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A NOTE ON THE GYROGROUPS OF ORDER 8

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Abstract

In this paper, we exhibit the complete list of gyrogroups of order 8 up to isomorphism. These gyrogroups will be used as examples or counter-examples to answer a few questions raised by some authors.

Keywords: finite gyrogroup, normal subgyrogroup, strong subgyrogroup, L-subgyrogroup.

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1. Introduction

Loosely speaking, a gyrogroup is a group-like structure whose operation is not, in general, associative. The algebra of gyrogroups has been intensively studied by several authors. It turns out that groups and gyrogroups share several common properties. However, certain properties are satisfied by groups but not by generic gyrogroups, as we will see in the sequel. Lemma 2.1 of [1] states that there are exactly six gyrogroups of order 8 (up to isomorphism). Here, we give the full list of gyrogroups of order 8 as well as their gyration tables. These gyrogroups appear as examples or counter-examples of some open questions raised recently. Basic definitions and notations used in the paper can be found in [3, 4, 10]. In the next section, we recall relevant definitions and notations for easy reference.

2. Preliminaries

A *gyrogroup* $(G, \oplus)$ consists of a non-empty set G, together with a binary operation $\oplus$ on G, satisfying the following properties: (i) there exists a (unique) two-sided identity e in G; (ii) each element a in G has a (unique) two-sided inverse denoted by $\ominus a$; (iii) for each pair (a,b) of elements in G, there exists a (unique) automorphism $\mathrm{gyr}\,[a,b]$ of $(G,\oplus)$ called a *gyroautomorphism* such that $a\oplus(b\oplus c)=(a\oplus b)\oplus\mathrm{gyr}\,[a,b](c)$ for all $c\in G$; and (iv) $\mathrm{gyr}\,[a\oplus b,b]=\mathrm{gyr}\,[a,b]$ for all $a,b\in G$ called the *left loop property.* It is evident that gyroautomorphisms remedy the absence of associativity of the binary operation in a gyrogroup. A gyrogroup is said to be *non-degenerate* if its operation is not associative, and in this case, it does not form a group.

Let G be a gyrogroup. A subset H of G is called a *subgyrogroup* if H is a gyrogroup under the operation inherited from G and $\mathrm{gyr}\,[a,b](H)=H$ for all $a,b\in H$. Let $a\in G$. The *cyclic subgyrogroup* generated by a is given by $\langle a\rangle=\{ma : m\in\mathbb{Z}\}$. In fact, $\langle a\rangle$ forms a cyclic group by Theorem 3.8 of [8]. The *order* of a is defined as the size of $\langle a\rangle$, denoted by $|a|$. The *left gyrotranslation* by a, denoted by L_a, is a map defined by the formula $L_a(x)=a\oplus x$ for all $x\in G$. By Theorem 10 of [9], L_a is a permutation of G. Let H be a subgyrogroup of G. The *left coset* $a\oplus H$ is defined as $a\oplus H=\{a\oplus h : h\in H\}$. Similarly, the *right coset* $H\oplus a$ is defined as $H\oplus a=\{h\oplus a : h\in H\}$. The *index* of H in G, denoted by $[G:H]$, is defined as the size of $G/H=\{a\oplus H : a\in G\}$. The *left nucleus* of G, denoted by $N_\ell(G)$, is defined as

$$N_\ell(G)=\{a\in G : \mathrm{gyr}\,[a,b]=\mathrm{I}_G \text{ for all } b\in G\}, \tag{1}$$

where I_G is the identity map on G. The *right nucleus* of G, denoted by $N_r(G)$, is defined as

$$N_r(G)=\{c\in G : \mathrm{gyr}\,[a,b](c)=c \text{ for all } a,b\in G\}. \tag{2}$$

Left and right nuclei of gyrogroups can be used to gain a better understanding of the structure of a gyrogroup; see, for instance, [5].

Let G be a gyrogroup. A map φ from G to a gyrogroup is said to be a *homomorphism* if $\varphi(a\oplus b)=\varphi(a)\oplus\varphi(b)$ for all $a,b\in G$. In this case, the *kernel* of φ is defined as the inverse image of $\{e\}$ under φ. A subgyrogroup H of G is called an *L-subgyrogroup* if $\mathrm{gyr}\,[a,h](H)=H$ for all $a\in G, h\in H$; is called a *strong subgyrogroup* if $\mathrm{gyr}\,[a,b](H)=H$ for all $a,b\in G$; is called a *normal subgyrogroup* if H is equal to the kernel of a homomorphism from G to a gyrogroup. In this case, we write $H\trianglelefteq G$. As in the proof of Proposition 35 of [4], any normal subgyrogroup is invariant under all the gyroautomorphisms and hence is strong. In fact, we have the following implications:

$$\text{“being normal”} \quad\Rightarrow\quad \text{“being strong”} \quad\Rightarrow\quad \text{“being an L-subgyrogroup”}.$$

In light of Theorem 3.8 of [5], $N_\ell(G)$ forms a normal associative subgyrogroup of G (that is, $N_\ell(G)$ is normal and forms a group under the gyrogroup operation of G). This fact will be used to prove that a certain gyrogroup is degenerate in the next section.

3. Properties of gyrogroups of order less than 8

It is clear that the trivial gyrogroup forms group. Recall that a gyrogroup of order p, where p is a prime, is degenerate (see, for instance, Theorem 6.2 of [8]) in the sense that it forms a group under the same operation. Hence, any gyrogroup of order n with $n \in \{2, 3, 5, 7\}$ is degenerate. Next, we give a gyrogroup-theoretic proof that every gyrogroup of order 4 is degenerate.

