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# ON WEAKLY SEMI $\delta$-PRIMARY IDEALS IN LATTICES

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## Abstract

In this paper, we have introduced semi-primary ideals and weakly semi-primary ideals in a lattice. We have also proved several results about these ideals and established the relationships of semi-primary ideals with other types of ideals. Furthermore, we have introduced semi-$\delta$-primary ideals, weakly semi-$\delta$-primary ideals, and dual zero in a lattice. We have obtained many properties and characterizations of semi-$\delta$-primary ideals. Additionally, we have defined strongly weakly semi-$\delta$-primary ideals in a lattice.

**Keywords:** expansion function, weakly $\delta$-primary ideal, semi $\delta$-primary ideal, weakly semi $\delta$-primary ideal, dual zero, semi primary ideal, weakly semi primary ideal, strongly weakly semi $\delta$-primary ideal.

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## 1. Introduction

The notion of a prime ideal is well-known in both ring theory and lattice theory. Anderson and Bataineh [1], as well as Anderson and Smith [2], introduced some generalizations of this concept. Another generalization, namely 2-absorbing ideals in a commutative ring, was introduced by Badawi [3].

The study of expansions of ideals and $\delta$-primary ideals for commutative rings was conducted by Zhao [8]. In [5], Fahid and Zhao introduced the concept of a 2-absorbing $\delta$-primary ideal in a commutative ring. Recently, the concept of a weakly 2-absorbing $\delta$-primary ideal in a commutative ring was studied by Badawi and Fahid [4].

Nimbhorkar and Nehete [6] studied $\delta$-primary ideals and weakly $\delta$-primary ideals in a lattice. They also investigated 2-absorbing $\delta$-primary ideals in a lattice [7].

In this paper, we define a semi $\delta$-primary ideal and study some of its properties. Additionally, we define weakly semi-primary ideals and semi-primary ideals in lattices. We investigate several properties of a semi $\delta$-primary ideal with respect to a homomorphism. Furthermore, we define the concept of a weakly semi $\delta$-primary ideal in a lattice and introduce the notion of a $\delta$-dual-zero. We also define strongly weakly semi $\delta$-primary ideals in a lattice.

Throughout this paper, $L$ denotes a lattice with a least element 0. It is known that $Id(L)$, the set of all ideals of a lattice $L$, forms a lattice under set inclusion.

## 2. Preliminaries

The following definitions are from Nimbhorkar and Nehete [6].

**Definition.** An expansion of ideals, or an ideal expansion, is a function $\delta : Id(L) \to Id(L)$, satisfying the conditions (i) $I \subseteq \delta(I)$ and (ii) $J \subseteq K$ implies $\delta(J) \subseteq \delta(K)$, for all $I, J, K \in Id(L)$.

**Example 1.** (1) The identity function $\delta_0 : Id(L) \to Id(L)$, where $\delta_0(I) = I$ for every $I \in Id(L)$, is an expansion of ideals.

(2) The function $\mathbf{B}$ that assigns the biggest ideal $L$ to each ideal is an expansion of ideals.

(3) For each proper ideal $P$, the mapping $\mathbf{M} : Id(L) \to Id(L)$, defined by $\mathbf{M}(P) = \cap\{I \in Id(L) \mid P \subseteq I, I \text{ is a maximal ideal other than } L\}$ and $\mathbf{M}(L) = L$. Then $\mathbf{M}$ is an expansion of ideals.

(4) For each ideal $I$ define $\delta_1(I) = \sqrt{I} = \bigcap\{P \in Id(L) \mid P \text{ is a prime ideal}, I \subseteq P\}$ is the radical of $I$. Then $\delta_1(I)$ is an expansion of ideals.

**Definition.** Let $\delta$ be an expansion of ideals of $L$. A proper ideal $I$ of $L$ is called $\delta$-primary if $a \wedge b \in I$, then $a \in I$ or $b \in \delta(I)$ for all $a, b \in L$.

**Definition.** For an expansion of ideals $\delta$, an ideal $P$ of $L$ is called weakly $\delta$-primary if $0 \neq a \wedge b \in P$ implies either $a \in P$ or $b \in \delta(P)$ for all $a, b \in L$.

**Definition** (See Nimbhorkar and Nehete [7])**.** Let $\delta$ be an expansion of ideals of $L$. A proper ideal $I$ of $L$ is called a 2-absorbing $\delta$-primary ideal if for $a, b, c \in L$, $a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $b \wedge c \in \delta(I)$ or $a \wedge c \in \delta(I)$.

**Definition** (See Nimbhorkar and Nehete [7])**.** Let $\delta$ be an expansion of ideals of $L$. A proper ideal $I$ of $L$ is called a weakly 2-absorbing $\delta$-primary ideal if for $a, b, c \in L$, $0 \neq a \wedge b \wedge c \in I$, then either $a \wedge b \in I$ or $b \wedge c \in \delta(I)$ or $a \wedge c \in \delta(I)$.

## 3. Weakly semi $\delta$-primary ideals

**Definition.** For an expansion of ideals $\delta$, a proper ideal $S$ of $L$ is called a semi $\delta$-primary if $a \wedge b \in S$ implies either $a \in \delta(S)$ or $b \in \delta(S)$ for all $a, b \in L$.

