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SPECIAL TYPE OF ADDITIVE MAPS IN PRIME RINGS WITH ANNIHILATING AND CENTRALIZING CONDITION

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Abstract

Let R be a prime ring with char $R \neq 2$ and $f(r_1, \ldots, r_n)$ be a non-central multilinear polynomial over $C(= Z(U))$, where U is the Utumi ring of quotients of R. Let I be a nonzero two sided ideal of R, L a non central Lie ideal of R and $\mathscr{F}$, $\mathscr{G}$ two generalized derivations of R. Denote the set $f(I) = \{f(r_1, \ldots, r_n) | r_1, \ldots, r_n \in I\}$. If for some $0 \neq a \in R$,

$$a[(\mathscr{F}^2 + \mathscr{G})(u), u] \in C$$

for all $u \in f(I)$ or $u \in L$, then possible forms of the maps are described.

This result improves the result proved by De Filippis *et al.* in [8] and Carini and Scudo in [6].

Keywords: prime ring, derivation, generalized derivation.

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1. Introduction

Let R be a prime ring with center $Z(R)$, U be its Utumi ring of quotients. C is the extended centroid of R which is basically center of U. By a derivation d on R, one usually means an additive mapping $d : R \to R$ such that for any $x, y \in R$, $d(xy) = d(x)y + xd(y)$. By a generalized derivation g on R, one usually means an additive mapping $g : R \to R$ such that for any $x, y \in R$, $g(xy) = g(x)y + xd(y)$ for some derivation d in R. Every derivation is a generalized derivation. Thus generalized derivation map is the generalization of the map derivation.

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For any $a, b \in R$, we denote $[a, b] = ab - ba$, which is called the commutator of a and b. The standard polynomial of four variables is $s_4(t_1, t_2, t_3, t_4) = \sum_{\sigma \in S_4} (-1)^\sigma t_{\sigma(1)} t_{\sigma(2)} t_{\sigma(3)} t_{\sigma(4)}$, where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or an odd permutation in symmetric group S_4. R satisfies s_4, we mean $s_4(t_1, t_2, t_3, t_4) = 0$ for all $t_1, t_2, t_3, t_4 \in R$. Let $f(r_1, \ldots, r_n)$ be a noncentral valued multilinear polynomial over C in n non-commuting variables.

Let S be a nonempty subset of R. Then $f(S)$ denotes the set of all evaluations of $f(x_1, \ldots, x_n)$ over S, that is, $f(S) = \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in S\}$. A mapping $\chi : R \to R$ is said to be commuting on S if $[\chi(s), s] = 0$ for all $s \in S$ and centralizing on S if $[\chi(s), s] \in Z(R)$ for all $s \in S$.

Let d, g be two derivations and $\mathscr{F}, \mathscr{G}$ two generalized derivations on a prime ring R. A well known result proved by Posner [26], says that if a nonzero centralizing derivation exists in a prime ring R, then the ring R must be commutative. After that, several authors have given their contributions to the theory extending Posner's [26] result in many directions (for instance, we refer to [1–4, 8, 11]).

In [15], authors of this paper studied the case when d^2 is commuting and centralizing on $f(I)$, where I is a non-zero right ideal of R.

In [11], De Filippis studied the case when $\mathscr{G}$ is commuting on $f(I)$, where I is a non-zero right ideal of R and then described forms of the maps.

In [22, Theorem 2.1], Lee *et al.* introduced a special type of additive map $d^2 + g$ and then initiated to study this type of map. They proved that if R is a $n!$-torsion free semiprime ring such that $[(d^2 + g)(s), s^n] = 0$ for all $s \in R$, then d and g are both commuting on R.

Further this special type of additive map was studied by Rehman and De Filippis in [27], replacing derivations with generalized derivation, that is, the map $\mathscr{F}^2 + \mathscr{G}$.

Inspired by the above cited results, in [8], De Filippis *et al.* studied the additive map $\mathscr{F}^2 + \mathscr{G}$ centralizing on $f(I)$, that is, $[(\mathscr{F}^2 + \mathscr{G})(f(I)), f(I)] = 0$, where I is a non-zero right ideal of R and then obtained forms of the maps.

There is also ongoing interest to investigate the above identities with left annihilating conditions.

In [10], De Filippis proved that if char $(R) \neq 2$ and $0 \neq a \in R$ such that $a[\mathscr{F}(f(R)), f(R)] = 0$, then one of the following holds:

(1) there exists $\alpha' \in C$ such that $\mathscr{F}(x) = \alpha' x$ for all $x \in R$,

(2) there exist $q' \in U$ and $\lambda' \in C$ such that $\mathscr{F}(x) = (q' + \lambda')x + xq'$ for all $x \in R$ and $f(r_1, \ldots, r_n)^2$ is central valued on R.

In [13, Corollary 2.7], Dhara *et al*. studied the above situation of [10] with central valued, that is, $a[\mathscr{F}(f(R)), f(R)] \in C$ and described the forms of the maps.

Carini and Scudo in [6], already proved that if char $(R) \neq 2$ and $0 \neq a \in R$ such that $a[\mathscr{F}^2(f(R)), f(R)] = 0$, then one of the following holds:

(1) there exists $\alpha' \in C$ such that $\mathscr{F}(x) = \alpha' x$, for all $x \in R$,

(2) there exists $a' \in U$ such that $\mathscr{F}(x) = a'x$, for all $x \in R$, with $a'^2 \in C$,

(3) there exists $a' \in U$ such that $\mathscr{F}(x) = xa'$, for all $x \in R$, with $a'^2 \in C$.

Recently in [12], Dhara *et al.* studied the above situation of [6] with central values, that is, $a[\mathscr{F}^2(f(R)), f(R)] \in C$.

