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ON THE GRADING OF QUOTIENT SEMIRINGS

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Abstract

This paper investigates the grading of a quotient semiring of a graded semiring and explores many relationships between the homogeneous components of both gradings. Furthermore, the relationship between the supports of the original and induced gradings is established.

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1. Introduction

The concept of semirings was first introduced by H. S. Vandiver in 1935 [25], and it has since garnered considerable attention from researchers across various fields of mathematics. Semirings naturally arise in numerous mathematical areas due to their versatile structure. In 2008, Sharma and Joseph [23] introduced the notion of grading of a semiring by a finite group and investigated several properties related to such gradings. Earlier, in 1969, P. J. Allen [2] introduced the concepts of Q-ideals and the quotient semirings of a semiring. The quotient semiring has the greatest importance, as there are many studies that explain its importance. Next, the grading of semiring plays an important role in studying its properties, as it saves us effort and time to obtain the results, as we do not need to deal with the semiring to study its properties. Rather, it is only sufficient to deal with one or more subsets of it (homogeneous components) to judge whether that semiring checks this property or not. But the problem was how we would grade the quotient semiring.

In this research paper, we solved that problem by grading the quotient semiring of a graded semiring. We demonstrate how there are relationships between the homogeneous components of the original graded semiring and those of the quotient semiring. To this end, we investigate several properties that help to clarify how such connections can be established. Particularly, when the semiring is assumed to be E-inversive [1], yoked [10, 23], inverse [12], additively regular [12], additively idempotent [1], additively cancellative [18], M-divisible [15, 16], divisible [15, 16], almost-divisible [15], or zero sum free [14] and when an element in the semiring is assumed to be infinite [7], M-divisible [15, 16], divisible [15, 16], almost-divisible [15], or additively idempotent [1]. Moreover, we explore the relationship between the zeroids [18] of both the semiring and its quotient semiring. As a future idea, we will extend our research on studying as many properties and relations between the semiring and the quotient semiring as possible. The problem of the grading of a quotient semiring has already been solved in this research paper.

2. PRELIMINARIES

This section introduces key definitions concerning semirings and quotient semirings, which will be frequently referenced throughout the text [1, 3, 5, 7, 8, 10, 12, 14, 15, 16, 17, 18, 19, 22, 23, 24].

Note that by a group (monoid, semigroup) we mean a multiplicative group (multiplicative monoid, multiplicative semigroup) unless otherwise specified. For the definition of commutative cancellative monoid, see [4, 21], and for the definition of yoked semigroup, see [20].

Definition 2.1 [3, 19]**.** A semiring is a nonempty set R with two associative operations $(+)$ and $(.)$ for which $(+)$ is commutative and there exists $0 \in R$ such that for each $\alpha \in R$, we get $\alpha 0 = 0\alpha = 0$ and $\alpha + 0 = \alpha$, and the multiplication operation is distributive over addition on both sides.

If $(.)$ is commutative, then R is called commutative. A subsemiring S of a semiring R with zero 0 is a nonempty subset of the semring R satisfies $\alpha+\beta, \alpha\beta \in S$ for all $\alpha, \beta \in S$ and $0 \in S$.

Definition 2.2 [18, 22]**.** A semiring R is said to be additively cancellative if for all $a, b, c \in R$, $a + b = a + c$ implies that $b = c$.

Definition 2.3 [18]**.** The zeroid $Z_r(R)$ of a semiring R is the set

$$Z_r(R) = \{x \in R | \exists y \in R, x + y = y\}$$

Definition 2.4 [1]**.** Denote by $E^+(R)$ the set of all additive idempotent elements in a semiring R where an additive idempotent element in R is such an element ζ for which $\zeta = \zeta + \zeta$, and if all elements in R are additive idempotent, then R is called additively idempotent. Now, if for each $\kappa \in R$ there exists an element $\psi \in R$ for which $\kappa + \psi \in E^+(R)$, then R is called E-inversive.

Definition 2.5 [10, 23]**.** In a semiring R if for every $\zeta, \kappa \in R$, there exists $\psi \in R$ for which $\zeta + \psi = \zeta$ or $\kappa + \psi = \zeta$, then R is called yoked.

Definition 2.6 [14]**.** A semiring R is said to be zero-sum free if for all $\zeta, \kappa \in R$, $\zeta + \kappa = 0$ implies that $\zeta = \kappa = 0$.

Definition 2.7 [12]**.** An element ζ in a semiring R is called additively regular if there exists unique element $\kappa \in R$ such that $\zeta = \zeta + \kappa + \zeta$. If every element in R is additively regular, then R is called additively regular.

Definition 2.8 [12]**.** An element ζ in a semiring R is called inverse if there exists unique element $\kappa \in R$ such that $\zeta = \zeta + \kappa + \zeta$ and $\kappa = \kappa + \zeta + \kappa$. If every element in R is inverse, then R is called inverse.

Definition 2.9 [7]**.** an element ζ in a semiring R is called infinite if $\zeta + \kappa = \zeta$ for all $\kappa \in R$.

Definition 2.10 [15, 16]**.** Let $(R, +)$ be a semigroup. Suppose $\phi \neq M \subseteq \mathbb{Z}^+$. An element $\alpha \in R$ is said to be M-divisible in R if for each $n \in M$ there exists $\beta \in R$ such that $\alpha = n\beta$.

A semigroup $(R, +)$ is said to be M-divisible (divisible, resp.) if every element of R is M- divisible ($\mathbb{Z}^+$-divisible, resp.) in R.

A semiring R is said to be additively M-divisible (additively divisible, resp.) if $(R, +)$ is an M-divisible semigroup (a divisible semigroup, resp.).

Definition 2.11 [15]**.** Let $(R, +)$ be a semigroup. An element $\alpha \in R$ is said to be almost-divisible in R if there is $\beta \in \mathbb{Z}^+a$ such that β is P-divisible in R for some infinite set of prime numbers P.

A semigroup $(R, +)$ is said to be almost-divisible if every element of R is almost- divisible in R.

A semiring R is said to be additively almost-divisible if $(R, +)$ is an almost-divisible semigroup.

Definition 2.12 [8, 17, 24]**.** A partitioning ideal (also called a Q-ideal) of a semiring R is an ideal I in R for which there exists a subset Q of the semiring R for which $R = \bigcup_{\zeta \in Q} (\zeta + I)$, and $(\zeta_1 + I) \cap (\zeta_2 + I) \neq \phi$ iff $\zeta_1 = \zeta_2$ for all $\zeta_1, \zeta_2 \in Q$.

Define $R/I = \{\zeta + I \mid \zeta \in Q\}$ and define on R/I an operation $(+)$ and an operation $(.)$ such that $(\zeta_1 + I) + (\zeta_2 + I) = \zeta_3 + I$ where the element ζ_3 is unique element in Q for which $(\zeta_1 + \zeta_2) + I \subseteq \zeta_3 + I$, and $(\zeta_1 + I)(\zeta_2 + I) = \zeta_4 + I$ where the element ζ_4 is unique element in Q for which $(\zeta_1\zeta_2) + I \subseteq \zeta_4 + I$, then we get that $(R/I, +, .)$ is the quotient semiring of R, and the zero element in (R/I) is unique element $\zeta' + I$ for which $0 + I \subseteq \zeta' + I$. Note that if Q is an additive submonoid of R, then $\zeta' = 0$.

Definition 2.13 [5, 23]**.** A graded semiring R by a group (a semigroup, a monoid) G is a semiring in which $R = \underset{t\in G}{\oplus} R_t$ where R_t is an additive submonoid of R and $R_tR_s \subseteq R_{ts}$ for all $t, s \in G$.

Denote (R, G) the grading of the semiring R by G. The support of the grading (R, G) is a subset of G denoted by $Supp(R, G)$ defined as follows $\{t \in G | R_t \neq \{0\}\}$. For each $t \in G$, R_t is called a homogeneous component of the grading (R, G). Also, if R is graded by a group or a monoid G with identity e, then R_e is a subsemiring of R.