**Definition.** For an expansion of ideals $\delta$, a proper ideal $W$ of $L$ is called a weakly semi $\delta$-primary if $0 \neq a \wedge b \in W$ implies either $a \in \delta(W)$ or $b \in \delta(W)$ for all $a, b \in L$.

**Definition.** If $\delta : Id(L) \to Id(L)$ such that $\delta(S) = \sqrt{S}$ for every proper ideal $S$ of $L$, then $\delta$ is an expansion function of ideals of $L$. In this case a proper ideal $S$ of $L$ is called a (weakly) semi primary if $(0 \neq a \wedge b \in S) a \wedge b \in S$ implies either $a \in \sqrt{S}$ or $b \in \sqrt{S}$ for all $a, b \in L$.

**Example 2.** Consider the lattice shown in Figure 1.

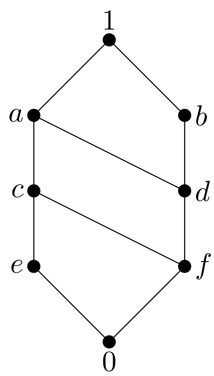


Figure 1

From Example 1, for the ideal $I = (e]$, $\delta_0(I) = I$, $\mathbf{M}(I) = (a]$ then $I$ is a semi $\delta$-primary ideal and weakly semi $\delta$-primary ideal of $L$. As $\delta_1(I) = \sqrt{I}$ so the ideal $I = (e]$ is also semi primary and weakly semi primary ideal of $L$. But the ideal $J = (f]$, where $\delta_0(J) = J$, $\delta_1(J) = \mathbf{M}(J) = J$ is not a semi $\delta$-primary ideal and weakly semi $\delta$-primary ideal also it is not semi primary and weakly semi primary ideal of $L$, as $c \wedge d = f \in J$ but neither $c \in \delta(J)$ nor $d \in \delta(J)$.

**Example 3.** Consider the ideal $W = (d]$ of the lattice as shown in figure 2, it is weakly prime, weakly primary, weakly semi $\delta$-primary and weakly semiprimary ideal of $L$. But the ideal $P = (c]$ of the given lattice $L$ is neither weakly prime nor weakly primary nor weakly semi $\delta$-primary nor weakly semiprimary, as $f \wedge g = c \in P$ but neither $f \in \delta(P)$ nor $g \in \delta(P)$.

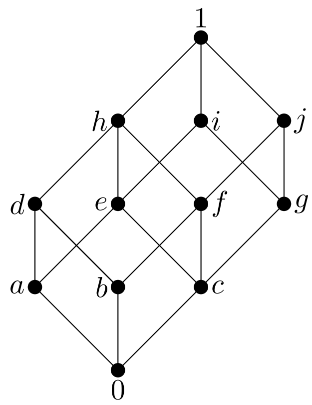


Figure 2

**Theorem 4.** *Let $P$ be a proper ideal of $L$ and let $\delta$ be an expansion function of ideals of $L$.*

(1) *If $P$ is a weakly $\delta$-primary ideal of $L$, then $P$ is a weakly semi $\delta$-primary ideal of $L$. In particular, if $P$ is a weakly primary ideal of $L$, then $P$ is a weakly semiprimary ideal of $L$.*

(2) *$\sqrt{\{0\}}$ is a weakly prime ideal of $L$ if and only if $\sqrt{\{0\}}$ is a weakly semiprimary ideal of $L$.*

**Lemma 5.** *Every semi $\delta$-primary ideal is weakly semi $\delta$-primary ideal of $L$.*

**Remark 6.** The following example shows that the converse of above Lemma 5 does not hold.

**Example 7.** Consider the ideal $P = (0]$ of the lattice shown in figure 3. Then $\delta(P) = \delta_1(P) = \mathbf{M}(P) = (i]$ then $P$ is a weakly semi $\delta$-primary ideal but not semi $\delta$-primary ideal of given lattice $L$ as $b \wedge d \in (0]$ but neither $b \in \delta(P)$ nor $d \in \delta(P)$.

We have one non-zero ideal which is a weakly semi $\delta$-primary ideal but not semi $\delta$-primary ideal. Consider the ideal $P = (b]$ of the lattice shown in figure 3. Then $\delta(P) = \delta_1(P) = \mathbf{M}(P) = (m]$ then $P$ is a weakly semi $\delta$-primary ideal but not semi $\delta$-primary ideal of given lattice $L$ as $c \wedge d \in P$ but neither $c \in \delta(P)$ nor $d \in \delta(P)$.

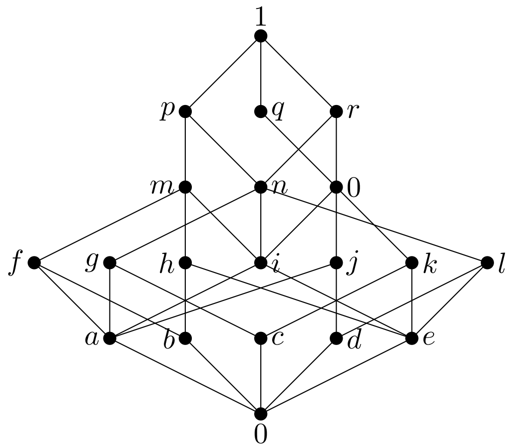


Figure 3

**Lemma 8.** *Every weakly prime ideal of $L$, is a weakly semiprimary ideal of $L$.*

**Remark 9.** The following is an example of a proper ideal of a lattice $L$ that is a weakly semiprimary ideal of $L$, but it is not a weakly prime.