Proposition 1. *If G is a gyrogroup of order 4, then G forms a group.*

Proof. In the case when G has an element of order 4, say a, $G = \langle a \rangle \cong \mathbb{Z}_4$ by Corollary 3.12 of [8]. Now, suppose that G has no elements of order 4. Let $c, d \in G \setminus \{e\}$ with $c \neq d$. Then $c \neq e$, $d \neq e$, $|c| = 2$, and $|d| = 2$. If $c \oplus d = e$, we would have $c = \ominus d = d$, a contradiction. Similarly, $c \oplus d \neq c$ and $c \oplus d \neq d$. Thus, $G = \{e, c, d, c \oplus d\}$, and $c \oplus d = d \oplus c$. By assumption, $|c \oplus d| = 2$. Hence, $L_c = (e\ c)(d\ c \oplus d)$, $L_d = (e\ d)(c\ c \oplus d)$, and $L_{c\oplus d} = (e\ c \oplus d)(c\ d)$. According to the composition law of left gyrotranslations (see part 3 of Theorem 10 of [9]),

$$\begin{aligned} \operatorname{gyr}[c, d] &= L_{c\oplus d}^{-1} \circ L_c \circ L_d \\ &= L_{\ominus(c\oplus d)} \circ L_c \circ L_d \\ &= L_{c\oplus d} \circ L_c \circ L_d \\ &= (e\ c \oplus d)(c\ d)(e\ c)(d\ c \oplus d)(e\ d)(c\ c \oplus d) \\ &= \mathrm{I}_G. \end{aligned}$$

It is clear that $\operatorname{gyr}[d, e] = \mathrm{I}_G$, and that $\operatorname{gyr}[d, d] = \mathrm{I}_G$. By the inversive symmetric property (see Theorem 2.34 of [10]), $\operatorname{gyr}[d, c] = \operatorname{gyr}[c, d]^{-1} = \mathrm{I}_G$. It follows from the right loop property (see Theorem 2.35 of [10]) that $\operatorname{gyr}[d, c \oplus d] = \operatorname{gyr}[d, c] = \mathrm{I}_G$. This shows that d lies in the left nucleus $N_\ell(G)$ of G. One obtains in a similar fashion that $c \in N_\ell(G)$. Thus, $c \oplus d \in N_\ell(G)$ by the closure property. Hence, $G = N_\ell(G)$, and so G forms a group by Theorem 3.1 of [5]. ■

In 2020, Smith asked a question "why there is no a non-degenerate gyrogroup of order 6?" via private communication. Here, we give a gyrogroup-theoretic proof that every gyrogroup of order 6 is necessarily a group. Recall that any gyrogroup of even order always contains an element of order 2 by Theorem 3.7 of [7]. To complete the goal, we first prove the following lemma.

Lemma 2. *If G is a gyrogroup of order 6, then G contains an element of order 3 or an element of order 6.*

Proof. Assume to the contrary that G has no elements of order 3 and has no elements of order 6. By Proposition 6.1 of [8], the order of any element in G is a divisor of 6. Hence, any non-identity element of G is of order 2. Let $a, b \in G \setminus \{e\}$ with $a \neq b$. By assumption, $|a| = 2$ and $|b| = 2$. Note that $a \oplus b \neq e$, $a \oplus b \neq a$, and $a \oplus b \neq b$ since otherwise $b = \ominus a = a$, $b = e$, and $a = e$, which are contradictions. For simplicity, the proof is divided into several parts.

Claim 1. $a \oplus b \neq b \oplus a$. Suppose to the contrary that $a \oplus b = b \oplus a$. Note that $|\{e, a, b, a \oplus b\}| = 4$. Hence, we can let c_1 and c_2 be distinct elements of G such that $c_1, c_2 \notin \{e, a, b, a \oplus b\}$. Note that $L_a(e) = a, L_a(a) = e, L_a(b) = a \oplus b$, and $L_a(a \oplus b) = b$ since $a = \ominus a$. This implies $L_a(c_1) = c_2$, noting that $L_a(c_1) \neq c_1$ since otherwise $a = e$. Similarly, $L_b(e) = b, L_b(b) = e$, $L_b(a) = b \oplus a = a \oplus b$, and $L_b(b \oplus a) = a$. This implies $L_b(c_1) = c_2$. It follows that $a \oplus c_1 = c_2 = b \oplus c_1$, which implies by the right cancellation law that $a = b$, a contradiction. Thus, $a \oplus b \neq b \oplus a$.

Claim 2. $G = \{e, a, b, a \oplus b, b \oplus a, a \oplus (b \oplus a)\}$. By Claim 1, $a \oplus b \neq b \oplus a$. By Theorem 4.1 of [7], $\langle a \rangle$ partitions G into disjoint right cosets. Note that $\langle a \rangle \oplus e = \{e, a\}$ and $\langle a \rangle \oplus b = \{b, a \oplus b\}$. Since $b \oplus a \notin \{e, a, b, a \oplus b\}$, it follows that $\langle a \rangle \oplus (b \oplus a)$ is the remaining right coset. Hence, $G = \{e, a, b, a \oplus b, b \oplus a, a \oplus (b \oplus a)\}$, as required.

To complete the proof, we show that a contradiction arises. As computed above, L_a and L_b decompose into products of disjoint transpositions as

$$L_a = (e\ a)(b\ a \oplus b)(b \oplus a\ a \oplus (b \oplus a)) \text{ and } L_b = (e\ b)(a\ b \oplus a)(a \oplus b\ a \oplus (b \oplus a)).$$

A direct computation shows that $L_a \circ L_b = (e\ a \oplus b\ b \oplus a)(a\ a \oplus (b \oplus a)\ b)$. Next, we consider $L_{a \oplus b}$. By assumption, $|a \oplus b| = 2$, and so $\ominus(a \oplus b) = a \oplus b$. This implies that $(a \oplus b) \oplus a \notin \{e, a, a \oplus b\}$. Thus, $(a \oplus b) \oplus a \in \{b, b \oplus a, a \oplus (b \oplus a)\}$. Note that $\mathrm{gyr}\,[a, b] = L_{a \oplus b}^{-1} \circ L_a \circ L_b = L_{a \oplus b} \circ L_a \circ L_b$. We show that the following cases lead to some contradictions.

Case 1. $(a \oplus b) \oplus a = b$. In this case,

$$L_{a \oplus b} = (e\ a \oplus b)(a\ b)(b \oplus a\ a \oplus (b \oplus a)).$$

Hence, $\mathrm{gyr}\,[a, b] = (e)(b)(a\ b \oplus a\ a \oplus b\ a \oplus (b \oplus a))$. In particular, $\mathrm{gyr}[a, b]^4 = \mathrm{I}_G$. Note that $\mathrm{gyr}\,[a, b](a) = b \oplus a$. Applying $\mathrm{gyr}[a, b]^3$ on both sides of the previous equation gives $a = b \oplus \mathrm{gyr}[a, b]^3(a)$. It follows that $\mathrm{gyr}[a, b]^3(a) = b \oplus a$. Since $\mathrm{gyr}[a, b]^3 = (e)(b)(a\ a \oplus (b \oplus a)\ a \oplus b\ b \oplus a)$, we obtain $b \oplus a = a \oplus (b \oplus a)$, which implies $a = e$, a contradiction.