Example 2.14. Let S be a semiring, G be a left zero semigroup with at least two elements g_1, g_2. Suppose R is the set of all 3×3 diagonal matrices over S. Then R with two binary operations $(+)$, the addition of matrices, and $(.)$, the multiplication of matrices, is a semiring. Now, Suppose

$$R_{g_1} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b \in S \right\}$$
$$R_{g_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \middle| \, c \in S \right\}$$
$$R_g = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \forall g \in G - \{g_1, g_2\}.$$

Then the family $\{R_t\}_{t\in G}$ forms a grading of R by G.

Example 2.15 [6]**.** Suppose $G = \{0, 1\}$. Then (G, min) is a monoid with identity element where all of its elements are idempotent and has an absorbing element 0. Now, let R be a semiring with unity 1_R. Define on $A = \{(a, b); a, b \in R\}$ two operations $(+)$ and $(.)$ as follows

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac + ad + bc, bd)$$

for all $(a,b),(c,d) \in A$, Then it can be easily seen that $(A,+,.)$ is a semiring with unity $(0_R, 1_R)$ and zero $(0_R, 0_R)$. Define

$$A_0 = \{(x, 0_R); x \in R\}$$
$$A_1 = \{(0_R, x); x \in R\}.$$

Then it can be easily seen that the family $\{A_0, A_1\}$ forms a grading of A by G.

3. The Grading of the Quotient Semiring R/I by a Grading of R

In this section, we prove the grading of a quotient semiring R/I where R is a graded semiring and I is a Q-ideal in R. Also, we give an example to support this theorem.

Theorem 3.1. *Suppose G is a left zero semigroup and suppose $Q = \underset{g\in G}{\oplus} Q_g$ is an additive submonoid of a semiring R in which Q_g is an additive submonoid of Q for all $g \in G$. Assume that I is a Q-ideal in R and the quotient semiring of R is R/I. Suppose for each element $g \in G$, the following condition*

$$r_g \in R_g \Longrightarrow \exists \alpha_g \in Q_g; r_g + I \subseteq \alpha_g + I (\forall r_g \in R_g). \tag{1}$$

If $R = \underset{g\in G}{\oplus} R_g$ is graded by G, then the family $\{(R/I)_g\}_{g\in G}$ forms a grading of R/I, such that

$$(R/I)_g = \{\alpha_g + I \in R/I | \exists r_g \in R_g; r_g + I \subseteq \alpha_g + I \ \&\ \alpha_g \in Q_g\}$$

for all $g \in G$.

Proof. We have $(R/I)_g \subseteq R/I$. Let g be an element in G. Since $0 + I \subseteq 0 + I$, $0 \in R_g$ and $0 \in Q_g$, then $0 + I \in (R/I)_g$. Therefore $(R/I)_g \neq \phi$. Now, suppose $\zeta + I, \beta + I$ are elements in $(R/I)_g$. Then $\zeta, \beta \in Q_g$ and there exist ψ_g and λ_g in R_g such that $\psi_g + I \subseteq \zeta + I$ and $\lambda_g + I \subseteq \beta + I$, also $\psi_g \in \psi_g + I \subseteq \zeta + I$, $\psi_g + \lambda_g \in R_g$, and $\zeta + \beta \in Q_g$. Therefore $(\psi_g + \lambda_g) + I \subseteq ((\psi_g + I) + \lambda_g) + I \subseteq (\psi_g + I) + (\lambda_g + I) \subseteq (\zeta + I) + (\beta + I) = (\zeta + \beta) + I$, $\psi_g + \lambda_g \in R_g$, and $\zeta + \beta \in Q_g$ follow that $(R/I)_g$ is an additive submonoid of R/I. Let g, h be two elements in G and let η be an element in $(R/I)_g (R/I)_h$. Then there exists $u + I \in (R/I)_g$ and there exists $v + I \in (R/I)_h$ for which $\eta = (u + I)(v + I)$. Now, we have

- $u + I \in (R/I)_g \Longrightarrow \exists x_g \in R_g \ ; \ x_g + I \subseteq u + I.$
- $v + I \in (R/I)_h \Longrightarrow \exists y_h \in R_h \ ; \ y_h + I \subseteq v + I.$

And also, we have $\eta = (u+I)(v+I) = \kappa + I$ where κ is unique element in Q for which $uv + I \subseteq \kappa + I$ (see [17, 24]). Now, we have $x_g y_h + I \subseteq (u+I)y_h + I \subseteq uy_h + Iy_h + I \subseteq uy_h + I \subseteq u(v+I) + I \subseteq uv + uI + I \subseteq uv + I \subseteq \kappa + I$ and $x_g y_h \in R_g R_h \subseteq R_{gh}$. Now, since $x_g y_h \in R_{gh}$, then by condition (1), there exists an element $\mu \in Q_{gh}$ for which $x_g y_h + I \subseteq \mu + I$. Since $x_g y_h + I \subseteq \kappa + I = \eta$, $x_g y_h + I \subseteq \mu + I$, and since I is a Q-ideal in R, we get $\kappa = \mu$, hence $\kappa \in Q_{gh}$ and $\eta \in (R/I)_{gh}$. Therefore $(R/I)_g (R/I)_h \subseteq (R/I)_{gh}$.

Now, we shall prove that $R/I = \bigoplus_{g\in G} (R/I)_g$. Let θ be an element in R/I. Then there exists $\zeta \in Q$ such that $\theta = \zeta + I$. Since $\zeta \in Q \subseteq R = \bigoplus_{g\in G} R_g$, we can write $\zeta = \sum_{g\in G} \gamma_g$ such that $\gamma_g \in R_g$. Thus $\theta = \zeta + I = \sum_{g\in G} \gamma_g + I$. On the other hand, for every $g \in G$, we have $\gamma_g \in R_g$ implies there exists $\zeta_g \in Q_g$ such that $\gamma_g \in \gamma_g + I \subseteq \zeta_g + I$ (Condition (1)). Therefore $\zeta_g + I \in (R/I)_g$ for all $g \in G$ and $\theta = \zeta + I = \sum_{g\in G} \gamma_g + I \subseteq \sum_{g\in G} (\zeta_g + I) + I \subseteq \sum_{g\in G} (\zeta_g + I) = \sum_{g\in G} \zeta_g + I$. Since I is a Q-ideal in R, and since ζ, $\sum_{g\in G} \zeta_g$ are elements in Q such that $\zeta + I \subseteq \zeta + I$ and $\zeta + I \subseteq \sum_{g\in G} \zeta_g + I$, we get $\theta = \zeta + I = \sum_{g\in G} \zeta_g + I = \sum_{g\in G} (\zeta_g + I)$. Therefore $R/I = \sum_{g\in G} (R/I)_g$. We have $\theta = \zeta + I = \sum_{g\in G} \gamma_g + I = \sum_{g\in G} \zeta_g + I$. Since $\sum_{g\in G} \gamma_g, \sum_{g\in G} \zeta_g \in Q$ and I is a Q-ideal, we get $\sum_{g\in G} \gamma_g = \sum_{g\in G} \zeta_g$. Assume that θ can also be written as follows $\theta = \sum_{g\in G} (\beta_g + I) = \sum_{g\in G} \beta_g + I$ such that $\beta_g + I \in (R/I)_g$ for all $g \in G$. Therefore $\theta = \sum_{g\in G} \gamma_g + I = \sum_{g\in G} \beta_g + I$. Since both $\sum_{g\in G} \gamma_g$, $\sum_{g\in G} \beta_g$ are elements in Q and I is Q-ideal, we get $\sum_{g\in G} \gamma_g = \sum_{g\in G} \beta_g$, hence $\sum_{g\in G} \zeta_g = \sum_{g\in G} \beta_g$ where $\zeta_g, \beta_g \in Q_g$ for all $g \in G$. It follows that $\zeta_g = \beta_g$ for all $g \in G$ (since $Q = \bigoplus_{g\in G} Q_g$). Therefore $\zeta_g + I = \beta_g + I$ for all $g \in G$. ∎

In the following, we give an example to support the above theorem.