In the present article our motivation is to examine the above situation of [8], with annihilator and centralizing conditions which improves and generalizes all the above results. More precisely, we prove the following theorems.

Theorem 1.1. *Let R be a prime ring with char $(R) \neq 2$ and $f(r_1, \ldots, r_n)$ be a non-central multilinear polynomial over $C(= Z(U))$, where U be the Utumi ring of quotients of R. Assume that I is a nonzero two sided ideal of R and $\mathscr{F}$, $\mathscr{G}$ are two generalized derivations of R. Denote the set $f(I) = \{f(r_1, \ldots, r_n) | r_1, \ldots, r_n \in I\}$. If for some $0 \neq a \in R$,*

$$a[(\mathscr{F}^2 + \mathscr{G})(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in C$$

for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) *there exist $b, p \in U$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = xp$ for all $x \in R$ with $b^2 + p \in C$,*

(2) *there exist $b, p \in U$ such that $\mathscr{F}(x) = bx$, $\mathscr{G}(x) = px$ for all $x \in R$ with $b^2 + p \in C$,*

(3) *$f(x_1, \ldots, x_n)^2$ is central valued and one of the following holds:*
    (a) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$, with $b^2 - p + q \in C$,*
    (b) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ and for all $x \in R$ with $b^2 + p - q \in C$,*

(4) *R satisfies s_4 and one of the following holds:*
    (a) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$, with $b^2 - p + q \in C$,*
    (b) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$ with $b^2 + p - q \in C$.*

Theorem 1.2. *Let R be a prime ring, L a noncentral Lie ideal of R and U the Utumi quotient ring of R, $C = Z(U)$. Suppose that $\mathscr{F}$ and $\mathscr{G}$ are two generalized derivations of R such that for some $0 \neq a \in R$,*

$$a[(\mathscr{F}^2 + \mathscr{G})(u), u] \in C$$

for all $u \in L$.

If char $(R) \neq 2$, then R satisfies s_4 and one of the following holds:

(1) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = xb$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$, with $b^2 - (p - q) \in C$,*

(2) *there exist $b, p, q \in U$ such that $\mathscr{F}(x) = bx$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$ with $b^2 + p - q \in C$.*

If char $(R) = 2$, then one of the following holds:

(1) *there exist $b, c, p, q \in U$ such that $\mathscr{F}(x) = bx + [p, x]$ and $\mathscr{G}(x) = cx + [q, x]$ for all $x \in R$ with $\mathscr{F}(b) + c, p^2 + q \in C$;*

(2) *R satisfies s_4.*

Example 1. Consider a ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in Z \right\}$, where Z is the set of all integers and a multilinear polynomial $f(x, y) = xy$ which is not central valued on R. Note that R is not prime ring as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We define maps $\mathscr{F}, \mathscr{G}, d, g : R \to R$ by $\mathscr{G} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}$, $g \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, $\mathscr{F} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 3b \\ 0 & 0 \end{pmatrix}$ and $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ 0 & 0 \end{pmatrix}$. Then $\mathscr{F}$ and $\mathscr{G}$ are generalized derivations of R associated to derivations d and g respectively. We see that for $0 \neq p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$,

$$p[(\mathscr{F}^2 + \mathscr{G})(f(x, y)), f(x, y)] = 0 \in Z(R)$$

for all $x, y \in R$. Since $\mathscr{F}$ is not in the form of $\mathscr{F}(x) = bx$ or $\mathscr{F}(x) = xb$ for all $x \in R$ and for some fixed $b \in R$, the primeness assumption is not superfluous in Theorem 1.1.

2. SOME RESULTS

Throughout this section, R always be a prime ring, I is two sided ideal of R and $f(r_1, \ldots, r_n)$ a noncentral valued multilinear polynomial over C. The C denotes the extended centroid of R which is the center of U.

The following facts are to be used frequently to prove our Theorem.

Fact 2.1. *Let us denote by $T = U *_C C\{X\}$, the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the countable set consisting of* *the noncommuting indeterminates* $x_1, x_2, \ldots$. *The elements of* T *are called generalized polynomials with coefficients in* U. *By a nontrivial generalized polynomial, we mean a nonzero element of* T. *For more details about these objects we refer to* [5, 18].

By [7], I, R *and* U *satisfy the same generalized polynomial identities* (*GPIs*) *with coefficients in* U.

Fact 2.2. *By* [23], I, R *and* U *satisfy the same differential identities.*

Fact 2.3 [4, Lemma 3]**.** *If there exist* $a, c, p, q \in U$ *such that*

$$(aX + Xc)X - X(pX + Xq) = 0$$

for all $X \in f(R)$, *then one of the following holds:*

(1) $a, q \in C$ *and* $q - a = c - p \in C$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $q - a = c - p \in C$;

(3) *char* $(R) = 2$ *and* R *satisfies* s_4.

Fact 2.4 (See [20, 23])**.** *Let* $Der(U)$ *be the set of all derivations on* U *and* D_{int} *be the set of all inner derivations on* U.

