring, where addition and multiplication are defined by $s/a+t/b = (sb+ta)/ab$ and $(s/a)(t/b) = st/ab$. Suppose that S is a commutative semiring, U is a multiplicatively closed subset and I is an ideal of S, then the set

$I_U = \{a/b : a \in I, b \in U\}$ is an ideal of S_U. The set I_U is called the localization of the ideal I at U [15].

A semifield is a commutative semiring in which the non-zero elements form a group under multiplication. Semiring S is referred to as "reduced semiring" if S contains no non-zero nilpotent elements. A semiring S is a semidomain if it is multiplicatively cancellative, i.e., if $ab = ac$ and $a \neq 0$, then $b = c$ for any $a, b, c \in S$. A semiring S is a valuation semiring if it is a semidomain and the set of ideals are totally ordered by inclusion [13]. A semiring S is said to be an entire semiring if $ab = 0$ implies either $a = 0$ or $b = 0$ for any $a, b \in S$ [13].

A proper ideal I of S is called a prime (resp., weakly prime) if for any $a, b \in S$, $ab \in I$ (resp., $0 \neq ab \in I$) implies either $a \in I$ or $b \in I$ [8]. A proper ideal I of a semiring S is primary if for all $x, y \in S$, we have that $xy \in I$ and $x \notin I$ implies $y^n \in I$ for some positive integer n. If I is primary, then $\sqrt{I} = P$ is a prime ideal of S [[2], Theorem 38]. In this case, we also say that I is a P-primary ideal of S. A prime ideal P of a semiring S is called to be a minimal prime ideal of S, if for every prime ideal I, $I \subseteq P$ implies either $I = \langle 0 \rangle$ or $I = P$. Let $P \supseteq I$ be ideals of a semiring S, where P is prime, then P is a minimal prime ideal of I if and only if for each $x \in P$, there is some $y \in S \setminus P$ and a nonnegative integer i such that $yx^i \in I$ [14].

A proper ideal M is maximal (resp. k-maximal) if $M \subseteq M' \subseteq S$ implies either $M = M'$ or $M' = S$ for any ideal (resp. k-ideal) M' of S. A semiring is said to be a local semiring if it has only one maximal ideal. A semiring S is a local semiring if and only if $S \setminus U(S)$ is an ideal of S. The Jacobson radical $Jac(S)$ of a semiring S is defined as the intersection of all maximal k-ideals of S. A prime ideal P of S is said to be a divided prime ideal of S if $P \subseteq \langle x \rangle$ for every $x \in S \setminus P$. If each prime ideal of S is divided, then S is called divided semiring [17]. For general references on semiring theory, one may refer to [9, 12] and [15].

A. Badawi introduced 2-absorbing (resp., weakly 2-absorbing) ideals in commutative rings as a generalization of prime ideals. The concept of 2-absorbing ideals in a commutative semiring was introduced by Darani [7]. A proper ideal I of a semiring S is called a 2-absorbing ideal if for any $a, b, c \in S$, $abc \in I$ implies $ab \in I$ or $bc \in I$ or $ca \in I$. The concept of 2-prime ideals of a ring was introduced by Beddani and Messirdi [5]. A proper ideal I of a semiring S is called 2-prime if for any $a, b \in S$ such that $ab \in I$, either $a^2 \in I$ or $b^2 \in I$, which was introduced by Khanra *et al.* [10]. The concept of 1-absorbing prime ideals which is another extension of prime ideals was introduced in [18]. The notions of 1-absorbing primary and weakly 1-absorbing primary ideals were investigated in [16]. A proper ideal I of a ring R is called a 1-absorbing prime (resp. weakly 1-absorbing prime) ideal if for all non-unit elements $a, b, c \in R$ such that $abc \in I$ (resp. $0 \neq abc \in I$), then either $ab \in I$ or $c \in I$ [11, 18]. In this paper, we generalize this notion of 1-absorbing prime ideals in commutative semiring with non-zero identity. Throughout the paper, unless otherwise stated, S stands for commutative semiring with non-zero identity and $\mathbb{Z}_0^+$ for the semiring of non-negative integers with usual binary addition and multiplication.

We shortly summarize the content of the paper. In the first section, we recall some essential preliminaries. In Section 2, we introduce 1-absorbing prime ideals as a generalization of prime ideals of a semiring. Some properties of 1-absorbing prime ideals are studied. We give a characterization of 1-absorbing prime ideals ($cf.$ Theorem 7). We show (cf. Theorem 12) that in a subtractive valuation semiring S, an ideal I is a 1-absorbing prime ideal of S if and only if either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S. In Section 3, we define weakly 1-absorbing prime ideal, as a generalization of weakly prime ideals. Some properties of 1-absorbing prime ideals are studied. We characterize a strong local semiring whose all proper ideals are 1-absorbing prime (cf. Theorem 24). At the end, we characterize 1-absorbing prime ideals in a decomposable semiring (cf. Theorem 26).

2. 1-absorbing Prime Ideals

Definition 1. A proper ideal I of a commutative semiring S is said to be a 1-absorbing prime ideal if whenever $abc \in I$ for some non-units $a, b, c \in S$, then either $ab \in I$ or $c \in I$.

It is clear that every prime ideal is 1-absorbing prime and every 1-absorbing prime ideal is 2-absorbing ideal, but the converse may not be true in general which can be seen from the following examples:.