Example 3.2. Suppose $R' = \bigoplus_{g\in G} R'_g$ is a graded semiring by a left zero semigroup G. Assume that

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in R' \right\}$$

$$R_g = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in R'_g \right\} \forall g \in G$$

$$Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \, a \in R' \right\}$$

$$Q_g = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in R'_g \right\} \forall g \in G$$
$$I = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle| a, b \in R' \right\}$$

1. $(R, +, .)$ is a graded semiring by G where $R = \underset{g\in G}{\oplus} R_g$.
2. $Q = \underset{g\in G}{\oplus} Q_g$ is an additive submonoid of the semiring R where for each $g \in G$, Q_g is an additive submonoid of Q, and I is a Q-ideal in R.
3. Assume that g is an element in G. For each $r_g \in R_g$, there exists $\alpha_g \in Q_g$ such that $r_g + I \subseteq \alpha_g + I$.

4. Homogeneous Components and Support Relations in a Graded Semiring and Its Graded Quotient Semiring

In this section, we examine several properties of the homogeneous components in the gradings of both the semiring and its quotient semiring. Also, we explore the relationship between the supports of these gradings. First, we begin this section with the following theorem.

Theorem 4.1. *Let $Q = \underset{g\in G}{\oplus} Q_g$ be an additive submonoid of a graded semiring $R = \underset{g\in G}{\oplus} R_g$ by a left zero semigroup G where Q_g is an additive submonoid of Q for all $g \in G$. Suppose I is a Q-ideal in R and R/I is the quotient semiring of R. Suppose for each $g \in G$ the following condition holds:*

$$r_g \in R_g \Longrightarrow \exists \alpha_g \in Q_g; r_g + I \subseteq \alpha_g + I (\forall r_g \in R_g). \tag{2}$$

By Theorem 3.1, the family $\{(R/I)_g\}_{g\in G}$ forms a grading of R/I such that

$$(R/I)_g = \{\alpha_g + I \in R/I | \exists r_g \in R_g; r_g + I \subseteq \alpha_g + I \ \&\ \alpha_g \in Q_g\}$$

for all $g \in G$. Then we have the following:

1. *For each $g \in G$.*
   - *(a) R_g is E-inversive $\Longrightarrow$ $(R/I)_g$ is E-inversive.*
   - *(b) R_g is yoked $\Longrightarrow$ $(R/I)_g$ is yoked.*
2. *Suppose $R_g \nsubseteq I$ for all $g \in Supp(R, G)$. Then $Supp(R, G) = Supp(R/I, G)$.*

3. *Let g be an element in G. Suppose Q_g is additively cancellative. Then*
   (a) *R_g is inverse $\implies$ $(R/I)_g$ is inverse.*
   (b) *R_g is additively regular $\implies$ $(R/I)_g$ is additively regular.*
4. *Let g be an element in G. Suppose $R_g \cap I - \{0\}$. Then*
$$R_g \text{ is zero-sum free} \iff (R/I)_g \text{ is zero-sum free.}$$
5. *Let g be an element in G. Suppose I is additively cancellative and suppose $\alpha + i = \alpha + j$ implies $i = j$ for all $\alpha \in Q_g$ and for all $i, j \in I$. Then*
$$(R/I)_g \text{ is additively cancellative} \implies R_g \text{ is additively cancellative.}$$
6. *Let g be an element in G and r be an infinite element in R_g. Then there exists an infinite element $a + I \in (R/I)_g$ such that $r + I \subseteq a + I$.*
7. *Let g be an element in G and r be an element in $Z_r(R_g)$. Then there exists an element $a + I \in Z_r((R/I)_g)$ such that $r + I \subseteq a + I$.*
8. *Let g be an element in G and r be an element in $E^+(R_g)$. Then there exists an element $a + I \in E^+((R/I)_g)$ such that $r + I \subseteq a + I$.*
9. *Let g be an element in G. Then*
$$R_g \text{ is additively idempotent} \implies (R/I)_g \text{ is additively idempotent.}$$
10. *Suppose $\phi \neq M \subseteq \mathbb{Z}^+$. Then we have the following:*
   (a) *Let g be an element in G and let r be an M-divisible element in R_g. Then there exists an M-divisible element $a + I \in (R/I)_g$ such that $r + I \subseteq a + I$.*
   (b) *Let g be an element in G. Then*
$$R_g \text{ is } M\text{-divisible} \implies (R/I)_g \text{ is } M\text{-divisible.}$$
   (c) *Let g be an element in G and let r be a divisible element in R_g. Then there exists a divisible element $a + I \in (R/I)_g$ such that $r + I \subseteq a + I$.*
   (d) *Let g be an element in G. Then*
$$R_g \text{ is divisible} \implies (R/I)_g \text{ is divisible.}$$
11. *Let g be an element in G and let r be an almost-divisible element in R_g. Then there exists $n \in \mathbb{Z}^+$ and there exists a P-divisibl element $a + I \in (R/I)_g$ such that $nr + I \subseteq a + I$ for some infinite set of prime numbers P.*

12. *Let g be an element in G and let r be an almost-divisible element in R_g. Then there exists almost-divisibl element $a+I \in (R/I)_g$ such that $r+I \subseteq a+I$.*

13. *Let g be an element in G. Then*

$$R_g \text{ is almost-divisible} \implies (R/I)_g \text{ is almost-divisible.}$$

Proof. First, we shall prove that R_g is a subsemiring of R and $(R/I)_g$ is a subsemiring of (R/I) for all $g \in G$. Suppose g is an element in G. Since R is graded by G, $R_gR_g \subseteq R_{gg} = R_g$. Since R is graded by G, R_g is an additive submonoid of R. Therefore R_g is a subsemiring of R. Similarly, we can prove that $(R/I)_g$ is a subsemiring of (R/I).

1. Suppose g is an element in G.

   (a) Suppose R_g is E-inversive, and $\psi + I$ is an element in $(R/I)_g$. Then $\psi \in Q_g$ and there exists $r_g \in R_g$ such that $r_g \in r_g + I \subseteq \psi + I$. Now, since $r_g \in R_g$ and R_g is E-inversive, there exists $\gamma \in R_g$ such that $\gamma + r_g \in E^+(R_g)$. Also, since $\gamma \in R_g$, there exists $\lambda \in Q_g$ such that $\gamma \in \gamma + I \subseteq \lambda + I$ (by Condition (2)). Therefore $\gamma + r_g \in (\gamma + r_g) + I \subseteq (\lambda + \psi) + I$. On the other hand, since $\gamma + r_g \in E^+(R_g)$, we have $(\gamma + r_g) + I = [(\gamma + r_g) + (\gamma + r_g)] + I = [(\gamma + r_g) + I] + [(\gamma + r_g) + I] \subseteq [(\lambda + \psi) + I] + [(\lambda + \psi) + I] = [(\lambda + \psi) + (\lambda + \psi)] + I$. Since $(\gamma + r_g) \in R_g$, by condition (2), there exists $\eta_g \in Q_g$ such that $(\gamma + r_g) + I \subseteq \eta_g + I$. Hence

$$\lambda + \psi \in Q_g \subseteq Q \quad \& \quad (\gamma + r_g) + I \subseteq (\lambda + \psi) + I \tag{3}$$

and

$$(\lambda + \psi) + (\lambda + \psi) \in Q_g \subseteq Q \quad \& \quad (\gamma + r_g) + I \subseteq [(\lambda + \psi) + (\lambda + \psi)] + I. \tag{4}$$

   Now, from (3), (4) and since I is Q-ideal in R, we get $(\lambda + \psi) + I = [(\lambda + \psi) + (\lambda + \alpha)] + I$. Since $(\lambda + \psi) + I = (\lambda + I) + (\psi + I)$ and $[(\lambda + \psi) + (\lambda + \psi)] + I = [(\lambda + I) + (\psi + I)] + [(\lambda + I) + (\psi + I)]$, then $(\lambda + I) + (\psi + I) = [(\lambda + I) + (\psi + I)] + [(\lambda + I) + (\psi + I)]$. Finally, we have $\gamma + I \subseteq \lambda + I$, $\gamma \in R_g$, and $\lambda \in Q_g$. It follows that $\lambda + I \in (R/I)_g$. Therefore $(R/I)_g$ is E-inversive.