By [20, Theorem 2] (*see also* [23, Theorem 1]), *we have the following result.*

Let $d_1, \ldots, d_m \in \mathrm{Der}(U)$ *and derivations words* Δ_j *are in the form*

$$\Delta_j = d_1^{s_{1,j}} d_2^{s_{2,j}} \cdots d_m^{s_{m,j}} \quad j = 1, \ldots, n$$

where

$$s = \max\{s_{i,j},\ i = 1, \ldots, m\ j = 1, \ldots, n\}.$$

If $d_1, \ldots, d_m$ *are linearly* $C-$*independent modulo* D_{int}, $s < p$, *with char*$(R) = p \neq 0$, *and* $\Phi\left(x_i^{\Delta_j}\right) = 0$ *be a differential identity on* R, *then* $\Phi(y_{ji}) = 0$ *is a GPI for* R, *where* y_{ji} *are distinct indeterminates.*

In particular, if derivation $d \notin D_{int}$ *and char* $(R) \neq 2$ *such that* R *satisfies*

$$\Phi\left(x_1, \ldots, x_n, x_1^d, \ldots, x_n^d, x_1^{d^2}, \ldots, x_n^{d^2}\right) = 0,$$

then R *satisfies GPI*

$$\Phi\big(x_1, \ldots, x_n, z_1, \ldots, z_n, \eta_1, \ldots, \eta_n\big) = 0.$$

Fact 2.5 [9, Lemma 1]**.** *Let* C *be an infinite field,* t *be a positive integer with* $t \geq 2$ *and* $R = M_t(C)$, *the algebra of all* $t \times t$ *matrices over* C. *Let* $B_1, \ldots, B_k$ *be not scalar matrices in* R. *Then there must exists at least one invertible matrix* $Q \in R$ *such that all the entries of the matrices* $QB_1Q^{-1}, \ldots, QB_kQ^{-1}$ *have non-zero values.*

Fact 2.6. *If R satisfies a nontrivial generalized polynomial identity (GPI) $\chi(r_1, \ldots, r_n) = 0$, then it is also satisfied by U by* [7]. *Let E be the algebraic closure of C. We know that if C is infinite, then $\chi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U \otimes_C E$. Since both of U and $U \otimes_C E$ are prime and centrally closed* (*see* [16, Theorems 2.5 and 3.5]), *we may replace R by U or R by $U \otimes_C E$ according to C finite or infinite and hence we may assume that R is centrally closed over C. Then by* [25], *R is a primitive ring having a nonzero socle $soc(R)$ and C is its associated division ring. By Jacobson's theorem* [19, p. 75], *R is isomorphic to a dense ring of linear transformations of a vector space V over C.*

Fact 2.7. *Let $X = \{x_1, x_2, \ldots\}$ be a countable set of consisting non-commuting indeterminates $x_1, x_2, \ldots$. We denote $T = U *_C C\{X\}$, the free product of the C-algebra U and the free C-algebra $C\{X\}$. Then the element of T are called the generalized polynomials.*

Then any element $m \in T$ of the form $m = q_0 y_1 q_1 y_2 q_2 \ldots y_n q_n$, where $q_0, q_1, \ldots, q_n \in U$ and $y_1, \ldots, y_n \in X$ is called a monomial.

Let B be a set of C-independent vectors of U. Each $f \in T$ can be represented in the form $f = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials and this representation is unique. Any generalized polynomial $f = \sum_i \alpha_i m_i$ is trivial, i.e., zero element in T if and only if $\alpha_i = 0$ for each i. For details we refer the reader to [7].

We shall use this simple criterion to prove that R satisfies a nontrivial generalized polynomial identity (*GPI*).

3. The Case: Inner generalized derivations

This section is dedicated, when all generalized derivations are inner.

Lemma 3.1. *Let R be a prime ring with char $(R) \neq 2$ and $a, a', a'', b, c, c', p' \in R$ such that*

$$[a'X^2 + a''XcX + aXp'X - 2aXbXc - aX^2c', y] = 0 \tag{3.1}$$

for all $X \in f(R)$ and $y \in R$. If $a \notin C$, $b \notin C$ and $c \notin C$, then (3.1) *is a non-trivial GPI for R.*

Proof. Let $a \notin C$, $b \notin C$ and $c \notin C$. By Fact 2.1, U satisfies (3.1). On contrary, we assume that (3.1) is a trivial GPI for U. Let $T = U *_C C\{r_1, \ldots, r_n, y\}$, the free product of U and $C\{r_1, \ldots, r_n, y\}$, the free C-algebra in noncommuting indeterminates $r_1, \ldots, r_n, y$. Let $f(r_1, \ldots, r_n) = X$. Then

$$[a'X^2 + a''XcX + aXp'X - 2aXbXc - aX^2c', y] = 0 \in T. \tag{3.2}$$

From above

$$y\left\{a'X^2 + a''XcX + aXp'X - 2aXbXc - aX^2c'\right\} = 0 \in T. \tag{3.3}$$

This implies that $\{c, c', 1\}$ is linearly C-dependent, other wise $aXc' = 0$ implying $a = 0$ or $c' = 0$, a contradiction. Then there exists $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 c + \alpha_2 c' + \alpha_3.1 = 0$. If $\alpha_2 = 0$, then $\alpha_1 \neq 0$, because of the C-independency of $\{c, c', 1\}$. So this fact implies that $c' = 0$, a contradiction. Thus $\alpha_2 \neq 0$ and hence $c' = \alpha + \beta c$, where $\alpha = -\alpha_2^{-1}\alpha_3, \beta = -\alpha_2^{-1}\alpha_1$. Then U satisfies

$$y\left\{a'X^2 + a''XcX + aXp'X - 2aXbXc - aX^2(\alpha + \beta c)\right\} = 0. \tag{3.4}$$

Since $c \notin C$, this implies that

$$y\left\{-2aXbXc - \beta aX^2c\right\} = 0 \tag{3.5}$$

that is, $y\left\{(2aXb + \beta aX)Xc\right\} = 0$. Again, since $b \notin C$, U satisfies $2yaXbXc = 0$, implying either $a = 0$ or $b = 0$ or $c = 0$, a contradiction. ■