Example 1. In semiring $S = \mathbb{Z}_0^+$, the ideal $I = 3\mathbb{Z}_0^+ \setminus \{3\}$ is a 1-absorbing prime ideal which is not a prime ideal.

Example 2. Let $S = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $I = 2\mathbb{Z}_0^+ \times 3\mathbb{Z}_0^+$. Then I is a 2-absorbing ideal, however I is not a 1-absorbing prime ideal of S. Indeed, $(2,5)(4,2)(7,3) \in I$ but neither $(2,5)(4,2) \in I$ nor $(7,3) \in I$.

Theorem 1. *Every 1-absorbing prime ideal of a semiring S is a 2-prime ideal of S.*

Proof. Suppose that I is a 1-absorbing prime ideal of S and $xy \in I$ for some elements $x, y \in S$. If either x or y is a unit, then I is a 2-prime ideal of S. So assume that x, y are non-unit elements of S. Then $x^2y \in I$, this implies either $x^2 \in I$ or $y \in I$. Thus either $x^2 \in I$ or $y^2 \in I$ and hence I is a 2-prime ideal of S. ∎

The converse of Theorem 1 may not be true, as shown in the following example.

Example 3. In semiring $S = \mathbb{Z}_0^+$, the ideal $I = 4\mathbb{Z}_0^+$ is a 2-prime ideal but I is not a 1-absorbing prime ideal, since $2 \cdot 3 \cdot 2 \in I$ but neither $2 \cdot 3 \in I$ nor $2 \in I$.

So the class of all 1-absorbing prime ideals of a semiring S is an intermediate class between the class of all prime ideals and the classes of 2-prime ideals, 2-absorbing ideals of S.

Theorem 2. *Let I be a* 1*-absorbing prime ideal of a semiring S. Then $x^2 \in I$ for every $x \in \sqrt{I}$ and $\sqrt{I}$ is a prime ideal of S.*

Proof. Suppose $x \in \sqrt{I}$. Then there exists a smallest positive integer n such that $x^n \in I$. For $n = 1$ or $n = 2$, it is clear. For $n \geq 3$, $x^n = xxx^{n-2} \in I$. That implies either $x^2 \in I$ or $x^{n-2} \in I$. Continuing this process, it is easy to conclude that $x^2 \in I$. Now consider $ab \in \sqrt{I}$ for some $a, b \in S$. If one of a or b is a unit, then there is nothing to prove. Assume that both a and b are non-units and $ab \in \sqrt{I}$. Then $(ab)^2 = aab^2 \in I$, which implies that either $a^2 \in I$ or $b^2 \in I$. That is either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a prime ideal of S. ■

Theorem 3. *Let I be a 2-prime ideal of S. If $(P^2 : x) \subseteq I$ for any $x \in P \setminus I$, where $\sqrt{I} = P$. Then I is a 1-absorbing prime ideal of S.*

Proof. Suppose that $xyz \in I$ and $xy \notin I$ for some non-units $x, y, z \in S$. Since I is a 2-prime ideal and $(xy)z \in I$, we obtain either $(xy)^2 \in I$ or $z^2 \in I$. If $(xy)^2 \in I$, then $xy \in P \setminus I$. But then $xy \in (P^2 : xy) \subseteq I$, which is a contradiction. So $z^2 \in I$, that is $z \in P$. If $z \in I$, then we are done. If $z \in P \setminus I$, then $z \in (P^2 : z) \subseteq I$, a contradiction. Therefore, I is a 1-absorbing prime ideal of S. ■

Theorem 4. *Suppose that a semiring S has a* 1*-absorbing prime ideal which is not a prime ideal. Then, S is a local semiring.*

Proof. Assume that I is a 1-absorbing prime ideal which is not a prime ideal of S. So there exist non-unit elements a, $b \in S$ such that $ab \in I$ but $a \notin I$, $b \notin I$. Now consider the set of all non-units $S \setminus U(S)$. Let $x, y \in S \setminus U(S)$. Then $xab \in I$ and $yab \in I$. Since I is a 1-absorbing prime ideal of S and $b \notin I$, we have $xa \in I$ and $ya \in I$. Hence $(x + y)a = xa + ya \in I$. If $(x + y)$ is a unit, then $a \in I$, which is a contradiction. Therefore, $x + y \in S \setminus U(S)$. Again for any $s \in S$ and $x \in S \setminus U(S)$, $sxab \in I$. Since I is a 1-absorbing prime ideal of S and $a, b \notin I$, $sx \in S \setminus U(S)$ and hence $S \setminus U(S)$ is an ideal of S. Therefore, S is a local semiring. ■

Corollary 1. *In a commutative non-local semiring, a proper ideal is a* 1*-absorbing prime if and only if it is prime.*

Theorem 5. *Let S be a local semiring with maximal ideal M. A proper ideal I of S is* 1*-absorbing prime ideal if and only if either I is a prime ideal or $M^2 \subseteq I$.*

Proof. Suppose that I is a 1-absorbing prime ideal of S. If I is not a prime ideal, then there exist non-units a, $b \in S$ such that $ab \in I$ but $a \notin I$ and $b \notin I$. Let x, $y \in M$. Here $xyab \in I$. Since I is a 1-absorbing prime ideal and $b \notin I$, we have $xya \in I$. Again $xy \in I$, since $a \notin I$. Therefore, $xy \in I$ for any $x, y \in M$. Hence $M^2 \subseteq I$.