   (b) Let $\psi + I, \lambda + I$ be two elements in $(R/I)_g$. Suppose that R_g is yoked. Then

- $\psi + I \in (R/I)_g \Longrightarrow \psi \in Q_g$ and $\exists \gamma_g \in R_g$; $\gamma_g + I \subseteq \psi + I$
- $\lambda + I \in (R/I)_g \Longrightarrow p \in Q_g$ and $\exists \theta_g \in R_g$; $\theta_g + I \subseteq \lambda + I$
- R_g is yoked and $\gamma_g, \theta_g \in R_g \Longrightarrow \exists r_g \in R_g$; $\gamma_g + r_g = \theta_g$ or $\theta_g + r_g = \gamma_g$

If $\gamma_g + r_g = \theta_g$, then $\theta_g + I \subseteq \lambda + I$ implies $(\gamma_g + r_g) + I \subseteq \lambda + I$. Since $r_g \in R_g$, there exists an element $\eta_g \in Q_g$ such that $r_g + I \subseteq \eta_g + I$ (by Condition (2)). Therefore $(\gamma_g + I) + (r_g + I) \subseteq (\psi + I) + (\eta_g + I)$. Hence $(\gamma_g + r_g) + I \subseteq (\psi + \eta_g) + I$. Since $(\gamma_g + r_g) + I \subseteq \lambda + I$, $(\gamma_g + r_g) + I \subseteq (\psi + \eta_g) + I$, and I is a Q-ideal, we get $(\psi + \eta_g) + I = (\psi + I) + (\eta_g + I) = \lambda + I$. Again, if $\theta_g + r_g = \gamma_g$, then in the similar way, we can find that $(\lambda + I) + (\eta_g + I) = \psi + I$ such that $\eta_g \in Q_g$ satisfies $r_g + I \subseteq \eta_g + I$. Finally, since $r_g + I \subseteq \eta_g + I$, $\eta_g \in Q_g$,and $r_g \in R_g$, we get $\eta_g + I \in (R/I)_g$. Hence $\eta_g + I \in (R/I)_g$ for which $(\lambda + I) + (\eta_g + I) = \psi + I$ or $(\psi + I) + (\eta_g + I) = \lambda + I$. Therefore $(R/I)_g$ is yoked.

2. Assume that $R_g \nsubseteq I$. Suppose g is an element in $Supp(R/I, G)$. Then $(R/I)_g \neq \{0_{R/I}\}$. This implies that there exists $0_{R/I} \neq \psi + I \in (R/I)_g$. Since $\psi + I \neq 0 + I$, $\psi \neq 0$. Again, since $\psi + I \in R/I)_g$, there exists $r_g \in R_g$ such that $r_g + I \subseteq \psi + I$ and $\psi \in Q_g$. We claim that $r_g \neq 0$. Suppose, to the contrary, that $r_g = 0$. Then $0 + I \subseteq \psi + I$. Now, since $0 + I \subseteq 0 + I$, $0 + I \subseteq \psi + I$, and I is a Q-ideal, we get $\psi = 0$, which contradicts the fact that $\psi \neq 0$. Hence $R_g \neq \{0\}$. Therefore $g \in Supp(R, G)$. Thus

$$Supp(R/I, G) \subseteq Supp(R, G). \tag{5}$$

Conversely, let g be an element in $Supp(R, G)$. Then $R_g \neq \{0\}$. On the other hand, let $r_g \in R_g - \{0\}$. Then there exists $t_g \in Q_g$ such that $r_g + I \subseteq t_g + I$ (by Condition (2)). Hence $t_g + I \in (R/I)_g$, $r_g \in r_g + I \subseteq t_g + I$, and $r_g \neq 0$. Suppose $(R/I)_g = \{0 + I\}$. Since $r_g \in t_g + I$ and $t_g + I \in (R/I)_g = \{0 + I\}$, $r_g \in I$. Therefore $R_g \subseteq I$, which contradicts the assumption that $R_g \nsubseteq I$. Therefore $(R/I)_g \neq \{0 + I\}$. Hence $g \in Supp(R/I, G)$. Therefore

$$Supp(R, G) \subseteq Supp(R/I, G) \tag{6}$$

Thus from (5) and (6), we get $Supp(R, G) = Supp(R/I, G)$.

3. (a) Suppose R_g is inversive and suppose $\kappa_g + I$ is an element in $(R/I)_g$. Since $\kappa_g + I \in (R/I)_g$, $\kappa_g \in Q_g$ and there exists $r_g \in R_g$ such that $r_g + I \subseteq \kappa_g + I$. Since $r_g \in R_g$ and R_g is inverse, there exists unique element $a_g \in R_g$ such that $r_g = r_g + a_g + r_g$ and $a_g = a_g + r_g + a_g$. Hence $(r_g + a_g + r_g) + I \subseteq \kappa_g + I$. Now, since $a_g \in R_g$, there exists an element $\zeta_g \in Q_g$ such that $a_g + I \subseteq \zeta_g + I$ (by Condition (2)). Hence

$(r_g + a_g + r_g) + I \subseteq (\kappa_g + \zeta_g + \kappa_g) + I$. Since $(r_g + a_g + r_g) + I \subseteq \kappa_g + I$, $(r_g + a_g + r_g) + I \subseteq (\kappa_g + \zeta_g + \kappa_g) + I$, and I is Q-ideal, we get $\kappa_g = \kappa_g + \zeta_g + \kappa_g$. Therefore $\kappa_g + I = (\kappa_g + I) + (\zeta_g + I) + (\kappa_g + I)$. Again, since $a_g = a_g + r_g + a_g$, then $a_g + I \subseteq \zeta_g + I$ implies $(a_g + r_g + a_g) + I \subseteq \zeta_g + I$. Also, since $(a_g + r_g + a_g) + I \subseteq (\zeta_g + \kappa_g + \zeta_g) + I$, $(a_g + r_g + a_g) + I \subseteq \zeta_g + I$, and since I is Q-ideal, we get $\zeta_g + \kappa_g + \zeta_g = \zeta_g$. Therefore $(\zeta_g + I) + (\kappa_g + I) + (\zeta_g + I) = \zeta_g + I$. Now, assume that $s_g + I \in (R/I)_g$ such that $\kappa_g + I = (\kappa_g + I) + (s_g + I) + (\kappa_g + I)$ and $s_g + I = (s_g + I) + (\kappa_g + I) + (s_g + I)$. Then $\kappa_g + I = (\kappa_g + s_g + \kappa_g) + I$ and $s_g + I = (s_g + \kappa_g + s_g) + I$. Since I is Q-ideal, we get $\kappa_g = \kappa_g + s_g + \kappa_g$ and $s_g = s_g + \kappa_g + s_g$. Finally, we have $\kappa_g + I = (\kappa_g + s_g + \kappa_g) + I = (\kappa_g + \zeta_g + \kappa_g) + I$. Since I is Q-ideal, we get $\kappa_g + s_g + \kappa_g = \kappa_g + \zeta_g + \kappa_g$. Again, since Q_g is additively cancellative, we get $s_g = \zeta_g$. Thus $s_g + I = \zeta_g + I$. Therefore $(R/I)_g$ is inverse. Similarly, we can prove (b).

4. Suppose R_g is zero-sum free. Let $a + I, b + I$ be two elements in $(R/I)_g$ such that $(a + I) + (b + I) = 0 + I$. Since $a + I, b + I \in (R/I)_g$, there exist $a_g, b_g \in R_g$ such that $a_g + I \subseteq a + I$ and $b_g + I \subseteq b + I$. Therefore $(a_g + I) + (b_g + I) \subseteq (a + I) + (b + I)$. Hence $(a_g + b_g) + I \subseteq (a + b) + I$. Now, since $(a + b) + I = (a + I) + (b + I) = 0 + I$, we get $(a_g + b_g) + I \subseteq 0 + I$. Hence $a_g + b_g \in R_g \cap I = \{0\}$. Therefore $a_g + b_g = 0$. Since R_g is zero-sum free, we get $a_g = b_g = 0$. Therefore $0 + I \subseteq a + I$ and $0 + I \subseteq b + I$. Now, since I is Q-ideal, we get $a + I = 0 + I$ and $b + I = 0 + I$. Thus $(R/I)_g$ is zero-sum free.