Case 2. $(a \oplus b) \oplus a = b \oplus a$. In this case,

$$L_{a\oplus b} = (e\ a \oplus b)(a\ b \oplus a)(b\ a \oplus (b \oplus a)).$$

Hence, $\operatorname{gyr}[a, b] = (e)(a \oplus (b \oplus a))(a\ b\ b \oplus a\ a \oplus b)$, and so $\operatorname{gyr}[a, b](b) = b \oplus a$. According to the gyrator identity (see part 10 of Theorem 2.10 of [10]), $b \oplus a = \ominus(a \oplus b) \oplus (a \oplus (b \oplus b)) = (a \oplus b) \oplus a$, which implies $b = a \oplus b$, a contradiction.

Case 3. $(a\oplus b)\oplus a = a\oplus(b\oplus a)$. Since $L_{a\oplus b} = (e\ a\oplus b)(a\ a\oplus(b\oplus a))(b\ b\oplus a)$, we obtain $\operatorname{gyr}[a, b] = (e)(a)(b\ a \oplus (b \oplus a)\ b \oplus a\ a \oplus b)$. Thus, $\operatorname{gyr}[a, b]^4 = \mathrm{I}_G$. As $\operatorname{gyr}[a, b](b) = a \oplus (b \oplus a)$, applying $\operatorname{gyr}[a, b]^3$ on both sides of the previous equation gives $b = a \oplus \operatorname{gyr}[a, b]^3(b \oplus a)$. Hence, $\operatorname{gyr}[a, b]^3(b \oplus a) = a \oplus b$. Since $\operatorname{gyr}[a, b]^3 = (e)(a)(b\ a \oplus b\ b \oplus a\ a \oplus (b \oplus a))$, it follows that $a \oplus b = a \oplus (b \oplus a)$, which implies $b = b \oplus a$, a contradiction.

This completes the proof. ∎

Lemma 2 implies that any gyrogroup of order 6 satisfies the Cauchy property: any gyrogroup of order 6 has an element of order 2 and an element of order 3. We are now in a position to prove the main theorem of this section.

Theorem 3. *If G is a gyrogroup of order 6, then G forms a group.*

Proof. In the case when G has an element of order 6, say g, we obtain that $G = \langle g\rangle \cong \mathbb{Z}_6$. Therefore, we assume that G has no elements of order 6. Hence, any non-identity element of G must be of order 2 or of order 3 as a consequence of Theorem 4.1 of [7].

By Theorem 3.7 of [7], G has an element of order 2, say a. By Lemma 2, G has an element of order 3, say b. Thus, $\langle b\rangle = \{e, b, 2b\}$, and $a \notin \langle b\rangle$. Note that $\langle b\rangle$ is an L-subgyrogroup of G by Theorem 4.4 of [7], and so $[G : \langle b\rangle] = 2$. Since $a \notin \langle b\rangle$, it follows that $a \oplus \langle b\rangle \neq e \oplus \langle b\rangle$. Thus, $(a \oplus \langle b\rangle) \cap (e \oplus \langle b\rangle) = \emptyset$. It follows that $G = \{e, b, 2b, a, a\oplus b, a\oplus 2b\}$. For simplicity, the proof is divided into several parts.

Claim 1. $\operatorname{gyr}[b, a](b) = b$. To prove the claim, note that all non-identity elements in $\langle b\rangle$ have order 3 and, by Theorem 4.4 of [7], any element of G of order 3 must be in $\langle b\rangle$. This implies either $\operatorname{gyr}[b, a](b) = b$ or $\operatorname{gyr}[b, a](b) = 2b$ since $|\operatorname{gyr}[b, a](b)| = |b| = 3$. If $\operatorname{gyr}[b, a](b) = 2b$, we would have $\operatorname{gyr}[b, a](2b) = b$ and would have $\operatorname{gyr}[a, b](b) = 2b$ since $\operatorname{gyr}[b, a]^{-1} = \operatorname{gyr}[a, b]$. This would imply

$$a \oplus 2b = a \oplus (b \oplus b) = (a \oplus b) \oplus \operatorname{gyr}[a, b](b) = (a \oplus b) \oplus 2b,$$

which would imply by the right cancellation law that $a = a \oplus b$, a contradiction. Thus, $\operatorname{gyr}[b, a](b) = b$. This also implies $\operatorname{gyr}[a, b](b) = b$.

Claim 2. $|b \oplus a| = 2$. Note that $b \oplus a \neq e$. Furthermore, by assumption, $|b \oplus a| \neq 6$. To prove the claim, suppose to the contrary that $|b \oplus a| = 3$. Then $b \oplus a = b$ or $b \oplus a = 2b$. In the former case, we obtain that $a = e$, a contradiction. In the latter case, we obtain that $a = b$, a contradiction. Thus, $|b \oplus a| = 2$. This also implies $\ominus(b \oplus a) = b \oplus a$.

Claim 3. $a \oplus b = 2b \oplus a$. To prove the claim, note first that $a \oplus \langle b\rangle = \langle b\rangle \oplus a$ for $\langle b\rangle \trianglelefteq G$ (see Proposition 39 of [4]). As noted above, $a \oplus b \neq a$. Hence, either $a \oplus b = b \oplus a$ or $a \oplus b = 2b \oplus a$. We show that $a \oplus b \neq b \oplus a$. Suppose to the contrary that $a \oplus b = b \oplus a$. Then $b = \ominus a \oplus (b \oplus a) = a \oplus (b \oplus a)$, which implies $L_b = L_{a\oplus(b\oplus a)} = L_a \circ L_b \circ L_a$ (see Equation (2.6) of [6]). Thus, $L_a \circ L_b = L_b \circ L_a$ since $L_a^{-1} = L_{\ominus a} = L_a$. Using the composition law, we obtain $L_{a\oplus b} \circ \mathrm{gyr}\,[a, b] = L_{b\oplus a} \circ \mathrm{gyr}\,[b, a]$, which implies $\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[b, a]$. Hence, $\mathrm{gyr}[b, a]^2 = \mathrm{I}_G$. By the gyrator identity and Claim 2, $\mathrm{gyr}\,[b, a](a) = \ominus(b\oplus a)\oplus(b\oplus(a\oplus a)) = (b\oplus a)\oplus b$. Hence, by Claim 1, $\mathrm{gyr}\,[b, a](a) = (a\oplus b)\oplus b = a\oplus(b\oplus \mathrm{gyr}\,[b, a](b)) = a\oplus 2b$. From the even property (see Theorem 2.34 of [10]), we obtain $\mathrm{gyr}\,[2b, a] = \mathrm{gyr}\,[\ominus b, \ominus a] = \mathrm{gyr}\,[b, a]$, which implies that