Lemma 3.2. *Let R be a prime ring with char $(R) \neq 2$ and $a, a', b, c, c', p' \in R$ such that*

$$\left[a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2c', y\right] = 0 \tag{3.6}$$

for all $X \in f(R)$ and $y \in R$. If $b \notin C$ and $c \notin C$, then (3.6) *is a non-trivial generalized polynomial identity for R.*

Proof. Let $b \notin C$ and $c \notin C$. By Fact 2.1, U satisfies (3.6). On contrary, we assume that (3.6) is a trivial GPI for U. Let $T = U *_C C\{r_1, \ldots, r_n, y\}$, the free product of U and $C\{r_1, \ldots, r_n, y\}$. Let $f(r_1, \ldots, r_n) = X$. Then

$$[a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2c', y] = 0 \in T. \tag{3.7}$$

From above

$$y\left\{a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2c'\right\} \tag{3.8}$$

is zero element in T. This implies that $\{c, c', 1\}$ is linearly C-dependent. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 c + \alpha_2 c' + \alpha_3.1 = 0$. If $\alpha_2 = 0$, then $c \in C$, a contradiction. Thus $\alpha_2 \neq 0$ and hence $c' = \alpha + \beta c$, where $\alpha = -\alpha_2^{-1}\alpha_3$ and $\beta = -\alpha_2^{-1}\alpha_1$. Then U satisfies

$$y\left\{a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2(\alpha + \beta c)\right\} = 0. \tag{3.9}$$

Since $c \notin C$, this implies that

$$y\left\{-2XbXc - \beta X^2c\right\} = 0 \tag{3.10}$$

that is, $y\left\{(2Xb + \beta X)Xc\right\} = 0$. Again, since $b \notin C$, U satisfies $2yXbXc = 0$, implying either $b = 0$ or $c = 0$, a contradiction. ■

Lemma 3.3. *Let C be a field, m be a positive integer with $m \geq 2$ and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C with char $R \neq 2$. If $a, a', a'', b, c, c', p' \in R$ such that R satisfies*

$$[a'X^2 + a''XcX + aXp'X - 2aXbXc - aX^2c', y] = 0 \tag{3.11}$$

for all $X \in f(R)$ and $y \in R$, then either $a \in C.I_m$ or $b \in C.I_m$ or $c \in C.I_m$.

Proof. We consider the following two cases.

Case 1. *When C is infinite field.*
On contrary, we assume that $a \notin C.I_m$, $b \notin C.I_m$ and $c \notin C.I_m$. We denote by e_{hl} the usual matrix unit, that is, 1 in (h, l)-entry and zero elsewhere.

By Fact 2.5, there exists an invertible matrix N such that all the entries of the matrices NaN^{-1}, NbN^{-1} and NcN^{-1} are nonzero. Let $\phi(x) = NxN^{-1}$, an inner automorphism on R. Then by hypothesis, for all $X \in f(R)$,

$$\begin{aligned} &[\phi(a')X^2 + \phi(a'')X\phi(c)X + \phi(a)X\phi(p')X \\ &-2\phi(a)X\phi(b)X\phi(c) - \phi(a)X^2\phi(c'), y] = 0. \end{aligned} \tag{3.12}$$

By [24], since $f(r_1, \ldots, r_n)$ is not central valued, there exist matrices $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, with $i \neq j$, where $\gamma \in C - \{0\}$. Thus we can substitute the value of X as e_{ij} in (3.12) and then we have $[\phi(a'')e_{ij}\phi(c)e_{ij} + \phi(a)e_{ij}\phi(p')e_{ij} - 2\phi(a)e_{ij}\phi(b)e_{ij}\phi(c), e_{ij}] = 0$. Left multiplying by e_{ij} yields

$$2e_{ij}\phi(a)e_{ij}\phi(b)e_{ij}\phi(c)e_{ij} = 0,$$

which is a contradiction, since all the entries of the matrices $\phi(a)$, $\phi(b)$ and $\phi(c)$ are non-zero.

Case 2. *When C is finite field.*
Let E be an infinite field such that $C \subseteq E$, that is, E is an extension of C. Let $\overline{R} = M_m(E) \cong R \otimes_C E$. Note that the multilinear polynomial $f(r_1, \ldots, r_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} &\Psi(r_1, \ldots, r_{n-1}, y) \\ &= [a'f(r_1, \ldots, r_n)^2 + a''f(r_1, \ldots, r_n)cf(r_1, \ldots, r_n) \\ &+ af(r_1, \ldots, r_n)p'f(r_1, \ldots, r_n) - 2af(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c \\ &- af(r_1, \ldots, r_n)^2c', y]. \end{aligned} \tag{3.13}$$

Then $\Psi(r_1, \ldots, r_{n-1}, y) = 0$ is a GPI for R.

Notice that $\Psi(r_1, \ldots, r_{n-1}, y)$ is a polynomial of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$ and degree 1 in the indeterminate y.

Now linearizing the identity $\Psi(r_1,\ldots,r_{n-1},y)=0$ with respect to variable r_1 (i.e., replacing r_1 with r_1+s_1), we get a polynomial identity for R

$$\Psi_1(r_1,\ldots,r_{n-1},s_1,y)=0$$

such that $\Psi_1(r_1,\ldots,r_{n-1},r_1,y) = 2\Psi(r_1,\ldots,r_{n-1},y)$. Continuing the process of linearization, we get a multilinear generalized polynomial identity of $2n+1$ indeterminates

$$\Psi_n(r_1,\ldots,r_n,s_1,\ldots,s_n,y)=0$$

such that

$$\Psi_n(r_1,\ldots,r_n,r_1,\ldots,r_n,y)=2^n\Psi(r_1,\ldots,r_n,y).$$

Since $\Psi_n(r_1,\ldots,r_n,s_1,\ldots,s_n,y)$ is the multilinear polynomial, we can write that

$$\Psi_n(r_1,\ldots,r_n,s_1,\ldots,s_n,y)=0$$

is a GPI for R and $\overline{R}$ too. Since $char(C)\neq 2$ we have $\Psi(r_1,\ldots,r_n,y)=0$ for all $r_1,\ldots,r_n,y\in\overline{R}$ and thus the conclusion follows by case-1 as above. ■

As a particular case of above Lemma 3.3, we have the following corollary.