   Conversely, suppose $(R/I)_g$ is zero-sum free. Let a_g, b_g be two elements in R_g such that $a_g + b_g = 0$. Sincce $a_g, b_g \in R_g$, there exist $a, b \in Q_g$ such that $a_g + I \subseteq a + I$ and $b_g + I \subseteq b + I$. Therefore there exist $a + I, b + I \in (R/I)_g$ such that $a_g + I \subseteq a + I$ and $b_g + I \subseteq b + I$. Thus $(a_g + b_g) + I \subseteq (a + b) + I$, which implies $0 + I \subseteq (a + b) + I$. Since I is Q-ideal, we get $(a + b) + I = 0 + I$. Therefore $(a + I) + (b + I) = 0 + I$. Now, since $(R/I)_g$ is zero-sum free, we get $a + I = 0 + I$ and $b + I = 0 + I$. Therefore $a_g \in a_g + I \subseteq I$ and $b_g \in b_g + I \subseteq I$. Since $R_g \cap I = \{0\}$, we get $a_g = b_g = 0$. Thus R_g is zero-sum free.

5. Suppose $(R/I)_g$ is additively cancellative. Let a_g, b_g, c_g be elements in R_g such that $a_g + b_g = a_g + c_g$. Since $a_g, b_g, c_g \in R_g$, there exist $a, b, c \in Q_g$ such that $a_g + I \subseteq a + I$, $b_g + I \subseteq b + I$, and $c_g + I \subseteq c + I$. Therefore there exist $a + I, b + I, c + I \in (R/I)_g$ such that $a_g + I \subseteq a + I$, $b_g + I \subseteq b + I$, and $c_g + I \subseteq c + I$. Since $(a_g + b_g) + I \subseteq (a + b) + I$, $(a_g + b_g) + I = (a_g + c_g) + I \subseteq (a + c) + I$, and I is Q-ideal, we get $(a + b) + I = (a + c) + I$. It follows that $(a + I) + (b + I) = (a + I) + (c + I)$. Therefore $b + I = c + I$,

since $(R/I)_g$ is additively cancellative. Now, since $b_g \in b_g + I \subseteq b + I$ and $c_g \in c_g + I \subseteq c + I = b + I$, there exist $i, j \in I$ such that $b_g = b + i$ and $c_g = b + j$. Since $a_g \in a + I$, there exists $k \in I$ such that $a_g = a + k$. Hence $a_g + b_g = a_g + b + i = ((a+k)+b)+i$ and $a_g + c_g = a_g + c + i = ((a+k)+c)+j$. Therefore $a_g + b_g = a + b + i + k$ and $a_g + c_g = a + c + j + k$. Since $a_g + b_g = a_g + c_g$ and since $\alpha + i = \alpha + j$ implies $i = j$ for all $\alpha \in Q_g$ and for all $i, j \in I$, we get $i + k = j + k$. It follows that $i = j$ (since I is additively cancellative). Now, since $i = j$, $b_g = b + i$, and $c_g = b + j$, we get $b_g = c_g$. Thus R_g is additively cancellative.

6. Since $r \in R_g$, there exists an element $a \in Q_g$ such that $r + I \subseteq a + I$. Let $b + I$ be an element in $(R/I)_g$. Since $b \in Q_g$, there exists $t \in R_g$ such that $t + I \subseteq b + I$. Therefore $(r + t) + I \subseteq (a + b) + I$. Now, since r is infinite, $r + I \subseteq (a + b) + I$. Therefore $r + I \subseteq ((a + b) + I) \cap (a + I)$. It follows that $a + I = (a + I) + (b + I)$ (since I is Q-ideal). Thus $a + I \in (R/I)_g$ is infinite.

7. Since $r \in Z_r(R_g)$, there exists an element $t \in R_g$ such that $r + t = t$. Again, since $r, t \in R_g$, there exist $a, b \in Q_g$ such that $r + I \subseteq a + I$ and $t + I \subseteq b + I$. Therefore $(r + t) + I \subseteq (a + b) + I$. Hence $t + I \subseteq (a + b) + I$. Now, we have $t + I \subseteq ((a + b) + I) \cap (b + I)$. Thus $(a + I) + (b + I) = b + I$ (since I is Q-ideal). Therefore $a + I \in Z_r((R/I)_g)$ and $r + I \subseteq a + I$.

8. Since $r \in E^+(R_g) \subseteq R_g$, $r + r = r$ and there exists an element $a \in Q_g$ such that $r + I \subseteq a + I$. Therefore $(r + r) + I = r + I \subseteq (a + a) + I$. Hence $r + I \subseteq (a + I) \cap ((a + a) + I)$. It follows that $a + I = (a + I) + (a + I)$ (since I is Q-ideal). Thus $a + I \in E^+((R/I)_g)$ and $r + I \subseteq a + I$.

9. Suppose R_g is additively idempotent. Let $a + I$ be an element in $(R/I)_g$. Since $a + I \in (R/I)_g$, there exists an element $r \in R_g$ such that $r + I \subseteq a + I$. Now, since $r \in R_g$ and R_g is additively idempotent, we get $r + r = r$. Therefore $(r+r)+I = r+I \subseteq (a+a)+I$. Hence $r+I \subseteq (a+I) \cap ((a+a)+I)$. It follows that $a + I = (a + I) + (a + I)$ (since I is Q-ideal). Thus $a + I \in E^+((R/I)_g)$. Therefore $(R/I)_g$ is additively idempotent.

10. (a) Let n be an element in M. Since r is M-divisible in R_g, there exists $t \in R_g$ such that $r = nt$. Again, since $r, t \in R_g$, there exist $a, b \in Q_g$ such that $r + I \subseteq a + I$ and $t + I \subseteq b + I$. Therefore $a + I, b + I \in (R/I)_g$, $n(t + I) = nt + I \subseteq a + I$, and $nt + I = n(t + I) \subseteq n(b + I) = nb + I$. Since I is Q-ideal, we get $a + I = n(b + I)$. Thus $a + I$ is M-divisible in $(R/I)_g$ and $r + I \subseteq a + I$.

    (b) Suppose R_g is M-divisible. Let g be an element in G and $a + I$ be an in $(R/I)_g$. Since $a + I \in (R/I)_g$, there exists $r \in R_g$ such that

$r + I \subseteq a + I$. Now, let n be an element in M. Since $r \in R_g$ and R_g is M-divisible, there exists $t \in R_g$ such that $r = nt$. Again, since $t \in R_g$, there exists $b \in Q_g$ such that $t + I \subseteq b + I$. It follows that $b + I \in (R/I)_g$. Now, we have $n(t + I) = nt + I \subseteq a + I$ and $nt + I = n(t + I) \subseteq n(b + I) = nb + I$. Since I is Q-ideal, we get $a + I = n(b + I)$. Thus $a + I$ is M-divisible in $(R/I)_g$. Thus $(R/I)_g$ is M-divisible. Similarly, we can prove (c) and (d).

11. Since r is almost-divisible in R_g, there exists $n \in \mathbb{Z}^+$ such that nr is P-divisible in R_g for some infinite set of prime numbers P. Now, since nr is P-divisible in R_g, there exists a P-divisible element $a + I \in (R/I)_g$ such that $nr + I \subseteq a + I$ (see (10) of Theorem 4.1).

12. By (11) of Theorem 4.1 and since r is almost-divisible, there exists $n \in \mathbb{Z}^+$ and there exists a P-divisible element $b+I \in (R/I)_g$ such that $nr+I \subseteq b+I$ for some infinite set of prime numbers P. Again, since $r \in R_g$, there exists $a+I \in (R/I)_g$ such that $r+I \subseteq a+I$. Therefore $nr+I \subseteq na+I = n(a+I)$. Since I is Q-ideal and $a, b \in Q_g$, we get $n(a+I) = b+I$. Therefore $n(a+I)$ is P-divisible for some infinite set of prime numbers P. Thus $a + I$ is almost-divisible in $(R/I)_g$ and $r + I \subseteq a + I$.