**Example 10.** Consider the ideal $P = (b]$ of the lattice shown in figure 3. Then $\sqrt{P} = (m]$ then $P$ is a weakly semiprimary ideal but it is not weakly prime as $f \wedge h \in P$ neither $f \notin P$ nor $h \notin P$.

**Lemma 11.** *Every $\delta$-primary ideal of $L$, is a weakly semi $\delta$-primary ideal of $L$.*

**Remark 12.** The following is an example of a proper ideal of a lattice $L$ which is a weakly semi $\delta$-primary but not a $\delta$-primary.

**Example 13.** Consider the ideal $P = (b]$ of the lattice shown in figure 3. Then $\delta(P) = \delta_1(P) = \mathbf{M}(P) = (m]$ then $P$ is a weakly semi $\delta$-primary ideal but not $\delta$-primary ideal of given lattice $L$ as $c \wedge d = 0 \in P$ but neither $c \in P$ nor $d \in \delta(P)$.

**Definition.** Let $\delta$ be an expansion function of ideals of a lattice $L$. Suppose that $P$ is a weakly semi $\delta$-primary ideal of $L$ and $a \in L$. Then a is called a dual-zero element of $P$ if $a \wedge b = 0$ for some $b \in L$ and neither $a \in \delta(P)$ nor $b \in \delta(P)$. (Note that $b$ is also a dual-zero element of $P$.)

**Example 14.** Consider the ideal $P = (b]$ of the lattice shown in figure 3. Then $\delta(P) = \delta_1(P) = \mathbf{M}(P) = (m]$ then $P$ is a weakly semi $\delta$-primary ideal. Then $c$ is called a dual-zero element of $P$ as $c \wedge d = 0 \in (P]$ but neither $c \in \delta(P)$ nor $d \in \delta(P)$.

**Lemma 15.** *Let $\delta$ be an expansion function of ideals of a lattice $L$. If $P$ is a weakly semi $\delta$-primary ideal of $L$ which is not semi $\delta$-primary ideal, then $P$ must have a dual-zero element of $L$.*

The following example shows that the converse of above Lemma 15 does not hold.

**Example 16.** Consider the ideal $P = (a]$ of the lattice shown in figure 3. $P$ have dual zero but $P$ need not be a weakly semi $\delta$-primary which is not a semi $\delta$-primary as $f \wedge g = a \in P$ but neither $f \in \delta(P)$ nor $g \in \delta(P)$, where $\delta(P) = \delta_1(P) = \mathbf{M}(P) = (a]$.

**Theorem 17.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$. If $a \in L$ is a dual-zero element of $P$, then $a \wedge P = \{0\}$.*

***Proof.*** Assume that $a \in L$ is a dual-zero element of $P$. Then $a \wedge b = 0$ for some $b \in L$ such that neither $a \in \delta(P)$ nor $b \in \delta(P)$. Thus, $a \wedge (b \vee p) = 0 \vee (a \wedge p) = (a \wedge p) \in P$ for $p \in P$. Suppose that $a \wedge p \neq 0$. Since $0 \neq a \wedge (b \vee p) = 0 \vee (a \wedge p) = (a \wedge p) \in P$ and $P$ is a weakly semi $\delta$-primary ideal of $L$, we conclude that $a \in \delta(P)$ or $(b \vee p) \in \delta(P)$, and hence $a \in \delta(P)$ or $b \in \delta(P)$, a contradiction. Thus, $a \wedge p = 0$. Hence $a \wedge P = \{0\}$. ■

**Theorem 18.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$ that is not semi $\delta$-primary ideal. Then $P^2 = \{0\}$, where $P^2 = \{a \wedge b : a \neq b; a, b \in P\}$.*

***Proof.*** Since $P$ is a weakly semi $\delta$-primary ideal of $L$ that is not a semi $\delta$-primary, we conclude that $P$ has a dual-zero element $a \in L$. Then $a \wedge b = 0$ and neither $a \in \delta(P)$ or $b \in \delta(P)$, we conclude that $b$ is a dual-zero element of $P$. Then by Theorem 17, for $i, j \in P$ we have $(a \vee i) \wedge (b \vee j) = P^2 \subseteq P$. Suppose that $P^2 \neq 0$. i.e., $i \wedge j \neq 0$. Since $0 \neq (a \vee i) \wedge (b \vee j) = i \wedge j \in P$ and $P$ is a weakly semi $\delta$-primary ideal of $L$, we conclude that $(a \vee i) \in \delta(P)$ or $(b \vee P) \in \delta(P)$, and hence $a \in \delta(P)$ or $b \in \delta(P)$, a contradiction. Therefore $P^2 = 0$. ■

**Remark 19.** The following example show that the converse of above Theorem 18 does not hold.

**Example 20.** Consider the ideal $P = (a]$ of the lattice as shown in figure 4. Then $P^2 = 0 \wedge a = 0$ but $P$ is not a weakly semi $\delta$-primary ideal of $L$ as $\delta(P) = \delta_1(P) = \mathbf{M}(P) = P$, $f \wedge g = a \in P$ but neither $f \in \delta(P)$ nor $g \in \delta(P)$.

In view of Theorem 18, we have the following result.