$$
\begin{aligned}
\mathrm{gyr}\,[b, a](a \oplus 2b) &= (a \oplus 2b) \oplus 2b \\
&= a \oplus (2b \oplus \mathrm{gyr}\,[2b, a](2b)) \\
&= a \oplus (2b \oplus \mathrm{gyr}\,[b, a](2b)) \\
&= a \oplus (2b \oplus 2b) \\
&= a \oplus b.
\end{aligned}
$$

This leads to a contradiction since $\mathrm{gyr}\,[b, a](a \oplus 2b) = \mathrm{gyr}[b, a]^2(a) = a$. This shows that $a \oplus b = 2b \oplus a$.

Claim 4. $\mathrm{gyr}\,[b, a](a) = a$. By Claim 3, $a \oplus b = 2b \oplus a$. Hence, it follows from the gyrator identity and Claim 2 that

$$
\begin{aligned}
\mathrm{gyr}\,[b, a](a) &= (b \oplus a) \oplus b \\
&= b \oplus (a \oplus \mathrm{gyr}\,[a, b](b)) \\
&= b \oplus (a \oplus b) \\
&= b \oplus (2b \oplus a) \\
&= b \oplus (\ominus b \oplus a) \\
&= a.
\end{aligned}
$$

Claim 5. *a and b belong to $N_\ell(G)$.* Since every element of G can be expressed in terms of a and b, together with the fact that $\mathrm{gyr}\,[b, a]$ leaves a and b fixed, it follows that $\mathrm{gyr}\,[b, a] = \mathrm{I}_G$. This implies $\mathrm{gyr}\,[a, b] = \mathrm{I}_G$. By parts 2 and 4 of Theorem 2.10 of [10], $\mathrm{gyr}\,[a, e] = \mathrm{gyr}\,[e, a]^{-1} = \mathrm{I}_G$, and $\mathrm{gyr}\,[a, a] = \mathrm{I}_G$. As proved above, $\mathrm{gyr}\,[a,2b] = \mathrm{gyr}\,[2b,a]^{-1} = \mathrm{gyr}\,[b,a]^{-1} = \mathrm{I}_G$. By the left and right loop properties,

$$\mathrm{gyr}\,[a, a\oplus b] = \mathrm{gyr}\,[a\oplus(a\oplus b), a\oplus b] = \mathrm{gyr}\,[b, a\oplus b] = \mathrm{gyr}\,[b,a] = \mathrm{I}_G.$$

Similarly, $\mathrm{gyr}\,[a, a\oplus 2b] = \mathrm{gyr}\,[2b,a] = \mathrm{I}_G$. This shows that $a \in N_\ell(G)$. As above, $\mathrm{gyr}\,[b,e] = \mathrm{I}_G$, $\mathrm{gyr}\,[b,b] = \mathrm{I}_G$, and $\mathrm{gyr}\,[b,a] = \mathrm{I}_G$. By Proposition 3.10 of [8], $\mathrm{gyr}\,[b,2b] = \mathrm{I}_G$. By the right loop property, $\mathrm{gyr}\,[b, a\oplus b] = \mathrm{gyr}\,[b,a] = \mathrm{I}_G$. Note that $(a\oplus 2b)\oplus b = a\oplus(2b\oplus \mathrm{gyr}\,[2b,a](b)) = a\oplus(2b\oplus b) = a$. Hence, by the right loop property, $\mathrm{gyr}\,[b, a\oplus 2b] = \mathrm{gyr}\,[b,a] = \mathrm{I}_G$. This shows that $b\in N_\ell(G)$.

We are now in a position to prove the main statement. By Claim 5, a and b lie in $N_\ell(G)$. Hence, $2b, a\oplus b$, and $a\oplus 2b$ lie in $N_\ell(G)$ by the closure property. It follows that $G = N_\ell(G)$, which implies that G forms a group. ■

As a consequence of Theorem 3 and the remark above, every non-degenerate gyrogroup has at least eight elements. We summarize this result as Corollary 4.

Corollary 4. *If G is a non-degenerate gyrogroup, then $|G| \geq 8$.*

Proof. This follows from the previous results, together with the fact that there is a concrete example of a non-degenerate gyrogroup of order 8 (see, for instance, Example 1 of [4]). ■

4. Properties of gyrogroups of order 8

In view of Lemma 2.1 of [1], there are precisely six gyrogroups of order 8 (up to isomorphism) for there are six non-isomorphic non-associative left Bol loops of order 8 (see [2]) and all of which have the A_ℓ-property. In this section, we give the complete list of gyrogroups of order 8 up to isomorphism and prove several related properties. The verification of the axioms for a gyrogroup is simply a matter of direct computation.

4.1. The gyrogroup $G_{8,1}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,1}$ are given in Table 1. In $G_{8,1}$, there are four non-trivial proper subgyrogroups, as shown in Table 2. The only non-trivial gyroautomorphism of $G_{8,1}$ is given in cycle decomposition by $\alpha = (1\ 3)(2\ 4)$.

Bao asked the following questions via private communication.

Question 5. *Is there a non-degenerate gyrogroup in which every subgyrogroup is normal?*

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	7	6	0	5	2	4	3
2	2	5	7	6	0	3	1	4
3	3	0	5	7	6	4	2	1
4	4	6	0	5	7	1	3	2
5	5	2	3	4	1	7	0	6
6	6	4	1	2	3	0	7	5
7	7	3	4	1	2	6	5	0

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	α	ι	α	α	α	ι
2	ι	α	ι	α	ι	α	α	ι
3	ι	ι	α	ι	α	α	α	ι
4	ι	α	ι	α	ι	α	α	ι
5	ι	α	α	α	α	ι	ι	ι
6	ι	α	α	α	α	ι	ι	ι
7	ι	ι	ι	ι	ι	ι	ι	ι

Table 1. The gyroaddition and gyration tables for the gyrogroup $G_{8,1}$, where ι is the identity automorphism and $\alpha = (1\ 3)(2\ 4)$.