13. Suppose R_g is almost-divisible. Let $a + I$ be an element in $(R/I)_g$. Then $a \in Q_g$ and there exists $r \in R_g$ such that $r + I \subseteq a + I$. Since R_g is almost-divisible, there exists an almost divisible element $b + I \in (R/I)_g$ such that $r + I \subseteq b + I$ (by (12) of Theorem 4.1). Since I is Q-ideal and $a, b \in Q_g$, we get $a + I = b + I$. Therefore $b + I$ is almost-divisible in $(R/I)_g$. Thus $(R/I)_g$ is almost-divisible.

■

Theorem 4.2. *Let $Q = \bigoplus_{g\in G} Q_g$ be an additive submonoid of a graded semiring $R = \bigoplus_{g\in G} R_g$ by a left zero semigroup G, where Q_g is an additive submonoid of Q for all $g \in G$. Suppose I is a Q-ideal in R and R/I is the quotient semiring of R and suppose for each $g \in G$ the following condition*

$$r_g \in R_g \Longrightarrow \exists \alpha_g \in Q_g; r_g + I \subseteq \alpha_g + I (\forall r_g \in R_g). \tag{7}$$

By Theorem 3.1, the family $\{(R/I)_g\}_{g\in G}$ forms a grading of R/I such that

$$(R/I)_g = \{\alpha_g + I \in R/I | \exists r_g \in R_g; r_g + I \subseteq \alpha_g + I \ \&\ \alpha_g \in Q_g\}$$

for all $g \in G$. Define the set $Q'_g = \{\alpha_g \in Q_g \mid \exists r_g \in R_g \ ; r_g + I \subseteq \alpha_g + I\}$ for all $g \in G$. Let g be an element in G. Then we have the following:

1. *$(Q'_g, +)$ is a submonoid of the monoid $(Q_g, +)$.*

2. *$\zeta + \kappa = 0$ implies $\zeta = \kappa = 0$ for all $\zeta, \kappa \in Q'_g \iff (R/I)_g$ is zero-sum free.*

3. *$(Q'_g, +)$ is yoked $\iff (R/I)_g$ is yoked.*

4. *$(Q'_g, +)$ is an idempotent semigroup $\iff (R/I)_g = E^+((R/I)_g)$.*

5. *Suppose $\phi \neq M \subseteq \mathbb{Z}^+$. Then we have the following:*

   (a) $(Q'_g, +)$ is an M-divisible semigroup $\iff (R/I)_g$ is M-divisible.

   (b) $(Q'_g, +)$ is a divisible semigroup $\iff (R/I)_g$ is divisible.

6. *$(Q'_g, +)$ is an almost-divisible semigroup $\iff (R/I)_g$ is almost-divisible.*

Proof. First, we have R_g is a subsemiring of R and $(R/I)_g$ is a subsemiring of (R/I) for all $g \in G$ (see the proof of Theorem 4.1).

1. We have $Q'_g \subseteq Q_g$. Since $0 + I = 0 + I$ where $0 \in Q_g$ and $0 \in R_g$, we get $0 \in Q'_g$. Hence $Q'_g \neq \phi$. Now, let γ, θ be two elements of Q'_g. Since $\gamma, \theta \in Q'_g$, then $\gamma, \theta \in Q_g$ and there exist $\psi, \lambda \in R_g$ such that $\psi + I \subseteq \gamma + I$ and $\lambda + I \subseteq \theta + I$. Which implies $(\psi + \lambda) + I \subseteq (\gamma + \theta) + I$, where $\psi + \lambda \in R_g$ and $\gamma + \theta \in Q_g$. Therefore $\gamma + \theta \in Q'_g$. Thus Q'_g is an additive submonoid of $(Q_g, +)$.

2. Suppose $\gamma + \theta = 0$ implies $\gamma = \theta = 0$ for all $\gamma, \theta \in Q'_g$. Assume that $\beta + I$ and $\alpha + I$ are two elements in $(R/I)_g$, where $(\beta + I) + (\alpha + I) = 0 + I$. Then $(\beta + \alpha) + I = 0 + I$. This implies $\beta + \alpha = 0$ (Since $0, \beta + \alpha \in Q_g \subseteq Q$ and I is a Q-ideal of R). Since $0 + I = (\beta + \alpha) + I$ where $\beta, \alpha \in Q_g, 0 \in R_g$ and $\beta + \alpha = 0$, we get $\beta + \alpha \in Q'_g$ and $\beta = \alpha = 0$. Hence $\beta + I = \alpha + I = 0 + I$. Therefore $(R/I)_g$ is a zero-sum free semiring.

   Conversely, suppose $(R/I)_g$ is a zero-sum free semiring. Let γ, θ be two elements of Q'_g such that $\gamma + \theta = 0$. Since Q'_g is an additive submonoid of Q_g, we get $\gamma + \theta \in Q'_g \subseteq Q_g$. Now, since $\gamma + \theta = 0$, we get $(\gamma + \theta) + I = 0 + I$. Hence $(\gamma + I) + (\theta + I) = 0 + I$. Also, since $\gamma, \theta \in Q'_g$, we get $\gamma, \theta \in Q_g$ and there exist $\psi, \lambda \in R_g$ such that $\psi + I \subseteq \gamma + I$ and $\lambda + I \subseteq \theta + I$. Hence $\gamma + I, \theta + I \in (R/I)_g$. Thus $\gamma + I, \theta + I \in (R/I)_g$, $(\gamma + I) + (\theta + I) = 0 + I$, and $(R/I)_g$ is zero-sum free. Therefore $\gamma + I = \theta + I = 0 + I$. Thus $\gamma = \theta = 0$.

3. Suppose $(Q'_g, +)$ is yoked. Let $\gamma + I, \theta + I$ be two elements of $(R/I)_g$. Then $\gamma, \theta \in Q_g$ and there exist $x, y \in R_g$ such that $x + I \subseteq \gamma + I$ and $y + I \subseteq \theta + I$. Hence $\gamma, \theta \in Q'_g$. Since $\gamma, \theta \in Q'_g$ and Q'_g is yoked, there exists $\kappa \in Q'_g$ such that $\gamma + \kappa = \theta$ or $\theta + \kappa = \gamma$. Therefore $(\gamma + I) + (\kappa + I) = \theta + I$ or $(\theta + I) + (\kappa + I) = \gamma + I$. Since $\kappa \in Q'_g$, $\kappa + I \in (R/I)_g$. Thus $(R/I)_g$ is yoked.

Conversely, suppose $(R/I)_g$ is yoked. Let γ, θ be two elements of Q'_g. Then $\gamma, \theta \in Q_g$ and there exist $x, y \in R_g$ such that $x+I \subseteq \gamma+I$ and $y+I \subseteq \theta+I$. Hence $\gamma+I, \theta+I \in (R/I)_g$. Since $(R/I)_g$ is yoked, there exists an element $\kappa+I \in (R/I)_g$ such that $(\gamma+I)+(\kappa+I) = \theta+I$ or $(\theta+I)+(\kappa+I) = \gamma+I$. Hence $(\gamma+\kappa)+I = \theta+I$ or $(\theta+\kappa)+I = \gamma+I$. Since I is Q-ideal, we get $\gamma+\kappa = \theta$ or $\theta+\kappa = \gamma$. Also, since $\kappa+I \in (R/I)_g$, we get $\kappa \in Q_g$ and there exists an element $z \in R_g$ such that $z+I \subseteq \kappa+I$. Thus $\kappa \in Q'_g$. Therefore $(Q'_g, +)$ is yoked.