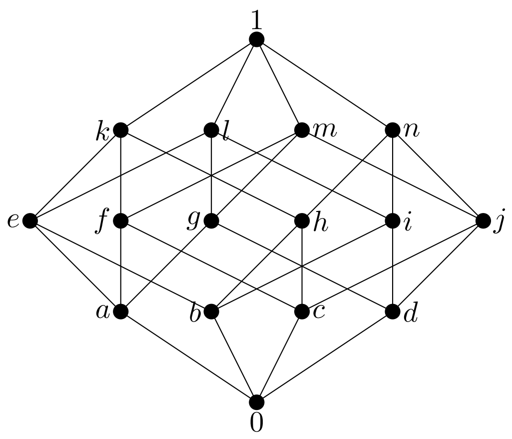


Figure 4

**Theorem 21.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semiprimary ideal of $L$ that is not semi primary. Then $P^2 = \{0\}$.*

The following example shows that a proper ideal $P$ of $L$ with the property $P^2 = 0$ need not be a weakly semiprimary ideal of $L$.

**Example 22.** Consider the ideal $P = (e]$ of the lattice as shown in figure 3. Then $P^2 = 0 \wedge e = 0$ but $P$ is not weakly semi primary ideal of $L$ as $\sqrt{P} = (i]$, $k \wedge l = e \in P$ but neither $k \in \sqrt{P}$ nor $l \in \sqrt{P}$.

**Theorem 23.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a proper ideal of $L$. If $\delta(P)$ is a weakly prime of $L$, then $P$ is a weakly semi $\delta$-primary ideal of $L$. In particular, if $\sqrt{P}$ is a weakly prime of $L$, then $P$ is a weakly semiprimary ideal of $L$.*

***Proof.*** Suppose that $0 \neq a \wedge b \in P$ for some $a, b \in L$. Hence, $0 \neq a \wedge b \in \delta(P)$. Since $\delta(P)$ is weakly prime, we conclude that $a \in \delta(P)$ or $b \in \delta(P)$. Thus, $P$ is a weakly semi $\delta$-primary ideal of $L$. ■

**Remark 24.** If $W$ is a weakly semi $\delta$-primary ideal of a lattice $L$, then $\delta(W)$ need not be a weakly prime ideal of $L$. We have the following example.

**Example 25.** Consider the ideal $W = (b]$ of lattice $L$ as shown in figure 3, which is weakly semi $\delta$-primary where $\delta_1(W) = \mathbf{M}(W) = (m]$. But $\delta(W) = \delta_1(W) = \mathbf{M}(W) = (m]$ is not weakly prime ideal of $L$ as $g \wedge j = a \in \delta(W)$ but neither $g \in \delta(W) = (m]$ nor $j \in \delta(W) = (m]$.

**Remark 26.** If $W$ is a weakly semiprimary ideal of a lattice $L$ that is not a semiprimary, then $\sqrt{\{W\}}$ need not be a weakly prime ideal of $L$. We have the following example.

**Example 27.** Consider the ideal $J = (d]$ of lattice $L$ as shown in figure 3, where $\sqrt{J} = (o]$, then $J$ is a weakly semiprimary ideal which not semiprimary as $b \wedge g = 0 \in J$ but neither $b \in \sqrt{J} = (o]$ nor $g \in \sqrt{J} = (o]$. But $\sqrt{J} = (o]$ is not weakly prime ideal of $L$ as $f \wedge n = a \in \sqrt{J}$ but neither $f \in \sqrt{J} = (o]$ nor $n \in \sqrt{J} = (o]$.

**Theorem 28.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$. Suppose that $\delta(P) = \delta(\{0\})$. Then the following statements are equivalent:*

(1) *$P$ is not a semi $\delta$-primary ideal.*

(2) *$\{0\}$ has a dual-zero element of $L$.*

***Proof.*** (1) $\Rightarrow$ (2): As $P$ is a weakly semi $\delta$-primary ideal of $L$ that is not semi $\delta$-primary, then there exists $a, b \in L$ such that $a \wedge b = 0$ and neither $a \in \delta(P)$ or $b \in \delta(P)$. Since $\delta(P) = \delta(0)$, we conclude that $a$ is a dual-zero element of $\{0\}$.

(2) $\Rightarrow$ (1): Suppose that $a$ is a dual-zero element of $\{0\}$. Since $\delta(P) = \delta(\{0\})$, so clearly $a$ is a dual-zero element of $P$. ■

In view of Theorem 28, we have following result.

**Theorem 29.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $I$ be a weakly semiprimary ideal of $L$. Suppose that $\delta(I) = \sqrt{\{0\}}$. Then the following statements are equivalent:*

(1) *$I$ is not a semiprimary ideal.*

(2) *$\{0\}$ has a dual-zero element of $L$.*

**Theorem 30.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$. If $Q \subseteq P$ and $\delta(P) = \delta(Q)$, then $Q$ is a weakly semi $\delta$-primary ideal of $L$.*

***Proof.*** Suppose that $0 \neq a \wedge b \in Q$ for some $a, b \in L$. Since $Q \subseteq P$, we have $0 \neq a \wedge b \in P$. Since $P$ is a weakly semi $\delta$-primary ideal of $L$, we see that $a \in \delta(P)$ or $b \in \delta(P)$. Since $\delta(P) = \delta(Q)$, we conclude that $a \in \delta(Q)$ or $b \in \delta(Q)$. Thus, $Q$ is a weakly semi $\delta$-primary ideal of $L$. ■

**Theorem 31.** *Let $\delta$ be an expansion function of ideals of $L$ such that $\delta(\{0\})$ is a semi $\delta$-primary ideal of $L$ and $\delta(\delta(\{0\})) = \delta(\{0\})$. Then the following statements hold:*

(1) $\delta(\{0\})$ *is a prime ideal of* $L$.