Conversely, if I is a prime ideal of S, then I is a 1-absorbing prime ideal of S. Consider $M^2 \subseteq I$ and $abc \in I$ for some non-units a, b, $c \in S$. Since a, b, c are non-units, so $a, b, c \in M$. Thus $ab \in M^2 \subseteq I$ and hence I is a 1-absorbing prime ideal of S. ■

Corollary 2. *In a local semiring S with unique maximal ideal M, if $M^2 = \langle 0 \rangle$, then every proper ideal of S is a 1-absorbing prime ideal.*

Theorem 6. *Let S be a semiring. Then, $\langle 0 \rangle$ is a 1-absorbing ideal of S if and only if S is an entire semiring or (S, M) is a local semiring such that $M^2 = \langle 0 \rangle$.*

Proof. Suppose that $\langle 0 \rangle$ is a 1-absorbing ideal of S and S is not an entire semiring. So $\langle 0 \rangle$ is a 1-absorbing prime ideal of S which is not a prime ideal. Thus, by Theorem 4 and Theorem 5, S is a local semiring with maximal ideal M such that $M^2 = 0$.

Conversely, if S is an entire semiring, then $\langle 0 \rangle$ is a prime ideal of S and so $\langle 0 \rangle$ is 1-absorbing prime ideal of S. If S is a local semiring with maximal ideal M such that $M^2 = 0$, then by Corollary 2, $\langle 0 \rangle$ is a 1-absorbing prime ideal of S. ■

Theorem 7. *Let I be a proper ideal of a semiring S, then the following are equivalent:*

(1) *I is a 1-absorbing prime ideal of S.*

(2) *If $abJ \subseteq I$ for non-units a,b $\in S$, then either $ab \in I$ or $J \subseteq I$.*

(3) *For any proper ideals K, L and non-unit element a of S, if $aKL \subseteq I$, then either $aK \subseteq I$ or $L \subseteq I$.*

(4) *If for any ideals J, K, L of S, $JKL \subseteq I$, then either $JK \subseteq I$ or $L \subseteq I$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that $abJ \subseteq I$ and $ab \notin I$. Consider $j \in J$. Then $abj \in abJ \subseteq I$. Since I is a 1-absorbing prime ideal of S, we have $j \in I$. This implies that $J \subseteq I$.

(2) $\Longrightarrow$ (3) Assume that $aKL \subseteq I$ and $aK \nsubseteq I$. Then there exists $k \in K$ such that $ak \notin I$. As $akL \subseteq I$, by (2), $L \subseteq I$. Therefore, either $aK \subseteq I$ or $L \subseteq I$.

(3) $\Longrightarrow$ (4) Suppose $JKL \subseteq I$ and $JK \nsubseteq I$ for some ideals J, K, L of S. Then there exists an element $j \in J$ such that $jK \nsubseteq I$. By (3), we have $L \subseteq I$. Therefore, either $JK \subseteq I$ or $L \subseteq I$.

(4) $\Longrightarrow$ (1) Assume that $abc \in I$ for some non-units $a, b, c \in S$. Consider $J = \langle a \rangle$, $K = \langle b \rangle$, $L = \langle c \rangle$. Then $JKL = abcS \subseteq I$. By (4), either $JK \subseteq I$ or $L \subseteq I$. So either $ab \in I$ or $c \in I$. Hence I is a 1-absorbing prime ideal of S. ■

Proposition 1. *Let I be a* 1*-absorbing prime ideal of a semiring S. Then for every non-unit $c \in S \setminus I$, $(I : c)$ is a prime ideal of S.*

Proof. Suppose $ab \in (I : c)$ for some $a, b \in S$ and non-unit $c \in S \setminus I$. Without loss of generality, we may assume that a and b are non-units. Since I is a 1-absorbing prime ideal of S and $acb \in I$, we have either $ac \in I$ or $b \in I$. This implies either $a \in (I : c)$ or $b \in I \subseteq (I : c)$. Therefore, $(I : c)$ is a prime ideal of S for every non-unit $c \in S \setminus I$. ∎

Theorem 8. *If I is a P-primary ideal of S such that $(P^2 : x) \subseteq I$ for all $x \in P \setminus I$, then I is a* 1*-absorbing prime ideal of S.*

Proof. Let $abc \in I$ for some non-units $a, b, c \in S$ and $ab \notin I$. If possible, let $c \notin I$. Since I is a P-primary ideal of S and $ab \notin I$, we have $c \in P$. Also $c \notin I$ implies $ab \in P$. So $abc \in P^2$. By the given hypothesis, $(P^2 : c) \subseteq I$. Thus $ab \in (P^2 : c) \subseteq I$, which is a contradiction. Hence $c \in I$ and thus I is a 1-absorbing prime ideal of S. ∎

Theorem 9. *Let P be a nonzero divided prime ideal of a semidomain S. Then, P^2 is a* 1*-absorbing prime ideal of S.*