4. Suppose $(Q'_g, +)$ is idempotent. Let $\psi+I$ be an element in $(R/I)_g$. Then $\psi \in Q_g$ and there exists $r \in R_g$ such that $r+I \subseteq \psi+I$. Hence $\psi \in Q'_g$. Since Q'_g is idempotent, we get $\psi+\psi = \psi$. Hence $(\psi+I)+(\psi+I) = \psi+I$. Thus $\psi+I \in E^+((R/I)_g)$. Therefore $(R/I)_g \subseteq E^+((R/I)_g)$. Since $E^+((R/I)_g) \subseteq (R/I)_g$, we get $(R/I)_g = E^+((R/I)_g)$.

   Conversely, suppose $(R/I)_g = E^+((R/I)_g)$. Let ψ be an element in Q'_g. Then $\psi \in Q_g$ and there exists an element $r \in R_g$ such that $r+I \subseteq \psi+I$. Hence $\psi+I \in (R/I)_g = E^+((R/I)_g)$. Therefore $(\psi+I)+(\psi+I) = \psi+I$. Thus $(\psi+\psi)+I = \psi+I$. Since I is Q-ideal, we get $\psi+\psi = \psi$. Therefore $(Q'_g, +)$ is idempotent.

5. (a) Suppose $(Q'_g, +)$ is M-divisible. Let $\psi+I$ be an element in $(R/I)_g$. Then $\psi \in Q_g$ and there exists $r \in R_g$ such that $r+I \subseteq \psi+I$. Hence $\psi \in Q'_g$. Let n be an element in M. Since Q'_g is M-divisible and $\psi \in Q'_g$, there exists $\zeta \in Q'_g$ such that $\psi = n\zeta$. Hence $\psi+I = n\zeta+I$. Thus $(\psi+I) = n(\zeta+I)$. Now, since $\zeta \in Q'_g$, $\zeta+I \in (R/I)_g$. Again, since $\zeta+I \in (R/I)_g$ and $(\psi+I) = n(\zeta+I)$, $\psi+I$ is M-divisible in $(R/I)_g$. Therefore $(R/I)_g$ is M-divisible.

   Conversely, suppose $(R/I)_g$ is M-divisible. Let ψ be an element in Q'_g. Then $\psi \in Q_g$ and there exists an element $r \in R_g$ such that $r+I \subseteq \psi+I$. Therefore $\psi+I \in (R/I)_g$. Now, let n be an element in M. Since $(R/I)_g$ is M-divisible and $\psi+I \in (R/I)_g$, there exists $\zeta+I \in (R/I)_g$ such that $\psi+I = n(\zeta+I) = n\zeta+I$. Since $\zeta+I \in (R/I)_g$, we get $\zeta \in Q_g$ and there exists $t \in R_g$ such that $t+I \subseteq \zeta+I$. Therefore $\zeta \in Q'_g$. Again, since I is Q-ideal, $\psi, n\zeta \in Q_g$, and $\psi+I = n(\zeta+I) = n\zeta+I$, we get $\psi = n\zeta$. Hence ψ is M-divisible in $(Q'_g, +)$. Thus $(Q'_g, +)$ is M-divisible. Similarly, we can prove (b).

6. Suppose $(Q'_g, +)$ is almost-divisible. Let $\psi+I$ be an element in $(R/I)_g$. Then $\psi \in Q_g$ and there exists $r \in R_g$ such that $r+I \subseteq \psi+I$. Hence $\psi \in Q'_g$. Since $(Q'_g, +)$ is almost-divisible, there exists $n \in \mathbb{Z}^+$ such that $n\psi$ is P-divisible in Q'_g for some infinite set of prime numbers P. Now, let p is an element in P. Since $n\psi$ is P-divisible in Q'_g, there exists $\zeta \in Q'_g$ such that

$n\psi = p\zeta$. Therefore $\zeta+I \in (R/I)_g$ and $n(\psi+I) = p(\zeta+I)$. Hence $n(\psi+I)$ is P-divisible in $(R/I)_g$ for some infinite set of prime numbers P. Thus $\psi+I$ is almost-divisible in $(R/I)_g$. Therefore $(R/I)_g$ is almost-divisible.

Conversely, suppose $(R/I)_g$ is almost-divisible. Let ψ be an element in Q'_g. Then $\psi \in Q_g$ and there exists an element $r \in R_g$ such that $r + I \subseteq \psi + I$. Therefore $\psi + I \in (R/I)_g$. Since $(R/I)_g$ is almost-divisible, there exists $n \in \mathbb{Z}^+$ such that $n(\psi + I)$ is P-divisible in $(R/I)_g$ for some infinite set of prime numbers P. Now. let p be an element in P. Since $n(\psi + I)$ is P-divisible in $(R/I)_g$, there exists $\zeta+I \in (R/I)_g$ such that $n(\psi+I) = p(\zeta+I)$. Therefore $n\psi+I = p\zeta+I$. Since $\zeta+I \in (R/I)_g$, we get $\zeta \in Q'_g$. Again, since I is Q-ideal, $n\psi+I = p\zeta+I$, and $n\psi, p\zeta \in Q_g$, we get $n\psi = p\zeta$. Hence $n\psi$ is P-divisible in $(Q'_g, +)$ for some infinite set of prime numbers P. Therefore ψ is almost-divisible in $(Q'_g, +)$. Thus $(Q'_g, +)$ is almost-divisible.

■

Finally, we prove the following theorem.

Theorem 4.3. *Let $Q = \bigoplus_{g\in G} Q_g$ be an additive submonoid of a graded semiring $R = \bigoplus_{g\in G} R_g$ by a left zero semigroup G, such that Q_g is an additive submonoid of Q for each $g \in G$. Suppose I is a Q-ideal in R and the quotient semiring of R is R/I. Suppose for each element $g \in G$, the following condition*

$$r_g \in R_g \Longrightarrow \exists \alpha_g \in Q_g; r_g + I \subseteq \alpha_g + I \forall r_g \in R_g \tag{8}$$

By Theorem 3.1, the family $\{(R/I)_g\}_{g\in G}$ forms a grading of R/I such that

$$(R/I)_g = \{\alpha_g + I \in R/I | \exists r_g \in R_g; r_g + I \subseteq \alpha_g + I \ \&\ \alpha_g \in Q_g\}$$

for all $g \in G$. Define the set $Q'_g = \{\alpha_g \in Q_g \ |\exists r_g \in R_g \ ;\ r_g + I \subseteq \alpha_g + I\}$ and suppose $Q'_g Q'_h \subseteq Q'_{gh}$ for all $g, h \in G$. Then we have the following:

1. *Q'_g is a subsemiring of Q_g for all $g \in G$.*
2. *$Q' = \sum_{g\in G} Q'_g$ is a graded subsemiring of R.*
3. *The mapping $f_g : Q'_g \longrightarrow (R/I)_g$ defined as $f_g(\alpha) = \alpha + I\ \forall \alpha \in Q'_g$ is a semirings isomorphism for all $g \in G$.*

Proof. 1. Suppose $g \in G$. Then by Theorem 4.2, we get $(Q'_g, +)$ is a submonoid of the monoid $(Q_g, +)$ where $Q \subseteq R$. Since G is a left zero semigroup and $Q'_g Q'_h \subseteq Q'_{gh}$ for all $g, h \in G$, we get $Q'_g Q'_g \subseteq Q'_{gg} = Q'_g$. Therefore Q'_g is a subsemiring of R.