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 7\rangle$	$\{0,7\}$	$\mathbb{Z}_2$	left nucleus
$\langle 1\rangle$	$\{0,1,3,7\}$	$\mathbb{Z}_4$	n/a
$\langle 2\rangle$	$\{0,2,4,7\}$	$\mathbb{Z}_4$	n/a
$\langle 5\rangle$	$\{0,5,6,7\}$	$\mathbb{Z}_4$	right nucleus

Table 2. The non-trivial proper subgyrogroups of $G_{8,1}$.

Question 6. *Is there a non-degenerate gyrogroup in which every subgyrogroup is strong?*

Questions 5 and 6 have the affirmative answer as we will see shortly. Note that $\langle 7\rangle \trianglelefteq G_{8,1}$ since $\langle 7\rangle$ is the left nucleus of $G_{8,1}$ (see Theorem 3.8 of [5]), and that $\langle 1\rangle, \langle 2\rangle$, and $\langle 5\rangle$ are all normal in $G_{8,1}$ since they are strong subgyrogroups of $G_{8,1}$ of index 2 (see Theorem 4.5 of [5]). This shows that every subgyrogroup of $G_{8,1}$ is normal, and so Question 5 has the affirmative answer. Also, note that $\langle 5\rangle$ is the right nucleus of $G_{8,1}$. The lattice of subgyrogroups of $G_{8,1}$ is depicted in Figure 1 and is the same as the lattice of subgroups of the quaternion of order 8.

4.2. The gyrogroup $G_{8,2}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,2}$ are given in Table 3. In $G_{8,2}$, there are six non-trivial proper subgyrogroups, as shown in Table 4. The only non-trivial gyroautomorphism of $G_{8,2}$ is given in cycle decomposition by $\beta = (4\ 5)(6\ 7)$.

Note that $\langle 1\rangle \trianglelefteq G_{8,2}$ since $\langle 1\rangle$ is the left nucleus of $G_{8,2}$, and that $\langle 4\rangle, \langle 6\rangle$, and $\langle 1, 2\rangle$ are all normal in $G_{8,2}$ since they are strong subgyrogroups of $G_{8,2}$ of index 2. Also, note that $\langle 1, 2\rangle$ is the right nucleus of $G_{8,2}$. Moreover, $\langle 2\rangle$ is a strong subgyrogroup of $G_{8,2}$ but is not normal in $G_{8,2}$. In fact, $4\oplus\langle 2\rangle = \{4, 7\}$ but $\langle 2\rangle\oplus 4 = \{4, 6\}$ (see Proposition 39 of [4]). Similarly, $\langle 3\rangle$ is a strong subgyrogroup

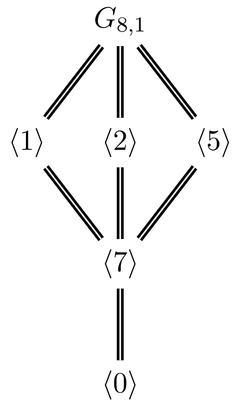

Figure 1. The lattice diagram of $G_{8,1}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	7	6	1	0	2	3
5	5	4	6	7	0	1	3	2
6	6	7	5	4	2	3	1	0
7	7	6	4	5	3	2	0	1

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	ι	ι	ι	ι	ι	ι
2	ι	ι	ι	ι	β	β	β	β
3	ι	ι	ι	ι	β	β	β	β
4	ι	ι	β	β	ι	ι	β	β
5	ι	ι	β	β	ι	ι	β	β
6	ι	ι	β	β	β	β	ι	ι
7	ι	ι	β	β	β	β	ι	ι

Table 3. The gyroaddition and gyration tables for the gyrogroup $G_{8,2}$, where ι is the identity automorphism and $\beta = (4\ 5)(6\ 7)$.

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 1 \rangle$	$\{0, 1\}$	$\mathbb{Z}_2$	left nucleus
$\langle 2 \rangle$	$\{0, 2\}$	$\mathbb{Z}_2$	n/a
$\langle 3 \rangle$	$\{0, 3\}$	$\mathbb{Z}_2$	n/a
$\langle 4 \rangle$	$\{0, 1, 4, 5\}$	$\mathbb{Z}_4$	n/a
$\langle 6 \rangle$	$\{0, 1, 6, 7\}$	$\mathbb{Z}_4$	n/a
$\langle 1, 2 \rangle$	$\{0, 1, 2, 3\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	right nucleus

Table 4. The non-trivial proper subgyrogroups of $G_{8,2}$.

of $G_{8,2}$ but is not normal in $G_{8,2}$. In fact, $4 \oplus \langle 3 \rangle = \{4, 6\}$ but $\langle 3 \rangle \oplus 4 = \{4, 7\}$. This in particular shows that every subgyrogroup of $G_{8,2}$ is strong. As non-normal strong subgyrogroups exist, the property of being a strong subgyrogroup is strictly weaker than the property of being a normal subgyrogroup. This also shows that Question 6 has the affirmative answer. The lattice of subgyrogroups of $G_{8,2}$ is depicted in Figure 2.

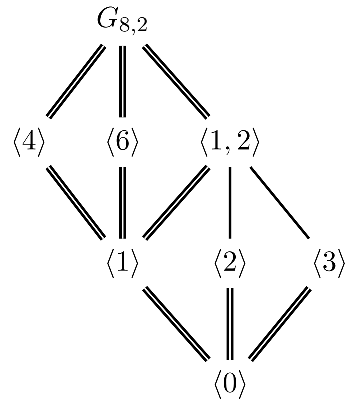

Figure 2. The lattice diagram of $G_{8,2}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

4.3. The gyrogroup $G_{8,3}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,3}$ are given in Table 5. In $G_{8,3}$, there are six non-trivial proper subgyrogroups, as shown in Table 6. The only non-trivial gyroautomorphism of $G_{8,3}$ is given in cycle decomposition by $\gamma = (4\ 6)(5\ 7)$. This gyrogroup is indeed the gyrogroup G_8 in Example 1 of [4].