Proof. First, we show that P^2 is a P-primary ideal of S. Let $xy \in P^2$ and $y \notin P$. Suppose $xy = \sum_{i=1}^{n} x_i y_i$ for some $x_i, y_i \in P$, $i = 1, 2, \ldots, n$. Since P is a divided prime ideal and $y \notin P$, we have $P \subseteq Sy$. Thus $x_i = s_i y$, where $s_i \in S$, for $i = 1, 2, \ldots, n$. Since P is a prime ideal and $y \notin P$, we have $s_i \in P$. Also since S is a semidomain, $xy = \sum_{i=1}^{n} s_i y y_i = (\sum_{i=1}^{n} s_i y_i)y$ implies $x = \sum_{i=1}^{n} s_i y_i$, that is $x \in P^2$. Thus P^2 is a P-primary ideal of S. Also, $(P^2 : x) \subseteq P^2$ for every $x \in P \setminus P^2$. So by Theorem 8, P^2 is a 1-absorbing prime ideal of S. ∎

Theorem 10. *In a semiring S, for any* 1*-absorbing prime ideal I of S, there exists exactly one minimal prime ideal containing I.*

Proof. If possible, let P_1 and P_2 be two minimal prime ideals containing I. Then there exist elements $a, b \in S$ such that $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. So there exist $p_2 \notin P_2$ and $p_1 \notin P_1$ such that $ap_2^n \in I$, $bp_1^n \in I$. Since p_1, $p_2 \notin I$ and I is a 1-absorbing prime ideal, $ap_2^n \in I$ implies $ap_2 \in I$ and $bp_1^n \in I$ implies $bp_1 \in I$. Thus $a^2 \in I$ and $b^2 \in I$, whence $a^2 \in I \subseteq P_2$ implies $a \in P_2$ and $b^2 \in I \subseteq P_1$ implies $b \in P_1$, which is a contradiction. Therefore, there exists one minimal prime ideal of S containing I. ∎

Theorem 11. *Let P be a non-zero divided prime ideal of a subtractive semiring S and I be an ideal of S such that $\sqrt{I} = P$. If I is a* 1*-absorbing prime ideal of S, then I is a P-primary ideal of S such that $P^2 \subseteq I$.*

Proof. Let $ab \in I$ for some $a, b \in S$ and $b \notin P$. Since P is a divided prime ideal of S, we have $P \subseteq Sb$. Thus $a = sb$ for some $s \in S$. So, we get $ab = b^2 s \in I$. Since $b^2 \notin I$ and I is a 1-absorbing prime ideal of S, we conclude that $s \in I$. Therefore $a = sb \in I$ and so I is a P-primary ideal of S. Now, let $x, y \in P$. Then by Theorem 2, $x^2, y^2 \in I$. Thus $x(x+y)y = x^2 y + xy^2 \in I$. Since I is a 1-absorbing prime ideal of S, we have either $x(x+y) = x^2 + xy \in I$ or $y \in I$. Since S is a subtractive semiring, $xy \in I$ and hence $P^2 \subseteq I$. ∎

Theorem 12. *Suppose that S is a subtractive valuation semiring and I is an ideal of S. Then I is a 1-absorbing prime ideal of S if and only if either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S.*

Proof. Suppose I is a 1-absorbing prime ideal of S. Since every subtractive valuation semiring is a divided semidomain [[6], Proposition 1.4], so I is a P-primary ideal of S such that $P^2 \subseteq I$, by Theorem 11. Therefore, by [[6], Theorem 1.2], either $I = P$ or $I = P^2$. Conversely, suppose that either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S. If $I = P$, then I is a 1-absorbing prime ideal of S. If $I = P^2$, then by Theorem 9, I is a 1-absorbing prime ideal of S. ∎

Theorem 13. *Let I be an irreducible subtractive ideal of S and $(I : x) = (I : x^2)$ for every $x \in S \setminus I$, then I is a 1-absorbing prime ideal of S.*

Proof. Let $abc \in I$ and $c \notin I$ for some non-units $a, b, c \in S$. By the assumption, $(I : c) = (I : c^2)$. If possible, let $ab \notin I$. Consider $r \in (I + (ab)) \cap (I + (c))$. Then $r = i_1 + abs_1 = i_2 + cs_2$ for some $i_1, i_2 \in I$ and $s_1, s_2 \in S$. Thus $rc = i_1 c + abcs_1 = i_2 c + c^2 s_2 \in I$. Since I is a subtractive ideal of S, we get $c^2 s_2 \in I$. So $cs_2 \in I$. Therefore, $r = i_2 + cs_2 \in I$. This shows that $(I + (ab)) \cap (I + (c)) \subseteq I$ and hence $(I + (ab)) \cap (I + (c)) = I$, a contradiction. Thus $ab \in I$ and so I is a 1-absorbing prime ideal of S. ∎

Theorem 14. *Let I be a 1-absorbing prime subtractive ideal of S with $\sqrt{I} = P$. If $I \neq P$, then $\Omega = \{(I : x) : x \in P \setminus I\}$ under set inclusion is a totally ordered set.*

Proof. If possible, let there exist $a, b \in P \setminus I$ such that neither $(I : a) \subseteq (I : b)$ nor $(I : b) \subseteq (I : a)$. Then there exist $x, y \in S \setminus I$ so that $x \in (I : a) \setminus (I : b)$ and $y \in (I : b) \setminus (I : a)$. Since $P \subseteq (I : c)$ for any $c \in S \setminus I$, $x \in (I : a) \setminus P$ and $y \in (I : b) \setminus P$. Thus $xy \notin P$, since P is a prime ideal of S. Consider $x(a+b)y = xay + xby \in I$. This implies $x(a+b) = xa + xb \in I$. Since I is a subtractive ideal and $xa \in I$, we get $xb \in I$, that is $x \in (I : b)$, which is a contradiction. Hence $\Omega = \{(I : x) : x \in P \setminus I\}$ under set inclusion is a totally ordered set. ∎