(2) *Suppose that* $P$ *is a weakly semi* $\delta$*-primary ideal of* $L$. *Then* $P$ *is a semi* $\delta$*-primary ideal of* $L$.

***Proof.*** (1) Let $x \wedge y \in \delta(\{0\})$ for some $x, y \in L$. Suppose that $x \notin \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\})$ is a semi $\delta$-primary ideal of $L$ and $x \notin \delta(\delta(\{0\}))$, it follows that $y \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus, $\delta(\{0\})$ is a prime ideal of $L$.

(2) Suppose that $P$ is not semi $\delta$-primary ideal. Clearly, $\delta(\{0\}) \subseteq \delta(P)$. Since $P^2 = 0$, by Theorem 18 and $\delta(\{0\})$ is a prime ideal of $L$, we have $P \subseteq \delta(\{0\})$. As $\delta(\delta(\{0\})) = \delta(\{0\})$, we have $\delta(P) \subseteq \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\}) \subseteq \delta(P)$ and $\delta(P) \subseteq \delta(\{0\})$, it follows that $\delta(\{0\}) = \delta(P)$ is a prime ideal of $L$. As $\delta(P)$ is prime, $P$ is a semi $\delta$-primary ideal of $L$, which is a contradiction. Thus, $P$ is semi $\delta$-primary. ■

**Theorem 32.** *Let* $\delta$ *be an expansion function of ideals of* $L$ *such that* $\delta(\{0\})$ *is a semi* $\delta$*-primary ideal of* $L$, $\sqrt{\{0\}} \subseteq \delta(\{0\})$ *and* $\delta(\delta(\{0\})) = \delta(\{0\})$ *then* $\delta(\{0\})$ *is a weakly prime ideal of* $L$.

***Proof.*** Let $0 \neq x \wedge y \in \delta(\{0\})$ for some $x, y \in L$. Suppose that $x \notin \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\})$ is a weakly semi $\delta$-primary ideal of $L$ and $x \notin \delta(\delta(\{0\}))$, it follows that $y \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus, $\delta(\{0\})$ is a weakly prime ideal of $L$. ■

**Lemma 33.** *Every* (*weakly*) $\delta$*-primary ideal is* 2*-absorbing* $\delta$*-primary ideal.*

**Remark 34.** The converse of above Lemma 33 does not hold. We have the following example.

**Example 35.** Consider the ideal $I = (e]$ of lattice as shown in Figure 2, is 2-absorbing $\delta$-primary, where $\delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$, but not $\delta$-primary. As $b \wedge g = 0 \in I$ but $b \notin \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$ and $g \notin I$. Also it is not weakly $\delta$-primary, as $h \wedge i = e \in I$ but $i \notin \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$ and $h \notin I$.

**Lemma 36.** *Every weakly semiprimary or semiprimary ideal is* 2*-absorbing* $\delta$*-primary ideal.*

**Remark 37.** The converse of above Lemma 36 does not hold. We have the following example.

**Example 38.** Consider the ideal $I = (e]$ of lattice as shown in Figure 3, is 2-absorbing $\delta$-primary, where $\sqrt{I} = I$, but not semiprimary and not a weakly semiprimary. As $k \wedge l = e \in I$ but $k \notin \sqrt{I} = I$ and $l \notin \sqrt{I}$.

**Lemma 39.** *Every weakly* $\delta$*-primary ideal or weakly semi* $\delta$*-primary is weakly* 2*-absorbing* $\delta$*-primary ideal.*

**Remark 40.** The converse of above Lemma 39 does not hold. We have the following example.

**Example 41.** Consider the ideal $I = (a]$ of lattice as shown in Figure 4, is weakly 2-absorbing $\delta$-primary, where $\delta(I) = \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$, but not 2-absorbing $\delta$-primary. As $h \wedge i \wedge m = 0 \in I$ but $h \wedge m = c \notin \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$, $m \wedge i = d \notin \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$ and $h \wedge i = b \notin I$. Also it is not weakly $\delta$-primary and not a weakly semi $\delta$-primary, as $f \wedge g = a \in I$ but $f \notin \delta(I) = \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$ and $g \notin \delta(I) = \delta_1(I) = \mathbf{M}(I) = \delta_0(I) = I$.

**Definition** (See Nimbhorkar and Nehete [6])**.** An expansion is said to be global if for any lattice homomorphism $f : L \to K$, $\delta(f^{-1}(I)) = f^{-1}(\delta(I))$ for all $I \in Id(K)$.

In following lemma, we prove that the inverse image of a weakly semi $\delta$-primary ideal of $L$ under a homomorphism is again a weakly semi $\delta$-primary ideal.