Corollary 3.4. *Let C be a field and m be a fixed positive integer with $m\geq 2$. Let $R=M_m(C)$ be the ring of all $m\times m$ matrices over C. If for some a,a', $a'',b,c,c',p'\in R$ such that*

$$\big[a'r^2+a''rcr+arp'r-2arbrc-ar^2c',y\big]=0$$

for all $r,y\in R$, then either $a\in C.I_m$ or $b\in C.I_m$ or $c\in C.I_m$.

Lemma 3.5. *Let R be a prime ring, $f(r_1,\ldots,r_n)$ a non-central multilinear polynomial over C and $a,a',a'',b,c,c',p'\in R$. If char $(R)\neq 2$ and*

$$\big[a'X^2+a''XcX+aXp'X-2aXbXc-aX^2c',y\big]=0 \tag{3.14}$$

for all $X\in f(R)$ and $y\in R$, then either $a\in C$ or $b\in C$ or $c\in C$.

Proof. By hypothesis and Fact 2.1,

$$\big[a'X^2+a''XcX+aXp'X-2aXbXc-aX^2c',y\big]=0 \tag{3.15}$$

for all $X\in f(U)$ and $y\in R$. By Lemma 3.1, above identity is a non-trivial GPI. Then by Fact 2.6, R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Let $\dim_C V=m$. By density of R, then $R\cong M_m(C)$. Given that $f(r_1,\ldots,r_n)$ is not central valued on R and therefore, R must be noncommutative. Hence $m\geq 2$. In this case, by Lemma 3.3, we get our conclusions.

Let $\dim_C V = \infty$. Since the set $f(R)$ is dense on R (Lemma 2 in [28]), from above, R satisfies

$$\big[a'r^2 + a''rcr + arp'r - 2arbrc - ar^2c', y\big] = 0. \tag{3.16}$$

In this case we want to prove that either $a \in C$ or $b \in C$ or $c \in C$. We know the fact that for any element $q \in R$, $[q, Soc(RC)] = (0)$ implies $q \in C$. Hence on contrary, we assume that $a \notin C$, $b \notin C$ and $c \notin C$. Hence, there exist $h_0, h_1, h_2 \in Soc(R)$ such that $[a, h_0] \neq 0$, $[b, h_1] \neq 0$ and $[c, h_2] \neq 0$. Now we show a number of contradiction. Since $\dim_C V = \infty$, for any idempotent $e \in Soc(R)$, we have $eRe \cong M_k(C)$ with $k = dim_C Ve$. By Litoff's theorem [17], there exists an idempotent $e \in Soc(R)$ such that $h_0, h_1, h_2, h_0a, ah_0, h_1b, bh_1, h_2c, ch_2 \in eRe$, where $eRe \cong M_k(C)$, $k = dim_C Ve$. Since R satisfies

$$e\big[a'(ere)^2 + a''erecere + aerep'ere - 2aereberec - a(ere)^2c', eye\big]e = 0, \tag{3.17}$$

the subring eRe satisfies

$$\big[ea'er^2 + ea''erecer + eaerep'er - 2eaereberece - eaer^2ec'e, y\big] = 0. \tag{3.18}$$

Then by Corollary 3.4, $eae \in eC$ or $ebe \in eC$ or $ece \in eC$. If $eae \in eC$, then

$$ah_0 = eah_0 = eaeh_0 = h_0eae = h_0ae = h_0a.$$

If $ebe \in eC$, then

$$bh_1 = ebh_1 = ebeh_1 = h_1ebe = h_1be = h_1b$$

and if $ece \in eC$, then

$$ch_2 = ech_2 = eceh_2 = h_2ece = h_2ce = h_2c.$$

In any case, we have contradiction with the choices of h_0, h_1 and h_2.

Thus we conclude that either $a \in C$ or $b \in C$ or $c \in C$. ■

Lemma 3.6. *Let R be a prime ring, $f(r_1, \ldots, r_n)$ a non-central multilinear polynomial over C and char $(R) \neq 2$, where C is the extended centroid of R. If a, a', $b, c, c', p' \in R$ such that R satisfies*

$$[a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2c', y] = 0 \tag{3.19}$$

for all $X \in f(R)$ and $y \in R$, then either $b \in C$ or $c \in C$.

Proof. By hypothesis

$$[a'X^2 + 2bXcX + Xp'X - 2XbXc - X^2c', y] = 0 \tag{3.20}$$

for all $X \in f(R)$ and $y \in R$. If this is a trivial GPI for R, then by Lemma 3.2, either $b \in C$ or $c \in C$. Now assume that (3.20) is a nontrivial GPI for R. Then by Fact 2.6, R is isomorphic to a dense ring of linear transformations of a vector space V over C. If $\dim_C V = m$, then $R \cong M_m(C)$ and $M_m(C)$ satisfies (3.20). By [24], since $f(r_1, \ldots, r_n)$ is not central valued, there exist matrices $r_1, \ldots, r_n \in M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, with $i \neq j$. Thus we can substitute a particular value of X with e_{ij} in (3.20), and then we have $[2be_{ij}ce_{ij} + e_{ij}p'e_{ij} - 2e_{ij}be_{ij}c, e_{ij}] = 0$. This implies $-4b_{ji}c_{ji} = 0$. Then by same argument as given in Lemma 3.3, either $b \in C$ or $c \in C$.