Theorem 15. *In a regular semiring S, every irreducible ideal of S is a 1-absorbing prime ideal of S.*

Proof. Let S be a regular semiring and I be an irreducible ideal of S. Let $abc \in I$ and $ab \notin I$ for some non-units $a, b, c \in S$. If possible, let $c \notin I$. Now consider the ideals $J = (I + (ab))$ and $K = (I + (c))$, properly containing I. Since I is an irreducible ideal of S, $J \cap K \nsubseteq I$. Thus, there exists $p \in S$ such that $p \in (I + (ab)) \cap (I + (c)) \setminus I$. So $p \in (I + (ab)(I + (c)) \setminus I$, by regularity of S [[9], Proposition 6.35]. Then, there are $p_1, p_2 \in I$ and $s_1, s_2 \in S$ such that $p = (p_1 + s_1 ab)(p_2 + s_2 c) = p_1 p_2 + p_1 s_2 c + s_1 ab p_2 + s_1 s_2 abc$. This implies that $p \in I$, which is a contradiction. Hence $c \in I$, and so I is a 1-absorbing prime ideal of S. ■

Theorem 16. *Let S be a commutative semiring and U be a multiplicative closed subset of S. If I is a 1-absorbing prime ideal of S with $U \cap I = \phi$, then I_U is a 1-absorbing prime ideal of S_U.*

Proof. Let $(a/s)(b/t)(c/r) \in I_U$ for some non-units $(a/s), (b/t), (c/r) \in S_U$, where $a, b, c \in S$ and $s, t, r \in U$. Then $(ua)bc \in I$ for some $u \in U$. So either $uab \in I$ or $c \in I$, as I is a 1-absorbing prime ideal of S. Thus either $(a/s)(b/t) \in I_U$ or $(c/r) \in I_U$. Therefore, I_U is a 1-absorbing prime ideal of S_U. ■

3. WEAKLY 1-ABSORBING PRIME IDEALS

Definition 2. A proper ideal I of a commutative semiring S is said to be a weakly 1-absorbing prime ideal if whenever $0 \neq abc \in I$ for some non-units $a, b, c \in S$, either $ab \in I$ or $c \in I$.

Clearly, every 1-absorbing prime ideal is also a weakly 1-absorbing prime ideal but the converse may be true in general which can be seen from the following examples.

Example 4. In the semiring $\mathbb{Z}_{15}$, the ideal $I = \{0\}$ is clearly a weakly 1-absorbing prime ideal of S but not 1-absorbing prime, as $0 = \bar{3} \cdot \bar{3} \cdot \bar{5} \in I$ but neither $\bar{3} \cdot \bar{3} \in I$ nor $\bar{5} \in I$.

Theorem 17. *If I is a weakly 1-absorbing prime of a reduced semiring S, then $\sqrt{I}$ is a weakly prime ideal of S.*

Proof. Suppose that $0 \neq ab \in \sqrt{I}$ for some non-units a, $b \in S$. Then $(ab)^n \in I$ for some positive integer n. Since S is a reduced semiring, so $0 \neq (ab)^n = aa^{n-1}b^n \in I$. Thus either $aa^{n-1} = a^n \in I$ or $b^n \in I$ which implies either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. So $\sqrt{I}$ is a weakly prime ideal of S. ■

Theorem 18. *If the zero ideal of a semiring S is a 1-absorbing prime, then every weakly 1-absorbing prime ideal is a 1-absorbing prime of S.*

Proof. Suppose that I is a weakly 1-absorbing prime ideal of S and $abc \in I$ for some non-units a, b, $c \in S$. If $0 \neq abc$, then either $ab \in I$ or $c \in I$, that is, I is a 1-absorbing prime ideal. If $abc = 0$ and since zero ideal is 1-absorbing prime, either $ab = 0 \in I$ or $c = 0 \in I$. Therefore, I is a 1-absorbing prime ideal of S. ■

Theorem 19. *Let S_1 and S_2 be two semirings and $f : S_1 \longrightarrow S_2$ be a monomorphism such that $f(a)$ is a non-unit in S_2 for every non-unit element a in S_1. Then the following statements hold:*

(1) *If I is a weakly 1-absorbing prime ideal of S_2, then $f^{-1}(I)$ is a weakly 1-absorbing prime ideal of S_1.*

(2) *If J is a weakly 1-absorbing prime k-ideal of S_1 with $Ker(f) \subseteq J$ and f is onto steady homomorphism, then $f(J)$ is a weakly 1-absorbing prime k-ideal of S_2.*

Proof. (1) Assume that $0 \neq abc \in f^{-1}(I)$ for some non-units $a, b, c \in S_1$. Since f is monomorphism, we have $0 \neq f(abc) = f(a)f(b)f(c) \in I$ for non-units $f(a), f(b), f(c) \in S_2$. As I is a weakly 1-absorbing prime ideal of S_2, so either $f(a)f(b) = f(ab) \in I$ or $f(c) \in I$. That is either $ab \in f^{-1}(I)$ or $c \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a weakly 1-absorbing prime ideal of S_1.