**Lemma 42.** *If $\delta$ is global and $f : L \to K$ is a lattice homomorphism, then for any weakly semi $\delta$-primary ideal $P$ of $K$, $f^{-1}(P)$ is a weakly semi $\delta$-primary ideal of $L$.*

***Proof.*** Let $x, y \in L$ with $x \wedge y \in f^{-1}(P)$ and $x \notin \delta(f^{-1}(P))$ then $f(x) \wedge f(y) \in P$ and $f(x) \notin \delta(P)$ but $P$ is a weakly semi $\delta$-primary then, we get $f(y) \in \delta(P)$, so $y \in f^{-1}(\delta(P)) = \delta(f^{-1}(P))$. Hence $f^{-1}(P)$ is weakly semi $\delta$-primary. ■

Next result gives a characterization for a weakly semi $\delta$-primary ideal.

**Lemma 43.** *Let $f : L \to K$ be a surjective lattice homomorphism, then an ideal $P$ of $L$ that contains ker(f) is a weakly semi $\delta$-primary ideal if and only if $f(P)$ is a weakly semi $\delta$-primary ideal of $K$.*

***Proof.*** First suppose that $f(P)$ is a weakly semi $\delta$-primary and $P$ contains $ker(f)$ we have $f^{-1}(f(P)) = P$. Then by Lemma 42, $P$ is weakly semi $\delta$-primary.

Conversely, suppose that $P$ is weakly semi $\delta$-primary. If $x, y \in P$ and $0 \neq x \wedge y \in f(P)$ and $x \notin \delta(f(P))$ then there exist $a, b \in L$ such that $f(a) = x$ and $f(b) = y$, then $f(a \wedge b) = f(a) \wedge f(b) = x \wedge y \in f(P)$ implies $a \wedge b \in f^{-1}(f(P)) = P$ and $f(a) = x \notin \delta(f(P)) = f(\delta(P))$ implies $a \notin \delta(P)$, so $b \in \delta(P)$ and hence $y = f(b) \in f(\delta(P))$. Since $\delta(P) = \delta(f^{-1}(f(P))) = f^{-1}(\delta(f(P)))$ which implies $f(\delta(P)) = \delta(f(P))$. Thus $f(P)$ is weakly semi $\delta$-primary. ■

## 4. WEAKLY SEMI $\delta$-PRIMARY IDEAL IN PRODUCT OF LATTICES

Let $L_1, L_2, \ldots, L_n$ where $n \geq 2$, be lattices with $1 \neq 0$. Assume that $\delta_1, \delta_2, \ldots, \delta_n$ are expansion functions of ideals of $L_1, L_2, \ldots, L_n$ respectively.

Let $L = L_1 \times L_2 \times \cdots \times L_n$. Define a function $\delta_\times : Id(L) \to Id(L)$ such that $\delta_\times(I_1 \times I_2 \times \cdots \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times \cdots \times \delta_n(I_n))$ for every $I_i \in Id(L_i)$, where $1 \leq i \leq n$. Clearly, $\delta_\times$ is an expansion function of ideals of $L$. Note that every ideals of $L$ is of the form $I_1 \times I_2 \times \cdots \times I_n$, where each $I_i$ is an ideal of $L_i$, for $1 \leq i \leq n$.

**Theorem 44.** *Let $L_1$ and $L_2$ be lattices with $1 \neq 0$. Let $L = L_1 \times L_2$ and $\delta_1$, $\delta_2$ and $\delta_\times$ be expansion function of ideals of $L_1, L_2$ and $L$, respectively. Let $P_1$ and $P_2$ be a proper ideal of $L_1$ and $L_2$, respectively $P = P_1 \times P_2$ is weakly semi $\delta$-primary ideal of $L$ then $P_1$ and $P_2$ are weakly semi $\delta$-primary ideal of $L_1$ and $L_2$, respectively.*

***Proof.*** Let $0 \neq x \wedge y \in P_1$ for some $a, b \in L_1$, then $0 \neq (x \wedge y, a) \in P_1 \times P_2$ for every $a \in L_2$. As $P_1 \times P_2$ is a weakly semi $\delta$-primary, we get either $(x, a) \in \delta_\times(P_1 \times P_2)$ or $(y, a) \in \delta_\times(P_1 \times P_2)$. It implies that $(x, a) \in \delta_1(P_1) \times \delta_2(P_2)$ or $(y, a) \in \delta_1(P_1) \times \delta_2(P_2)$. Thus we get either $x \in \delta_1(P_1)$ or $y \in \delta_1(P_1)$. Hence $P_1$ is weakly semi $\delta$-primary ideal of $L_1$.

Similarly, we can show $P_2$ is weakly semi $\delta$-primary ideal of $L_2$. ■

**Remark 45.** The Converse of above Theorem 44 does not hold.

The following example shows that the converse of Theorem 44 does not hold.

**Example 46.** Consider the lattices $L_1$, $L_2$ as shown in figure 5. We note that the ideals $P_1 = (x]$ and $P_2 = (0]$ of $L_1$ and $L_2$ are weakly semi $\delta_1$-primary and $\delta_2$-primary ideals respectively, where $\delta_1$, $\delta_2$ and $\delta_\times$ are the expansion function on $L_1$, $L_2$ and $L$, respectively.

Consider the ideal $P_1 \times P_2 = ((x, 0)]$. Consider $(y, 1) \wedge (z, 0) = (x.0) \in P_1 \times P_2$, but neither $(y, 1) \notin \delta_\times(P_1 \times P_2) = ((x, 0)]$ and $(z, 0) \notin \delta_\times(P_1 \times P_2) = ((x, 0)]$. Thus $P_1 \times P_2 = ((x, 0)]$ is not a weakly semi $\delta_\times$ primary element in $L_1 \times L_2$.