If $\dim_C V = \infty$, we have for any idempotent $e \in Soc(R)$, $eRe \cong M_k(C)$, with $k = dim_C Ve$. Let $b \notin C$ and $c \notin C$. Then there exist $h_1, h_2 \in Soc(R)$ such that $[b, h_1] \neq 0$ and $[c, h_2] \neq 0$ for some $h_1, h_2 \in Soc(R)$. By Litoff's theorem [17] there exists an idempotent $e \in Soc(R)$ such that $h_1, h_2, h_1b, bh_1, h_2c, ch_2$ are all in eRe. Moreover, if $k = dim_C Ve$, then $eRe \cong M_k(C)$. Since V is infinite dimensional over C, the set $f(R)$ is dense on R ([28, Lemma 2]) and hence by hypothesis, R satisfies the GPI

$$\left[a'x^2 + 2bxcx + xp'x - 2xbxc - x^2c', y\right] = 0.$$

Now replacing x with exe and y with eye we have

$$e[a'(exe)^2 + 2b(exe)c(exe) + (exe)p'(exe) - 2(exe)b(exe)c - (exe)^2c', (eye)\big]e = 0.$$

Thus the subring eRe satisfies the GPI

$$\left[(ea'e)x^2 + 2(ebe)x(ece)x + x(ep'e)x - 2x(ebe)x(ece) - x^2(ec'e), y\right] = 0.$$

As above of finite dimensional case, we have either $ebe \in eC$ or $ece \in eC$. If $ebe \in eC$, then

$$bh_1 = ebh_1 = ebeh_1 = h_1ebe = h_1be = h_1b,$$

a contradiction and if $ece \in eC$, then

$$ch_2 = ech_2 = eceh_2 = h_2ece = h_2ce = h_2c,$$

a contradiction. Therefore, we conclude that either $b \in C$ or $c \in C$. ■

Proposition 3.7. *Let R be a noncommutative prime ring, $f(r_1, \ldots, r_n)$ be a multilinear polynomial over C, which is not central valued on R, where C is the* *extended centroid of R. Suppose char $(R) \neq 2$ and I a nonzero two sided ideal of R. If for some $b, c, p, q \in U$, $\mathscr{F}(x) = bx + xc$, $\mathscr{G}(x) = px + xq$ for all $x \in R$ are two inner generalized derivations of R such that*

$$a[(\mathscr{F}^2 + \mathscr{G})(f(r)), f(r)] \in C$$

holds for all $r = (r_1, \ldots, r_n) \in I^n$, then one of the following holds:

(1) *$\mathscr{F}(x) = x(b+c)$ and $\mathscr{G}(x) = x(p+q)$ for all $x \in R$ with $(b+c)^2 + p + q \in C$;*

(2) *$\mathscr{F}(x) = (b+c)x$, $\mathscr{G}(x) = (p+q)x$ for all $x \in R$ with $(b+c)^2 + p + q \in C$;*

(3) *$f(x_1, \ldots, x_n)^2$ is central valued and one of the following holds:*

   (a) *$\mathscr{F}(x) = x(b+c)$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$, with $(b+c)^2 - (p-q) \in C$;*

   (b) *$\mathscr{F}(x) = (b+c)x$ and $\mathscr{G}(x) = px + xq$ and for all $x \in R$ with $(b+c)^2 + p - q \in C$;*

(4) *R satisfies s_4 and one of the following holds:*

   (a) *$\mathscr{F}(x) = x(b+c)$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$, with $(b+c)^2 - (p-q) \in C$;*

   (b) *$\mathscr{F}(x) = (b+c)x$ and $\mathscr{G}(x) = px + xq$ for all $x \in R$ with $(b+c)^2 + p - q \in C$.*

Proof. Since I, R and U satisfy same GPIs (see [7]), by hypothesis we have

$$a\big[(b^2 + p)X + 2bXc + X(c^2 + q), X\big] \in C \tag{3.21}$$

for all $X \in f(U)$.

We re-write it as

$$\begin{aligned} & a(b^2 + p)X^2 + 2abXcX + aX(c^2 + q - b^2 - p)X \\ & -2aXbXc - aX^2(c^2 + q) \in C \end{aligned} \tag{3.22}$$

for all $X \in f(U)$. By Lemma 3.5, either $a \in C$ or $b \in C$ or $c \in C$.

If $0 \neq a \in C$, then (3.22) reduces to

$$\begin{aligned} & \big(b^2 + p)X^2 + 2bXcX + X(c^2 + q - b^2 - p\big)X \\ & -2XbXc - X^2(c^2 + q) \in C \end{aligned} \tag{3.23}$$

for all $X \in f(U)$. In this case by Lemma 3.6, either $b \in C$ or $c \in C$.

Thus we have proved that either $b \in C$ or $c \in C$. Therefore, we examine these two situation in the below mentioned cases.

Case 1. $b \in C$. Equation (3.21) reduces to

$$a[(b^2+p)X + X(2bc+c^2+q), X] \in C \tag{3.24}$$

for all $X \in f(U)$.

By [13, Corollary 2.7], one of the following holds:

(1) $f(R)^2 \in C$ and $(b^2+p) - (2bc+c^2+q) \in C$, i.e., $p - q - (b+c)^2 \in C$. Therefore, form of the map will be $\mathscr{F}(x) = x(b+c)$ for all $x \in R$, which gives our conclusion (1).

(2) $b^2+p, 2bc+c^2+q \in C$. Since $b \in C$, we have $p \in C$. Therefore, $\mathscr{F}(x) = x(b+c)$ and $\mathscr{G}(x) = x(p+q)$ for all $x \in R$ with $(b+c)^2 + p + q \in C$.