(2) Clearly, $f(J)$ is a k-ideal of S_2 as J is a k-ideal of S_1. Now consider, $0 \neq abc \in f(J)$ for some non-units $a, b, c \in S_2$. There exist non-units $x, y, z \in S_1$ such that $f(x) = a$, $f(y) = b$, $f(z) = c$. So $0 \neq abc = f(x)f(y)f(z) = f(xyz) \in f(J)$. Thus $f(xyz) = f(j)$ for some $j \in J$. Since f is steady, we have $yxz + k_1 = j + k_2$ for some $k_1, k_2 \in Ker(f)$. As J is a k-ideal of S_1 and $Ker(f) \subseteq J$, we obtain $0 \neq xyz \in J$. Since J is a weakly 1-absorbing prime ideal of S_1, either $xy \in J$ or $z \in J$. This implies either $ab = f(xy) \in f(J)$ or $c = f(z) \in f(J)$. Hence, $f(J)$ is a weakly 1-absorbing prime k-ideal of S_2. ■

Definition 3. Let I be a weakly 1-absorbing prime ideal of S and $a, b, c \in S$ are non-units of S. We call (a, b, c) is a 1-triple zero of I if $abc = 0$, $ab \notin I$ and $c \notin I$.

Theorem 20. *Let I be a subtractive weakly 1-absorbing prime ideal of S and (a, b, c) be a 1-triple zero of I. Then, $abI = 0$ and if $a, b \notin (I : c)$, then $I^3 = 0$.*

Proof. Suppose that I is a subtractive weakly prime ideal of S and (a, b, c) is a 1-triple zero of I. So $abc = 0$ and $ab \notin I$, $c \notin I$. If possible, let $abI \neq 0$. Then there exists $i \in I$ such that $0 \neq abi$. This implies that $0 \neq ab(c + i) = abc + abi \in I$. Also $c + i$ is a non-unit, because $(c + i)$ is a unit implies $ab \in I$. Since I is weakly 1-absorbing prime and $ab \notin I$, so $c + i \in I$. As I is a subtractive ideal of S, we get $c \in I$, which is a contradiction. Therefore, $abI = 0$.

We first show that if $a, b \notin (I : c)$, then $bcI = caI = aI^2 = bI^2 = cI^2 = 0$. Since $a, b \notin (I : c)$, $ac \notin I$ and $bc \notin I$. Suppose $bcI \neq 0$. Then there exists $i \in I$ such that $bci \neq 0$. Since $abc = 0$, we have $0 \neq bci = (a + i)bc \in I$. Here $(a + i)$ is a non-unit, because if $a + i$ is a unit, then $bc \in I$, which is a contradiction. Since I is weakly 1-absorbing prime, either $(a + i)b \in I$ or $c \in I$, that is either $ab \in I$ or $c \in I$, a contradiction, as I is a subtractive ideal of S. Therefore $bcI = 0$. Similarly, $caI = 0$. Now, we will show that $aI^2 = 0$. Suppose that $ai_1i_2 \neq 0$ for some $i_i, i_2 \in I$. Since $abc = 0$ and $abI = acI = 0$, we have $a(b + i_1)(c + i_2) = abc + abi_2 + aci_1 + ai_1i_2 = ai_1i_2 \neq 0$. If $c + i_2$ is a unit, then $a(b + i_1) \in I$, and so $ab \in I$. Similarly, if $b + i_1$ is a unit, then $ac \in I$, both contradict the hypothesis. Since I is a weakly 1-absorbing prime ideal, either $a(b+i_1) \in I$ or $c+i_2 \in I$. Thus $ab \in I$ or $c \in I$ as I is a subtractive ideal of S, which is a contradiction. So $aI^2 = 0$. Similarly, one can show that $bI^2 = 0$ and $cI^2 = 0$. Now, assume that $I^3 \neq 0$. So there exist $i_1, i_2, i_3 \in I$ such that $0 \neq i_1i_2i_3 \in I$. Since $abI = bcI = caI = aI^2 = bI^2 = cI^2 = 0$, we have $0 \neq i_1i_2i_3 = (a+i_1)(b+i_2)(c+i_3) \in I$ and $(a+i_1)$, $(b+i_2)$, $(c+i_3)$ are non-units. So either $(a + i_1)(b + i_2) \in I$ or $c + i_3 \in I$. Since I is a subtractive ideal, we have either $ab \in I$ or $c \in I$, which is a contradiction. Therefore, $I^3 = 0$. ∎

Definition 4 ([1], Proposition 8.21)**.** An ideal I of a semiring S is called a Q-ideal if there exists a subset Q of S such that: 1) $S = \cup\{q + I : q \in Q\}$, 2) If q_1, $q_2 \in Q$, then $(q_1 + I) \cap (q_2 + I) \neq \phi \iff q_1 = q_2$. Let I be a Q-ideal of a semiring S. Then $S/I_Q = \{q + I : q \in Q\}$ forms a semiring under the following addition "$\oplus$" and multiplication "$\odot$", $(q_1 + I) \odot (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is unique such that $q_1q_2 + I \subseteq q_4 + I$. This semiring S/I_Q is called the quotient semiring of S by I and denoted by $(S/I_Q, \oplus, \odot)$ or just S/I_Q.

Theorem 21. *Let I be a Q-ideal and J be a k-ideal of a semiring S, so that $I \subseteq J$. If J is a weakly 1-absorbing prime ideal of S, then $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q.*

Let I be a weakly 1-absorbing prime ideal of S such that $u(S/I) = \{x + I : x \in u(S) \cap Q\}$. If $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q, then J is a weakly 1-absorbing prime ideal of S.