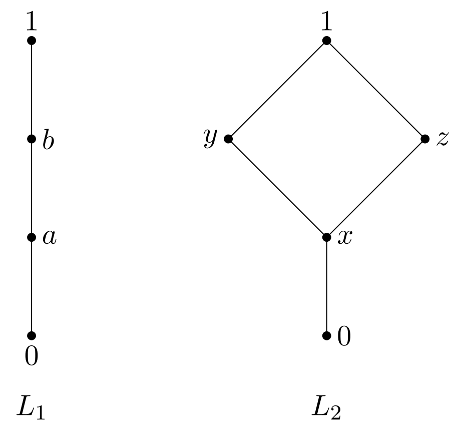


Figure 5

**Theorem 47.** *Let $L_1$ and $L_2$ be lattices with $1 \neq 0$. Let $L = L_1 \times L_2$ and $\delta_1$, $\delta_2$ and $\delta_\times$ be expansion function of elements of $L_1, L_2$ and $L$ respectively. Let $P$ be a proper ideal of $L_1$. Then the following statements are equivalent.*

(i) *$P \times L_2$ is a weakly semi $\delta_\times$-primary ideal of $L$.*

(ii) *$P \times L_2$ is a semi $\delta_\times$-primary ideal of $L$.*

(iii) *$P$ is a semi $\delta_1$-primary ideal of $L_1$.*

***Proof.*** $(i) \Rightarrow (ii)$ Let $Q = P \times L_2$ be a proper ideal of $L$. Then $Q^2 \neq \{(0,0)\}$. Hence $Q = P \times L_2$ is a semi $\delta_\times$-primary ideal of $L$, by Theorem 18.

$(ii) \Rightarrow (iii)$ Suppose that $P$ is not a $\delta_1$-semiprimary ideal in $L_1$. Then there exist $a, b \in L_1$ such that $a \wedge b \in P$ but neither $a \in \delta_1(P)$ nor $b \in \delta_1(P)$. Since $(a,1)(b,1) = (a \wedge b, 1) \in P \times L_2$. As $P \times L_2$ is a semi $\delta_\times$-primary ideal of $L_1 \times L_2$. We get either $(a,1) \in= \delta_\times(P \times L_2) = \delta_1(P) \times \delta_2(L_2)$ or $(b,1) \in \delta_\times(P \times L_2) = \delta_1(P) \times \delta_2(L_2)$, a contradiction . Thus $P_1$ is a semi $\delta_1$-primary ideal of $L_1$.

$(iii) \Rightarrow (i)$ Suppose that $P \times L_2$ is not a weakly semi $\delta_\times$-primary ideal of $L$ then there exist $(0,0) \neq (x,1) \wedge (y,1) \in P \times L_2$ but neither $(x,1) \in \delta_\times(P \times L_2) = \delta_1(P) \times \delta_2(L_2)$ nor $(y, L_2) \in \delta_\times(P \times L_2) = \delta_1(P) \times \delta_2(L_2)$. This implies $x \wedge y \in P$ we get $x \in \delta_1(P)$ nor $y \in \delta_1(P)$, a contradiction to $P$ is a semi $\delta_1$-primary ideal of $L_1$. Thus $P \times L_2$ is a weakly semi $\delta_\times$-primary ideal of $L$. ∎

**Theorem 48.** *Let $L_1$ and $L_2$ be the lattices with $1 \neq 0$. Let $L = L_1 \times L_2$ and $\delta_1$, $\delta_2$ and $\delta_\times$ be expansion function of ideals of $L_1, L_2$ and $L$ respectively such that $\delta_2(Q) = L_2$ for some ideal $Q$ of $L_2$ if and only if $Q = L_2$. Let $P = P_1 \times P_2$ be a proper ideal of $L$, where $P_1$ and $P_2$ are some ideals of $L_1$ and $L_2$, respectively. Suppose that $\delta_1(P_1) \neq L_1$. Then the following statements are equivalent:*

(1) *$P$ is a weakly semi $\delta_\times$-primary ideal of $L$.*

(2) *$P = (0,0)$ or $P = P_1 \times L_2$ is a semi $\delta_\times$-primary ideal of $L$ and hence $P_1$ is a semi $\delta_1$-primary ideal of $L_1$.*

***Proof.*** (1) $\Rightarrow$ (2) Suppose that $(0,0) \neq P = P_1 \times P_2$ is a weakly semi $\delta_\times$-primary ideal of $L$. Then there exists $(0,0) \neq (x,y) \in P$ such that $x \in P_1$ and $y \in P_2$. Since $P$ is a weakly semi $\delta_\times$-primary ideal of $L$ and $(0,0) \neq (x,1)(1,y) = (x,y) \in P$, we conclude that $(x,1) \in \delta_\times(P) = \delta_1(P_1) \times \delta_2(P_2)$ or $(1,y) \in \delta_\times(P) = \delta_1(P_1) \times \delta_2(P_2)$. As $\delta_1(P_1) \neq L_1$, we get $(1,y) \notin \delta_\times(P)$. Thus $(x,1) \in \delta_\times(P)$, and hence $1 \in \delta_2(P_2)$. Since $1 \in \delta_2(P_2)$, we see that $\delta_2(P_2) = L_2$, and hence $P_2 = L_2$ by hypothesis. Therefore, $P = P \times L_2$ is a semi $\delta_\times$-primary ideal of $L$ by Theorem 48.