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	3	0	2	7	4	5	6
2	2	0	3	1	5	6	7	4
3	3	2	1	0	6	7	4	5
4	4	5	7	6	3	2	0	1
5	5	6	4	7	2	0	1	3
6	6	7	5	4	0	1	3	2
7	7	4	6	5	1	3	2	0

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	ι	ι	γ	γ	γ	γ
2	ι	ι	ι	ι	γ	γ	γ	γ
3	ι	ι	ι	ι	ι	ι	ι	ι
4	ι	γ	γ	ι	ι	γ	ι	γ
5	ι	γ	γ	ι	γ	ι	γ	ι
6	ι	γ	γ	ι	ι	γ	ι	γ
7	ι	γ	γ	ι	γ	ι	γ	ι

Table 5. The gyroaddition and gyration tables for the gyrogroup $G_{8,3}$, where ι is the identity automorphism and $\gamma = (4\ 6)(5\ 7)$.

Note that $\langle 3\rangle \trianglelefteq G_{8,3}$ since $\langle 3\rangle$ is the left nucleus of $G_{8,3}$, and that $\langle 2\rangle, \langle 4\rangle$, and $\langle 3, 5\rangle$ are all normal in $G_{8,3}$ since they are strong subgyrogroups of $G_{8,3}$ of

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 3\rangle$	$\{0,3\}$	$\mathbb{Z}_2$	left nucleus
$\langle 5\rangle$	$\{0,5\}$	$\mathbb{Z}_2$	n/a
$\langle 7\rangle$	$\{0,7\}$	$\mathbb{Z}_2$	n/a
$\langle 2\rangle$	$\{0,1,2,3\}$	$\mathbb{Z}_4$	right nucleus
$\langle 4\rangle$	$\{0,3,4,6\}$	$\mathbb{Z}_4$	n/a
$\langle 3,5\rangle$	$\{0,3,5,7\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a

Table 6. The non-trivial proper subgyrogroups of $G_{8,3}$.

index 2. Also, note that $\langle 2\rangle$ is the right nucleus of $G_{8,3}$. Moreover, $\langle 5\rangle$ is a strong subgyrogroup of $G_{8,3}$ but is not normal in $G_{8,3}$. In fact, $1\oplus\langle 5\rangle=\{1,4\}$ but $\langle 5\rangle\oplus 1=\{1,6\}$. Similarly, $\langle 7\rangle$ is a strong subgyrogroup of $G_{8,3}$ but is not normal in $G_{8,3}$. In fact, $1\oplus\langle 7\rangle=\{1,6\}$ but $\langle 7\rangle\oplus 1=\{1,4\}$. The lattice of subgyrogroups of $G_{8,3}$ is depicted in Figure 3. We remark that $G_{8,2}$ and $G_{8,3}$ are not isomorphic because their right nuclei are not isomorphic: the right nucleus of $G_{8,2}$ is isomorphic to the Klein 4-group, whereas the right nucleus of $G_{8,3}$ is isomorphic to the cyclic group of order 4.

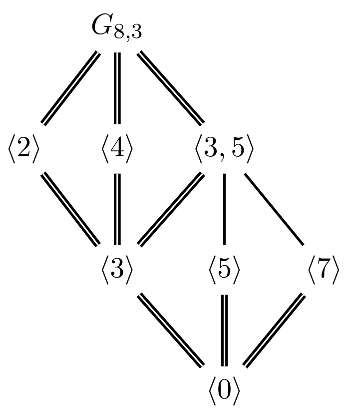

Figure 3. The lattice diagram of $G_{8,3}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

4.4. The gyrogroup $G_{8,4}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,4}$ are given in Table 7. In $G_{8,4}$, there are 10 non-trivial proper subgyrogroups, as shown in Table 8. The only non-trivial gyroautomorphism of $G_{8,4}$ is given in cycle decomposition by $\delta=(4\ 7)(5\ 6)$.

Note that $\langle 3\rangle\trianglelefteq G_{8,4}$ since $\langle 3\rangle$ is the left nucleus of $G_{8,4}$, and that $\langle 1,2\rangle,\langle 3,4\rangle$,

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	6	7	4	5
2	2	3	0	1	5	4	7	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	ι	ι	δ	δ	δ	δ
2	ι	ι	ι	ι	δ	δ	δ	δ
3	ι	ι	ι	ι	ι	ι	ι	ι
4	ι	δ	δ	ι	ι	δ	δ	ι
5	ι	δ	δ	ι	δ	ι	ι	δ
6	ι	δ	δ	ι	δ	ι	ι	δ
7	ι	δ	δ	ι	ι	δ	δ	ι

Table 7. The gyroaddition and gyration tables for the gyrogroup $G_{8,4}$, where ι is the identity automorphism and $\delta = (4\ 7)(5\ 6)$.

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 1\rangle$	$\{0,1\}$	$\mathbb{Z}_2$	n/a
$\langle 2\rangle$	$\{0,2\}$	$\mathbb{Z}_2$	n/a
$\langle 3\rangle$	$\{0,3\}$	$\mathbb{Z}_2$	left nucleus
$\langle 4\rangle$	$\{0,4\}$	$\mathbb{Z}_2$	n/a
$\langle 5\rangle$	$\{0,5\}$	$\mathbb{Z}_2$	n/a
$\langle 6\rangle$	$\{0,6\}$	$\mathbb{Z}_2$	n/a
$\langle 7\rangle$	$\{0,7\}$	$\mathbb{Z}_2$	n/a
$\langle 1,2\rangle$	$\{0,1,2,3\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	right nucleus
$\langle 3,4\rangle$	$\{0,3,4,7\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a
$\langle 3,5\rangle$	$\{0,3,5,6\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a

Table 8. The non-trivial proper subgyrogroups of $G_{8,4}$.

and $\langle 3,5\rangle$ are all normal in $G_{8,4}$ since they are strong subgyrogroups of $G_{8,4}$ of index 2. Also, note that $\langle 1,2\rangle$ is the right nucleus of $G_{8,4}$. Moreover, $\langle 1\rangle$ is not normal in $G_{8,4}$ since $4\oplus\langle 1\rangle \neq \langle 1\rangle\oplus 4$; $\langle 2\rangle$ is not normal in $G_{8,4}$ since $4\oplus\langle 2\rangle \neq \langle 2\rangle\oplus 4$; $\langle 4\rangle$ is not normal in $G_{8,4}$ since $1\oplus\langle 4\rangle \neq \langle 4\rangle\oplus 1$; $\langle 5\rangle$ is not normal in $G_{8,4}$ since $1\oplus\langle 5\rangle \neq \langle 5\rangle\oplus 1$; $\langle 6\rangle$ is not normal in $G_{8,4}$ since $1\oplus\langle 6\rangle \neq \langle 6\rangle\oplus 1$; and $\langle 7\rangle$ is not normal in $G_{8,4}$ since $1\oplus\langle 7\rangle \neq \langle 7\rangle\oplus 1$. Also, note that every element of $G_{8,4}$ is of order 2. The lattice of subgyrogroups of $G_{8,4}$ is depicted in Figure 4.