Proof. Suppose that J is a weakly 1-absorbing prime ideal of S and $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in J/I_{(Q\cap J)}$ for some non-units $q_1 + I$, $q_2 + I$, $q_3 + I \in S/I_Q$ where $q_1, q_2, q_3 \in Q$. So there exists a unique element $q_4 \in J \cap Q$ such that $q_1q_2q_3 + I \subseteq q_4 + I \in J/I_{(Q\cap J)}$. It follows that q_1, q_2, q_3 are non-units in S and $0 \neq q_1q_2q_3 \in J$. Since J is a weakly 1-absorbing prime ideal of S, we get $q_1q_2 \in J$ or $q_3 \in J$. Therefore if $q_3 \in J$, then $q_3 + I \in J/I_{(Q\cap J)}$. Otherwise if $q_1q_2 \in J$, then $(q_1 + I) \odot (q_2 + I) = q_5 + I$ where q_5 is the unique element in Q and $q_1q_2 + r = q_5 + s$

for some $r, s \in I$. Since J is a k-ideal, $q_5 \in J$. So $(q_1 + I) \odot (q_2 + I) \in J/I_{(Q \cap J)}$. Hence $J/I_{(Q \cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q.

Let $0 \neq abc \in J$ for some non-units $a, b, c \in S$. If $abc \in I$, then either $ab \in I \subseteq J$ or $c \in I \subseteq J$. Assume that $abc \notin I$. Since I is a Q-ideal, there exist $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Thus $a = q_1 + i_1$, $b = q_2 + i_2$, $c = q_3 + i_3$ for some $i_1, i_2, i_3 \in I$. So $abc = (q_1 + i_1)(q_2 + i_2)(q_3 + i_3)$ $= q_1q_2q_3 + i_1q_2q_3 + i_2q_1q_3 + i_3q_1q_2 + i_1i_3q_2 + i_1i_2q_3 + i_2i_3q_1 + i_1i_2i_3$. Since J is a k-ideal of S, we have $q_1q_2q_3 \in J$. Also $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ for some unique element $q_4 \in Q$, where $q_1q_2q_3 + I \subseteq q_4 + I$. Thus $q_1q_2q_3 + i_4 = q_4 + i_5 \in J$. Since J is a k-ideal, $q_4 \in J \cap Q$ and $q_4 + I \in J/I_{(Q \cap J)}$. By hypothesis, $q_1 + I, q_2 + I, q_3 + I$ are non-units elements of S/I_Q and $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in J/I_{(Q \cap J)}$. If there exists a unique element $q \in Q$ such that $q + I$ is a zero element of S/I_Q and $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I$. Thus $q_1q_2q_3 + r = q + s$ for some $r, s \in I$, as J is a k-ideal, $q_1q_2q_3 \in I$. So $abc \in I$, which is a contradiction. Hence $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in J/I_{(Q \cap J)}$. Since $J/I_{(Q \cap J)}$ is a weakly 1-absorbing prime ideal, we conclude that either $q_1q_2 + I \in J/I_{(Q \cap J)}$ or $q_3 + I \in J/I_{(Q \cap J)}$. $(q_1 + I) \odot (q_2 + I) \in J/I_{(Q \cap J)}$ implies $ab \in J$ or $q_3 + I \in J/I$ implies $c \in J$. Therefore, J is a weakly 1-absorbing prime ideal of S. ∎

Theorem 22. *Let S be a local semiring and I be a subtractive ideal of S. If I is a weakly 1-absorbing prime ideal of S, then either I is a 1-absorbing prime ideal of S or $I^3 = 0$.*

Proof. Suppose that $I^3 \neq 0$. Then there exists $x, y, z \in I$ such that $xyz \neq 0$. If possible, let I is not a weakly 1-absorbing prime ideal of S. So there exist non-units a, b, $c \in S$ such that $abc = 0$ and $ab \notin I$, $c \notin I$.

Since S is a local semiring, set of non-units forms an ideal. So $(a+x)$, $(b+y)$, $(c + z)$ are non-units. Consider $(a + x)(b + y)(c + z) = abc + bcx + acy + abz + ayz + bxz + cxy + xyz \in I$.

We claim that $abz = bcx = acy = 0$. If $abz \neq 0$, then $0 \neq abz = ab(c+z) \in I$. Since I is a weakly 1-absorbing prime ideal, either $ab \in I$ or $c + z \in I$. As I is a subtractive ideal of S, either $ab \in I$ or $c \in I$, which is a contradiction. Hence we may assume that $abz = 0$. In a similar way, one can show $bcx = acy = 0$. Also $ayz = bxz = cxy = 0$. If suppose $ayz \neq 0$, then we have $0 \neq ayz = ayz + abz + acy + abc = a(y + b)(z + c)$. Hence either $a(b + y) \in I$ or $c + z \in I$. Since I is a subtractive ideal, so in both cases we arrive at a contradiction. Thus $ayz = 0$. In a similar way, $bxz = cxy = 0$.