(3) R satisfies s_4 and $(b^2+p) - (2bc+c^2+q) \in C$ i.e., $p - q - (b+c)^2 \in C$. In this case $\mathscr{F}(x) = x(b+c)$ for all $x \in R$, which gives our conclusion (3).

Case 2. $c \in C$. In this case by (3.21),

$$a[((b+c)^2+p)X + Xq, X] \in C \tag{3.25}$$

for all $X \in f(U)$. By [13, Corollary 2.7], one of the following holds:

(1) $f(R)^2 \in C$ and $(b+c)^2 + p - q \in C$. Hence form of the map will be $\mathscr{F}(x) = (b+c)x$ for all $x \in R$, which gives our conclusion (2).

(2) $(b+c)^2 + p), q \in C$. Thus $\mathscr{F}(x) = (b+c)x$, $\mathscr{G}(x) = (p+q)x$ for all $x \in R$ with $(b+c)^2 + p + q \in C$.

(3) R satisfies s_4 and $(b^2+p+2bc) - (c^2+q) \in C$, i.e., $(b+c)^2 + p - q \in C$. In this case $\mathscr{F}(x) = (b+c)x$ for all $x \in R$, and thus conclusion (4) follows. ∎

4. PROOF OF THEOREM 1.1.

In all that follows, let R be a prime ring, $f(r_1, \ldots, r_n)$ a noncentral multilinear polynomial over C, char $(R) \neq 2$, where C is the extended centroid of R and U is the Utumi ring of quotients of R. By [21, Theorem 3], $\mathscr{F}(x) = bx + d(x)$, $\mathscr{G}(x) = cx + \delta(x)$ for some $b, c \in U$ and d, δ are two derivations of U. Then $\mathscr{F}^2(x) = \mathscr{F}(\mathscr{F}(x)) = \mathscr{F}(b)x + 2bd(x) + d^2(x)$.

By hypothesis, we have

$$a\big[\mathscr{F}(b)f(r) + 2bd(f(r)) + d^2(f(r)) + cf(r) + \delta(f(r)), f(r)\big] \in C$$

for all $r = (r_1, \ldots, r_n) \in I^n$. By Fact 2.1 and Fact 2.2, we have

$$a\big[\mathscr{F}(b)f(r) + 2bd(f(r)) + d^2(f(r)) + cf(r) + \delta(f(r)), f(r)\big] \in C \tag{4.1}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

If d and δ both are inner, by Proposition 3.7, conclusion follows. Thus we need to consider the cases when d and δ are not simultaneously inner. Thus the following three cases may occur.

Case 1. *d is inner, δ is outer.*
Assume for some $p \in U$, $d(x) = [p, x]$ for all $x \in R$. By (4.1), U satisfies

$$a\Big[\mathscr{F}(b)f(r) + 2b[p, f(r)] + [p, [p, f(r)]] + cf(r) + \delta(f(r)), f(r)\Big] \in C. \tag{4.2}$$

By Fact 2.4, we can replace $\delta(r_i)$ by t_i for $i = 1, \ldots, n$ in (4.2) and then U satisfies blended component

$$a\left[\sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)\right] \in C. \tag{4.3}$$

Replacing y_i by $[q', r_i]$ for some $q' \notin C$, we have that

$$a\Big[\,[q', f(r_1, \ldots, r_n)], f(r_1, \ldots, r_n)\Big] \in C$$

for all $r_1, \ldots, r_n \in U$. Then by [13, Corollary 2.7], $q' \in C$, a contradiction.

Case 2. *δ is inner, d is outer.* Assume for some $q \in U$, $\delta(x) = [q, x]$ for all $x \in R$. By (4.1), for all $r = (r_1, \ldots, r_n) \in U^n$,

$$a\Big[\mathscr{F}(b)f(r) + 2bd(f(r)) + d^2(f(r)) + cf(r) + [q, f(r)], f(r)\Big] \in C. \tag{4.4}$$

Since d is outer, by Fact 2.4, we can replace $d(r_i)$ by y_i for $i = 1, \ldots, n$ and $d^2(r_i)$ by t_i for $i = 1, \ldots, n$ in (4.4) and then U satisfies blended component

$$a\left[\sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)\right] \in C. \tag{4.5}$$

This equation is same as (4.3) and so it leads to a contradiction as above.

Case 3. *d, δ all are outer.* Assume first that d and δ are linearly C-independent modulo inner derivations of U. Then by applying Fact 2.4, we can replace $\delta(r_i)$ by t_i for $i = 1, \ldots, n$ and $d(r_i)$ by x_i for $i = 1, \ldots, n$ in (4.1). By this substitution, we have the blended component

$$a\left[\sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)\right] \in C \tag{4.6}$$

satisfied by U. This equation is same as (4.3). Thus by same argument we arrive to a contradiction.

Assume next that d and δ are linearly C-dependent modulo inner derivations of U. Then there exist some $\alpha_1, \beta_1 \in C$ and $q' \in U$ such that $\alpha_1 d + \beta_1 \delta = ad'_q$. Since d is outer, $\beta_1 \neq 0$ and hence $\delta(x) = \lambda d(x) + [q, x]$, where $\lambda = -\alpha_1 \beta_1^{-1}$ and $q = \beta_1^{-1} q'$.