(2) $\Rightarrow$ (1) Obvious. ∎

## 5. Strongly weakly semi $\delta$-primary ideal

**Definition.** Let $\delta$ be an expansion function of ideals of a lattice $L$. A proper ideal $P$ of $L$ is called a strongly weakly semi $\delta$-primary ideal of $L$ if whenever $\{0\} \neq IJ \subseteq P$ for some ideals $I, J$ of $L$, we have $I \subseteq \delta(P)$ or $J \subseteq \delta(P)$. Hence, a proper ideal $P$ of $L$ is called a strongly weakly semiprimary ideal of $L$ if whenever $\{0\} \neq IJ \subseteq P$ for some ideals $I, J$ of $L$, we have $I \subseteq \sqrt{P}$ or $J \subseteq \sqrt{P}$.

**Theorem 49.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$. Suppose that $X \wedge Y \subseteq P$ for some ideals $X, Y$ of $L$, and that $x \wedge y = 0$ for some $x \in X$ and $y \in Y$ such that neither $x \in \delta(P)$ nor $y \in \delta(P)$. Then $X \wedge Y = \{0\}$.*

***Proof.*** We have to show that $x \wedge Y = y \wedge X = \{0\}$. Suppose that $x \wedge Y \neq \{0\}$. Then $0 \neq x \wedge z \in P$ for some $z \in Y$. Since $P$ is a weakly semi $\delta$-primary ideal of $L$ and $x \notin \delta(P)$, we conclude that $z \in \delta(P)$. Hence, $0 \neq x \wedge (y \vee z) = x \wedge z \in P$. Thus, $x \in \delta(P)$ or $(y \vee z) \in \delta(P)$. Since $z \in \delta(P)$, we see that $x \in \delta(P))$ or $y \in \delta(P)$, a contradiction. Thus, $x \wedge Y = \{0\}$. Similarly, $y \wedge X = \{0\}$. Now suppose that $X \wedge Y \neq \{0\}$. Then there is an element $r \in X$ and there is an element $s \in Y$ such that $0 \neq r \wedge s \in P$. Since $P$ is a weakly semi $\delta$-primary ideal of $L$, we conclude that $r \in \delta(P)$ or $s \in \delta(P)$. We consider three cases.

*Case* I. Suppose that $r \in \delta(P)$ or $s \notin \delta(P)$. Since $x \wedge Y = \{0\}$, we obtain $0 \neq s \wedge (r \vee x) = s \wedge r \in P$, and thus we conclude that $s \in \delta(P)$ or $(r \vee x) \in \delta(P)$. Since $r \in \delta(P)$, we have $s \in \delta(P)$ or $x \in \delta(P)$, a contradiction.

*Case* II. Suppose that $r \notin \delta(P)$ or $s \in \delta(P)$. Since $y \wedge X = \{0\}$, we have $0 \neq r \wedge (s \vee y) = r \wedge s \in P$. Hence we conclude that $r \in \delta(P)$ or $(s \vee y) \in \delta(P)$. As $s \in \delta(P)$, we have $r \in \delta(P)$ or $y \in \delta(P)$, a contradiction.

*Case* III. Suppose that $r \in \delta(P)$ or $s \in \delta(P)$. Since $x \wedge X = y \wedge Y = \{0\}$, we can obtain $0 \neq (y \vee s) \wedge (r \vee x) = sr \in P$. Hence $y \vee s \in \delta(P)$ or $r \vee x \in \delta(P)$. As $r, s \in \delta(P)$, we have $x \in \delta(P)$ or $y \in \delta(P)$, a contradiction. Thus, $X \wedge Y = \{0\}$. ■

**Theorem 50.** *Let $\delta$ be an expansion function of ideals of a lattice $L$ and $P$ be a weakly semi $\delta$-primary ideal of $L$. Suppose that $\{0\} \neq X \wedge Y \subseteq P$ for some ideals $X, Y$ of $L$. Then $X \subseteq \delta(P)$ or $Y \subseteq \delta(P)$ (i.e., $P$ is a strongly weakly semi $\delta$-primary ideal of $L$).*

***Proof.*** Since $X \wedge Y \neq \{0\}$, by Theorem 49 we conclude that whenever $x \wedge y \in P$ for some $x \in X$ and $y \in Y$, we obtain $x \in \delta(P)$ or $y \in \delta(P)$. Assume that $\{0\} \neq X \wedge Y \subseteq P$ and $X \nsubseteq \delta(P)$. Then there is an $a \in X$ but $a \notin \delta(P)$. Let $b \in Y$. Since $a \wedge b \in X \wedge Y \subseteq P$, $\{0\} \neq X \wedge Y$ and $x \notin \delta(P)$, then we get $y \in \delta(P)$ by Theorem 49. Hence, $Y \subseteq \delta(P)$. ■

In view of above theorem, we have the following result.

**Corollary 51.** *Let $P$ be a weakly semiprimary ideal of $L$. We suppose that $\{0\} \neq\in X \wedge Y \subseteq P$ for some ideals $X, Y$ of $L$. Then $X \nsubseteq \sqrt{P}$, or $Y \nsubseteq \sqrt{P}$ (i.e., $P$ is a strongly weakly semiprimary ideal of $L$).*

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