2. Since $0 \in Q'_g \subseteq R$, we get $0 \in Q' \subseteq R$. Let γ, θ be two elements of Q'. Then $\gamma = \sum_{g\in G} \psi_g$, $\theta = \sum_{g\in G} \lambda_g$ such that $\psi_g, \lambda_g \in Q'_g$ for all $g \in G$. Hence $\gamma + \theta = \sum_{g\in G} \psi_g + \sum_{g\in G} \lambda_g = \sum_{g\in G} (\psi_g + \lambda_g)$. Now, since $\psi_g + \lambda_g \in Q'_g$ for all $g \in G$, we get $\gamma + \theta$ is an element in Q'. Also, since $(Q'_g, +)$ is an additive submonoied of $(Q_g, +)$ and $Q'_g Q'_h \subseteq Q'_{gh}$ for all $g, h \in G$, we get $\gamma\theta = (\sum_{g\in G} \psi_g)(\sum_{g\in G} \lambda_g) = \sum_{t\in G}\sum_{g\in G} \psi_t \lambda_g = \sum_{g\in G} \eta_g$ such that $\eta_g \in Q'_g$ for all $g \in G$. Thus $\gamma\theta \in Q'$. Therefore Q' is a subsemiring of R. Now, we shall prove that Q' is graded by G. Suppose x is an element in Q'. Write $x = \sum_{g\in G} x_g$ and $x = \sum_{g\in G} y_g$ where $x_g, y_g \in Q'_g$ for all $g \in G$. Then $\sum_{g\in G} x_g = \sum_{g\in G} y_g$. It follows that $\sum_{g\in G} (x_g + I) = \sum_{g\in G} (y_g + I)$. Since $x_g + I, y_g + I \in (R/I)_g$ for all $g \in G$ and $(R/I) = \bigoplus_{g\in G} (R/I)_g$, we get $x_g + I = y_g + I$ for all $g \in G$. Therefore $x_g = y_g$ for all $g \in G$, since I is Q-ideal. Now, since $(Q'_g, +)$ is a submonoid of $(Q_g, +)$ for all $g \in G$ and $Q'_g \subseteq Q'$ for all $g \in G$, we get $(Q'_g, +)$ is a submonoid of $(Q', +)$ for all $g \in G$. Since $Q'_g Q'_h \subseteq Q'_{gh}$ for all $g, h \in G$, we get the subsemiring $Q' = \bigoplus_{g\in G} Q'_g$ of R is graded by G.

3. Assume that g is an element in G. Let ζ, κ be two elements of Q'_g. Then
   (a) $f_g(\zeta + \kappa) = (\zeta + \kappa) + I = (\zeta + I) + (\kappa + I) = f_g(\zeta) + f_g(\kappa)$.
   (b) By the definition of R/I, we have $(\zeta + I)(\kappa + I) = \psi + I$ such that ψ is unique element in Q such that $\zeta\kappa + I \subseteq \psi + I$. Since $\zeta\kappa + I \subseteq \psi + I$, $\zeta\kappa + I \subseteq \zeta\kappa + I$, $\zeta\kappa \in Q_{gg} = Q_g \subseteq Q$, and I is Q-ideal, we get $\zeta\kappa + I = \psi + I = (\zeta + I)(\kappa + I)$. It follows that $f_g(\zeta\kappa) = \zeta\kappa + I = (\zeta + I)(\kappa + I) = f_g(\zeta) f_g(\kappa)$.
   (c) Suppose $f_g(\zeta) = f_g(\kappa)$. This implies $\zeta + I = \kappa + I$. Since I is Q-ideal, we get $\zeta = \kappa$.
   (d) Suppose η is an element in $(R/I)_g$. Then there exists $\alpha \in Q_g$ such that $\eta = \alpha + I$ and there exists $r \in R_g$ such that $r + I \subseteq \alpha + I$. Thus $\alpha \in Q'_g$ for which $f_g(\alpha) = \alpha + I = \eta$.

   Thus, from (a), (b), (c), and (d), we get f_g is an isomorphism.

■

Note that, in Theorem 4.2 and Theorem 4.3, we have $(R/I)_g = \{\alpha_g + I \mid \alpha_g \in Q'_g\}$ for all $g \in G$.

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REFERENCES

[1] M. Abu Shamla, On Some Types of Ideals in Semirings, M. Sc. Thesis (The Islamic University of Gaza, Palestine, 2008).

[2] P.J. Allen, *A fundamental theorem of homomorphisms for semirings*, Proc. Amer. Math. Soc. **21(2)**(1969) 412–416. https://doi.org/10.2307/2037016

[3] P.J. Allen, H.S. Kim and J. Neggers, *Ideal theory in graded semirings*, Appl. Math. Inform. Sci. **7**(2013) 87–91. https://digitalcommons.aaru.edu.jo/amis/vol07/iss1/9

[4] H. Chen, *Characterizations of normal cancellative monoids*, AIMS Math. **9(1)**(2024) 302–318. https://doi.org/10.3934/math.2024018

[5] M.E. Darwish, N. Alnader and H. Qrewi, On the Augmented Graduation for Some of Semirings and its Rings of Differences, University of Aleppo Research Journal (Basic Sciences Series, 2021) No. 149.

[6] M.E. Darwish, A Study on the Graduation of Semirings and about Graded Semirings, M. Sc. Thesis (Aleppo University, Aleppo, Syria, 2022).

[7] T.K. Dutta, S.K. Sardar and S. Goswami, Operations on Fuzzy Ideals of Γ-semirings, arXiv preprint arXiv:1101.4791 (2011). https://doi.org/10.48550/arXiv.1101.4791

[8] A.S. Ebrahimi, *The ideal theory in quotients of commutative semirings*, Glas. Math. **42**(2007) 301–308. https://doi.org/10.3336/gm.42.2.05

[9] C. Faith, Algebra: rings, modules and categories I, Vol. 190 (Springer Science & Business Media, 2012).

[10] J.S. Golan, Semirings and their Applications (Springer Science & Business Media, 2013).

[11] U. Hebisch and H.J. Weinert, *Semirings without zero divisors*, Math. Pannon **1**(1990) 73–94.

[12] M.A. Javed, M. Aslam and M. Hussain, *On condition* ($A2$) *of Bandlet and Petrich for inverse semirings*, Int. Math. Forum **7(59)** (2012) 2903–2914.

[13] N. Kimura, *The Structure of Idempotent Semigroups.* I, Pacific J. Math. **8(2)** (1958) 257–275.

[14] D. Kirsten, The support of a recognizable series over a zero-sum free, commutative semiring is recognizable, Developments in Language Theory: 13th International Conference, DLT 2009, Stuttgart, Germany, June 30–July 3, 2009, Proceedings 13, Springer Berlin Heidelberg (2009) 326–333. https://doi.org/10.1007/978-3-642-02737-6_26

[15] M. Korbelář, *Torsion and divisibility in finitely generated commutative semirings*, Semigroup Forum **95(2)** (2017) 293–302. https://doi.org/10.1007/s00233-016-9827-4

[16] M. Korbelář and G. Landsmann, *One-generated semirings and additive divisibility*, J. Algebra and Its Appl. **16(2)** (2017) 1750038. https://doi.org/10.1142/S0219498817500384

[17] D.R. LaTorre, *A note on quotient semirings*, Proc. Amer. Math. Soc. **24(3)** (1970) 463–465. https://doi.org/10.2307/2037388

[18] I. Melnyk, *On quasi-prime differential semiring ideals*, Nauk. Visnyk Uzhgorod. Univ. Ser. Math. and informat. **37(2)** (2020) 63–69. https://doi.org/10.24144/2616-7700.2020.2

[19] R.R. Nazari and S. Ghalandarzadeh, *Multiplication semimodules*, Acta Univ. Sapientiae, Mathematica **11(1)** (2019) 172–185. https://doi.org/10.2478/ausm-2019-0014

[20] V. Ponomarenko, *Arithmetic of semigroup semirings*, Ukrainskyi Matematychnyi Zhurnal **67(2)** (2015) 213–229.

[21] K. Saito, *Inversion formula for the growth function of a cancellative monoid*, J. Algebra **385** (2013) 314–332. https://doi.org/10.1016/j.jalgebra.2013.01.037

[22] R.P. Sharma and T.R. Sharma, *G-prime ideals in semirings and their skew group semirings*, Commun. Algebra **34(12)** (2006) 4459–4465. https://doi.org/10.1080/00927870600938738

[23] R.P. Sharma and R. Joseph, *Prime ideals of group graded semirings and their smash products*, Vietnam J. Math. **36(4)** (2008) 415–426.

[24] R.P. Sharma, M. Dadhwal and R. Sharma, *Relationships between a semiring R and quotient semiring* $R_Q(I)$, J. Combin., Inform. & System Sci. **41(4)** (2016).

[25] H.S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40(12)** (1934) 914–920.

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