4.5. The gyrogroup $G_{8,5}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,5}$ are given in Table 9. In $G_{8,5}$, there are eight non-trivial proper subgyrogroups, as shown in Table 10. The only non-trivial gyroautomorphism of $G_{8,5}$ is given in cycle decomposition by $\epsilon = (1\ 2)(3\ 4)$.

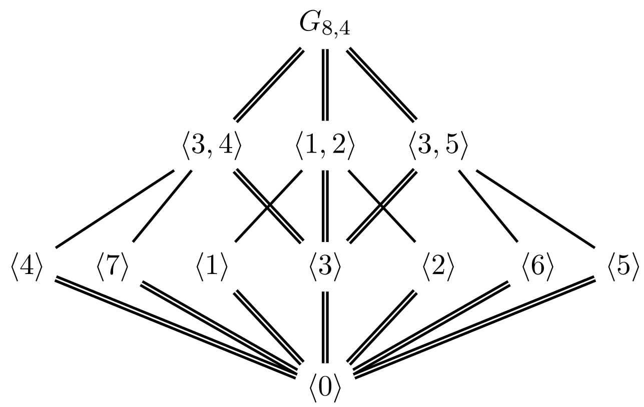

Figure 4. The lattice diagram of $G_{8,4}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	5	7	6	2	4	3
2	2	5	0	6	7	1	3	4
3	3	6	7	5	0	4	2	1
4	4	7	6	0	5	3	1	2
5	5	2	1	4	3	0	7	6
6	6	4	3	2	1	7	0	5
7	7	3	4	1	2	6	5	0

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	ι	ϵ	ϵ	ι	ϵ	ϵ
2	ι	ι	ι	ϵ	ϵ	ι	ϵ	ϵ
3	ι	ϵ	ϵ	ι	ι	ι	ϵ	ϵ
4	ι	ϵ	ϵ	ι	ι	ι	ϵ	ϵ
5	ι	ι	ι	ι	ι	ι	ι	ι
6	ι	ϵ	ϵ	ϵ	ϵ	ι	ι	ι
7	ι	ϵ	ϵ	ϵ	ϵ	ι	ι	ι

Table 9. The gyroaddition and gyration tables for the gyrogroup $G_{8,5}$, where ι is the identity automorphism and $\epsilon = (1\ 2)(3\ 4)$.

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 1\rangle$	$\{0,1\}$	$\mathbb{Z}_2$	n/a
$\langle 2\rangle$	$\{0,2\}$	$\mathbb{Z}_2$	n/a
$\langle 5\rangle$	$\{0,5\}$	$\mathbb{Z}_2$	left nucleus
$\langle 6\rangle$	$\{0,6\}$	$\mathbb{Z}_2$	n/a
$\langle 7\rangle$	$\{0,7\}$	$\mathbb{Z}_2$	n/a
$\langle 3\rangle$	$\{0,3,4,5\}$	$\mathbb{Z}_4$	n/a
$\langle 1,2\rangle$	$\{0,1,2,5\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a
$\langle 5,6\rangle$	$\{0,5,6,7\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	right nucleus

Table 10. The non-trivial proper subgyrogroups of $G_{8,5}$.

Recall that if a subgroup Ξ of a group Γ satisfies the condition that $g\Xi = \Xi g$

for all $g \in \Gamma$, then Ξ is normal in Γ. Therefore, Wattanapan asked the following question via private communication.

Question 7. *Let G be a gyrogroup, and let N be a subgyrogroup of G. Does the condition that $a \oplus N = N \oplus a$ for all $a \in G$ imply normality of N?*

We will see shortly that Question 7 has the negative answer. Note that $\langle 5\rangle \trianglelefteq G_{8,5}$ since $\langle 5\rangle$ is the left nucleus of $G_{8,5}$, and that $\langle 1,2\rangle, \langle 3\rangle$, and $\langle 5,6\rangle$ are all normal in $G_{8,5}$ since they are strong subgyrogroups of $G_{8,5}$ of index 2. Also, note that $\langle 5,6\rangle$ is the right nucleus of $G_{8,5}$. Moreover, $\langle 6\rangle$ is a strong subgyrogroup of $G_{8,5}$ satisfying the property that $a \oplus \langle 6\rangle = \langle 6\rangle \oplus a$ for all $a \in G_{8,5}$ but is not normal in $G_{8,5}$. In fact, the operation of cosets $1 \oplus \langle 6\rangle$ and $2 \oplus \langle 6\rangle$ depends on the choice of representatives: $(1 \oplus 2) \oplus \langle 6\rangle = \{5,7\}$ and $(4 \oplus 2) \oplus \langle 6\rangle = \{0,6\}$, whereas $1 \oplus \langle 6\rangle = 4 \oplus \langle 6\rangle$ (see Theorem 31 of [4]). Similarly, $\langle 7\rangle$ is a strong subgyrogroup of $G_{8,5}$ satisfying the property that $a \oplus \langle 7\rangle = \langle 7\rangle \oplus a$ for all $a \in G_{8,5}$ but is not normal in $G_{8,5}$. In fact, the operation of cosets $1 \oplus \langle 7\rangle$ and $4 \oplus \langle 7\rangle$ depends on the choice of representatives: $(1 \oplus 4) \oplus \langle 7\rangle = \{5,6\}$ and $(3 \oplus 4) \oplus \langle 7\rangle = \{0,7\}$, whereas $1 \oplus \langle 7\rangle = 3 \oplus \langle 7\rangle$. Moreover, $\langle 1\rangle$ is not normal in $G_{8,5}$ since $3 \oplus \langle 1\rangle = \{3,6\}$ but $\langle 1\rangle \oplus 3 = \{3,7\}$. Similarly, $\langle 2\rangle$ is not normal in $G_{8,5}$ since $4 \oplus \langle 2\rangle = \{4,6\}$ but $\langle 2\rangle \oplus 4 = \{4,7\}$. This in particular shows that the condition that $a \oplus N = N \oplus a$ for all elements a in a gyrogroup does not imply normality of N. Therefore, Question 7 has the negative answer. The lattice of subgyrogroups of $G_{8,5}$ is depicted in Figure 5.