From (4.1), we obtain

$$a\Big[F(b)f(r) + 2bd(f(r)) + d^2(f(r)) + cf(r) + \lambda d(f(r)) + [q, f(r)], f(r)\Big] \in C. \tag{4.7}$$

Again by applying Fact 2.4, we can replace $d(r_i)$ by y_i for $i = 1, \ldots, n$ and $d^2(r_i)$ by t_i for $i = 1, \ldots, n$ in (4.7) and then U satisfies blended components

$$a\Big[\sum_i f(r_1, \ldots, t_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big] \in C. \tag{4.8}$$

This equation is same as (4.3) and hence we have contradiction as before.

5. PROOF OF THEOREM 1.2.

In all that follows, we assume that R is a prime ring with char $(R) \neq 2$, U the Utumi ring of quotients of R and $C = Z(U)$ the extended centroid of R. By [21, Theorem 3], $\mathscr{F}(x) = bx + d(x)$, $\mathscr{G}(x) = cx + \delta(x)$ for some $b, c \in U$ and d, δ are derivations of U.

If char $(R) = 2$ and R satisfies s_4, then we have our conclusion (5).

Thus we assume that either char $(R) \neq 2$ or R does not satisfy s_4. Then by [14, Remark 1], there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence by hypothesis, we have

$$a\Big[\mathscr{F}(b)[s,t] + 2bd([s,t]) + d^2([s,t]) + c[s,t] + \delta([s,t]), [s,t]\Big] \in C$$

for all $s, t \in I$. If char $(R) \neq 2$, then by Theorem 1.1, we have our conclusions.

Thus we assume that char $(R) = 2$. Then R can not satisfy s_4. By Fact 2.1 and Fact 2.2, we have

$$a\Big[\mathscr{F}(b)[s,t] + d^2([s,t]) + c[s,t] + \delta([s,t]), [s,t]\Big] \in C \tag{5.1}$$

for all $s, t \in U$.

Moreover, if $d(x) = [p, x]$ and $\delta(x) = [q, x]$ are all inner derivations, then from above

$$a\Big[\mathscr{F}(b)[s,t] + [p^2, [s,t]] + c[s,t] + [q, [s,t]], [s,t]\Big] \in C$$

that is

$$a\Big[(\mathscr{F}(b)+p^2+c+q)[s,t]-[s,t](p^2+q),[s,t]\Big]\in C$$

for all $s,t\in U$. This can be written as

$$a\Big[\mathscr{F}(b)+p^2+c+q,[s,t]\Big][s,t]-[s,t]\Big[p^2+q,[s,t]\Big]\in C$$

for all $s,t\in U$.

By [14, Theorem 2.7], $\mathscr{F}(b)+p^2+c+q\in C$ and $p^2+q\in C$, i.e., $\mathscr{F}(b)+c, p^2+q\in C$.

Thus the following three cases may occur.

Case 1. *d are inner, δ is outer.*
Let for some $p\in U$, $d(x)=[p,x]$ for all $x\in R$. By (5.1), U satisfies

$$a\Big[\mathscr{F}(b)[s,t]+[p^2,[s,t]]+c[s,t]+\delta([s,t]),[s,t]\Big]\in C. \tag{5.2}$$

By Fact 2.4, we can replace $\delta([s,t])$ by $[x,t]+[s,y]$ in (5.2) and then U satisfies blended component

$$a\Big[[x,t]+[s,y],[s,t]\Big]\in C. \tag{5.3}$$

Replacing x by $[q,s]$ and y by $[q,t]$ for some $q\notin C$, we have that

$$a\Big[[q,[s,t]],[s,t]\Big]\in C$$

for all $s,t\in U$. Then by [14, Theorem 2.7], $q\in C$, a contradiction.

Case 2. *δ is inner, d is outer.*
Let for some $q\in U$, $\delta(x)=[q,x]$ for all $x\in R$. By (5.1), U satisfies

$$a\Big[\mathscr{F}(b)[s,t]+[d^2(s),t]+[s,d^2(t)]+c[s,t]+[q,[s,t]],[s,t]\Big]\in C. \tag{5.4}$$

Since d is outer, by Fact 2.4, we can replace $d^2(s)$ by x and $d^2(t)$ by y and then U satisfies blended component

$$a\Big[[x,t]+[s,y],[s,t]\Big]\in C. \tag{5.5}$$

This is same as (5.3) and hence a contradiction follows.

Case 3. *d,δ all are outer.*
Assume first that, d and δ are linearly C-independent modulo inner derivations

of U. Then by Fact 2.4, we can replace $d^2([s,t])$ by $[x,t]+[s,y]$ and $\delta([s,t])$ by $[u,t]+[s,v]$ in (5.1) and then U satisfies blended components

$$a\Big[[x,t]+[s,y],[s,t]\Big]\in C. \tag{5.6}$$

This is same equation as (5.3) and hence it leads to a contradiction as above.

Assume next that, d and δ are linearly C-dependent modulo inner derivations of U. Then there exist $\alpha',\beta'\in C$, $q'\in U$ such that $\alpha' d+\beta'\delta = ad'_q$. Since d is outer, $\beta'\neq 0$ and hence, we can write $\delta(x)=\lambda d(x)+[q,x]$, where $\lambda=-\alpha'\beta'^{-1}$ and $q=\beta'^{-1}q'$.

From (5.1), we obtain

$$a\Big[\mathscr{F}(b)[s,t]+d^2([s,t])+c[s,t]+\lambda d([s,t])+[q,[s,t]],[s,t]\Big]\in C. \tag{5.7}$$

By Fact 2.4, we can replace $d([s,t])$ by $[u,t]+[s,v]$ and $d^2([s,t])$ by $[x,t]+[s,y]$ in (5.7) and then U satisfies blended components

$$a\Big[[x,t]+[s,y],[s,t]\Big]\in C. \tag{5.8}$$

This is same equation as (5.3) and then by same argument we have a contradiction. Thus the Theorem is proved. □

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