Therefore, $0 \neq xyz = (a + x)(b + y)(c + z) \in I$ but $ab \notin I$, $c \notin I$ and I is a subtractive ideal of S, implies neither $(a + x)(b + y) \in I$ nor $c + z \in I$, which contradicts our hypothesis that I is a weakly 1-absorbing prime ideal of S. Therefore, I is a 1-absorbing prime ideal of S. ∎

Theorem 23. *Let S be a semiring and $Jac(S)$ be the Jacobson radical of S. If $Jac(S)$ is a strong ideal of S, then for any $a, b, c \in Jac(S)$, the ideal $I = \langle abc \rangle$ is a weakly 1-absorbing prime ideal of S if and only if $abc = 0$.*

Proof. If $abc = 0$ for some $a, b, c \in Jac(S)$, then clearly $I = \langle abc \rangle$ is a weakly 1-absorbing prime ideal of S. Conversely, assume that I is a weakly 1-absorbing prime ideal of S. If possible, let $abc \neq 0$. Since $abc \in I$, we have either $ab \in I$ or $c \in I$, that is, either $ab = abcx$ or $c = abcy$ for some $x, y \in S$. Suppose $ab = abcx$. As $c \in Jac(S)$ and $Jac(S)$ is a strong ideal, so there exists $c' \in Jac(S)$ such that $c + c' = 0$. This implies $ab(1 + c'x) = 0$ and by [[3], Lemma 3.4], $(1 + c'x)$ is a semiunit of S. So there exist $s, t \in S$ such that $1 + s(1 + c'x) = t(1 + c'x)$. Thus $ab = 0$ and so $abc = 0$, which is a contradiction. Also by similar arguments, if $c \in I$, we get $c = 0$, a contradiction. Therefore $abc = 0$. ■

Theorem 24. *Let S be a strong local semiring with maximal ideal M. Then every proper ideal of S is a weakly 1-absorbing prime ideal if and only if $M^3 = \{0\}$.*

Proof. Suppose that every proper ideal of S is weakly 1-absorbing prime and $a, b, c \in M$. Then the ideal $I = \langle abc \rangle$ is a weakly 1-absorbing prime. By Theorem 23, $abc = 0$ and so $M^3 = \{0.\}$ Conversely, suppose that $M^3 = \{0\}$ and I is a nonzero proper ideal of S. So there do not exist any non-units $a, b, c \in S$ such that $0 \neq abc \in I$. Hence, I is a weakly 1-absorbing prime ideal of S. ■

Theorem 25. *Let $S = S_1 \times S_2$, where S_1 and S_2 be commutative semirings. If I_1 is a proper ideal of S_1. Then the following are equivalent:*

(1) *I_1 is a 1-absorbing prime ideal of S_1.*

(2) *$I_1 \times S_2$ is a 1-absorbing prime ideal of S.*

(3) *$I_1 \times S_2$ is a weakly 1-absorbing prime ideal of S.*

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are trivial.

(3) $\implies$ (1) Suppose that $I_1 \times S_2$ is a weakly 1-absorbing prime ideal of S and $abc \in I_1$ for some non-units a, b, $c \in S_1$. Then $0 \neq (abc, 1) = (a, 1)(b, 1)(c, 1) \in I_1$. and $(a, 1)$, $(b, 1)$, $(c, 1)$ are non-units of S. Thus either $(a, 1)(b, 1) = (ab, 1) \in I_1 \times S_2$ or $(c, 1) \in I_1 \times S_2$, that is either $ab \in I_1$ or $c \in I_1$. Therefore, I_1 is a 1-absorbing prime ideal of S_1. ■

Theorem 26. *Let $S = S_1 \times S_2$, where S_1 and S_2 be commutative semirings but not semifields. If I_1 and I_2 are nonzero ideals of S_1 and S_2 respectively, then the following are equivalent:*

(1) *$I_1 \times I_2$ is a weakly 1-absorbing prime ideal of S.*

(2) *Either I_1 is a prime ideal of S_1 and $I_2 = S_2$ or I_2 is a prime ideal of S_2 and $I_1 = S_1$.*

(3) *$I_1 \times I_2$ is a 1-absorbing prime ideal of S.*

(4) *$I_1 \times I_2$ is a prime ideal of S.*

Proof. (1) $\implies$ (2) Suppose that $I_1 \times I_2$ is a weakly 1-absorbing prime ideal of S and $0 \neq (a, 0) \in I_1 \times I_2$ for some non zero $a \in I_1$. Then $0 \neq (a, 0) = (1, 0)(1, 0)(a, 1) \in I_1 \times I_2$. Therefore, either $(1, 0)(1, 0) = (1, 0) \in I_1 \times I_2$ or $(a, 1) \in I_1 \times I_2$. So either $1 \in I_1$ or $1 \in I_2$, that is, either $I_1 = S_1$ or $I_2 = S_2$.

Consider $I_2 = S_2$ and $ab \in I_1$ for some non-units $a, b \in S_1$. Since S_2 is not a semifield, there exists a non-unit $0 \neq z \in S_2$. Consider $0 \neq (ab, z) = (a, 1)(1, z)(b, 1) \in I_1 \times I_2$, then either $(a, z) = (a, 1)(1, z) \in I_1 \times I_2$ or $(b, 1) \in I_1 \times I_2$. Thus either $a \in I_1$ or $b \in I_2$. Therefore, I_1 is a prime ideal of S_1. Similarly, I_2 is a prime ideal of S_2 when $I_1 = S_1$.

(2) $\implies$ (3) follows from Theorem 25.

(3) $\implies$ (4) is clear from the Corollary 1.

(4) $\implies$ (1) is trivial. ■

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