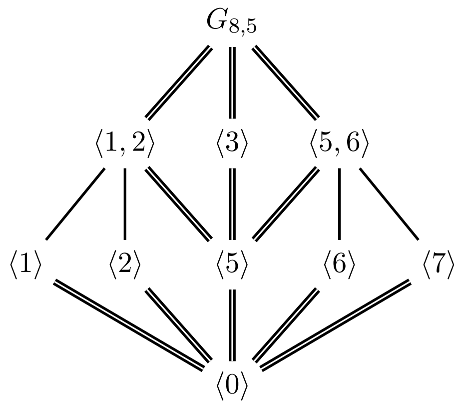

Figure 5. The lattice diagram of $G_{8,5}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

4.6. The gyrogroup $G_{8,6}$

The gyroaddition and gyration tables for the gyrogroup $G_{8,6}$ are given in Table 11. In $G_{8,6}$, there are eight non-trivial proper subgyrogroups, as shown in Table 12.

The only non-trivial gyroautomorphism of $G_{8,6}$ is given in cycle decomposition by $\zeta = (1\ 3)(2\ 4)$.

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	5	7	6	2	4	3
2	2	6	0	5	7	3	1	4
3	3	7	6	0	5	4	2	1
4	4	5	7	6	0	1	3	2
5	5	2	3	4	1	7	0	6
6	6	4	1	2	3	0	7	5
7	7	3	4	1	2	6	5	0

gyr	0	1	2	3	4	5	6	7
0	ι	ι	ι	ι	ι	ι	ι	ι
1	ι	ι	ζ	ι	ζ	ζ	ζ	ι
2	ι	ζ	ι	ζ	ι	ζ	ζ	ι
3	ι	ι	ζ	ι	ζ	ζ	ζ	ι
4	ι	ζ	ι	ζ	ι	ζ	ζ	ι
5	ι	ζ	ζ	ζ	ζ	ι	ι	ι
6	ι	ζ	ζ	ζ	ζ	ι	ι	ι
7	ι	ι	ι	ι	ι	ι	ι	ι

Table 11. The gyroaddition and gyration tables for the gyrogroup $G_{8,6}$, where ι is the identity automorphism and $\zeta = (1\ 3)(2\ 4)$.

Subgyrogroup	Elements	Isomorphism Type	Remark
$\langle 1\rangle$	$\{0,1\}$	$\mathbb{Z}_2$	n/a
$\langle 2\rangle$	$\{0,2\}$	$\mathbb{Z}_2$	n/a
$\langle 3\rangle$	$\{0,3\}$	$\mathbb{Z}_2$	n/a
$\langle 4\rangle$	$\{0,4\}$	$\mathbb{Z}_2$	n/a
$\langle 7\rangle$	$\{0,7\}$	$\mathbb{Z}_2$	left nucleus
$\langle 5\rangle$	$\{0,5,6,7\}$	$\mathbb{Z}_4$	right nucleus
$\langle 1,3\rangle$	$\{0,1,3,7\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a
$\langle 2,4\rangle$	$\{0,2,4,7\}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	n/a

Table 12. The non-trivial proper subgyrogroups of $G_{8,6}$.

Note that $\langle 7\rangle \trianglelefteq G_{8,6}$ since $\langle 7\rangle$ is the left nucleus of $G_{8,6}$, and that $\langle 5\rangle$, $\langle 1,3\rangle$, and $\langle 2,4\rangle$ are all normal in $G_{8,6}$ since they are strong subgyrogroups of $G_{8,6}$ of index 2. Also, note that $\langle 5\rangle$ is the right nucleus of $G_{8,6}$. Moreover, $\langle 1\rangle$ is not normal in $G_{8,6}$ since $2\oplus\langle 1\rangle \neq \langle 1\rangle\oplus 2$; $\langle 2\rangle$ is not normal in $G_{8,6}$ since $1\oplus\langle 2\rangle \neq \langle 2\rangle\oplus 1$; $\langle 3\rangle$ is not normal in $G_{8,6}$ since $2\oplus\langle 3\rangle \neq \langle 3\rangle\oplus 2$; and $\langle 4\rangle$ is not normal in $G_{8,6}$ since $1\oplus\langle 4\rangle \neq \langle 4\rangle\oplus 1$. The lattice of subgyrogroups of $G_{8,6}$ is depicted in Figure 6. We remark that $G_{8,5}$ and $G_{8,6}$ are not isomorphic because their right nuclei are not isomorphic: the right nucleus of $G_{8,5}$ is isomorphic to the Klein 4-group, whereas the right nucleus of $G_{8,6}$ is isomorphic to the cyclic group of order 4.

In summary, we present the complete list of pairwise non-isomorphic gyrogroups of order 8, determine their subgyrogroup structures, and draw their lattice diagrams of subgyrogroups. Finally, with these examples of gyrogroups, we can answer a few recent questions raised by Smith, Bao, and Wattanapan.

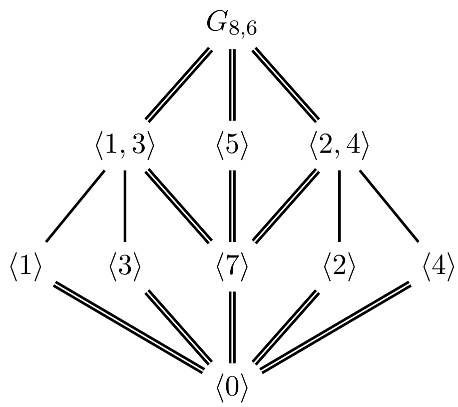

Figure 6. The lattice diagram of $G_{8,6}$, where double lines indicate being a normal subgyrogroup in the whole gyrogroup.

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