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m-ORDERED SEMIGROUPS

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Abstract

In this paper we define and study m-ordered semigroups. In particular, idempotents and subsemigroups of m-ordered semigroups are studied and is established a characterization of inverse semigroups that, under natural order, are m-ordered semigroups.

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1. Introduction

An ordered semigroup is a semigroup together with a compatible order. For standard notation in semigroup theory, we refer the reader to [4] and [1]. In particular, for a semigroup S, $E(S)$ denotes the set of idempotents of S and $V(x)$ denotes the set of inverses of an element $x \in S$. In an ordered semigroup S, for $x \in S$, $x^{\uparrow}$ denotes the set $\{y \in S : x \leq y\}$. Recall that the natural order on a regular semigroup S, represented by $\leq_n$, is defined by, for all $x, y \in S$,

$$x \leq_n y \Leftrightarrow x = ey = yf \text{ for some } e, f \in E(S).$$

Restricted to the set of idempotents $E(S)$, it simplifies to

$$\forall e, f \in E(S), \quad e \leq_n f \Leftrightarrow e = ef = fe.$$

In the case of an inverse semigroup S, we have that, for all $x, y \in S$,

$$\begin{aligned} x \leq_n y &\Leftrightarrow x = ey \quad \text{for some } e \in E(S) \\ &\Leftrightarrow x = yf \quad \text{for some } f \in E(S). \end{aligned}$$

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In [2], the study is dedicated to the class of regular ordered semigroups S such that, for every $x \in S$, there exists $x^* = \max\{y \in S : xyx \leq x\}$. These semigroups are called principally ordered semigroups. Motivated by this study, we consider the class of regular semigroups S such that, for every $x \in S$, there exists $\min\{y \in S : xyx \leq x\}$.

Definition. An m-ordered semigroup is a regular ordered semigroup S such that, for every $x \in S$, there exists

$$x^+ = \min\{y \in S : xyx \leq x\}.$$

In what follows, S is, unless otherwise is specified, an m-ordered semigroup.

2. Basic properties

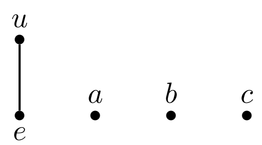
We begin our study of m-ordered semigroups by establishing some basic properties for the operation $x \mapsto x^+$. Considering the definition of this operation, it is immediate the following result.

Theorem 1. *Let S be an m-ordered semigroup. Then, for every $x \in S$,*

$$xx^+x \leq x.$$

In [2], they have proved that in a principally ordered semigroup T, for $x \in T$, $x = xx^*x$ and therefore $x^\circ = x^*xx^*$ is the largest inverse of x. However, in an m-ordered semigroup S, it is not necessarily true that, for every $x \in S$, $x = xx^+x$, as the following example shows.

Example 2. Let $S = (\{e, a, b, c, u\}, \cdot)$ be the Klein 4-group $(\{e, a, b, c\}, \cdot)$ with an additional identity u adjoined. Consider the order defined in S by the following Hasse diagram



Simple calculations prove that $x^+ = x$ for $x \in \{e, a, b, c\}$, and $u^+ = e$. Therefore S is an m-ordered semigroup. Moreover, $uu^+u = e \neq u$.

About the operation $x \mapsto x^+$ defined in an m-ordered semigroup S, it is interesting to observe that, for every $x \in S$, x^+ is a minimal element of S.

Theorem 3. *Let S be an m-ordered semigroup. If $y \leq x^+$, then $y = x^+$ for every $x, y \in S$.*

Proof. If $y \leq x^+$, we have $xyx \leq xx^+x \leq x$. Thus $x^+ \leq y$ and so $y = x^+$. ∎

Let $x \in S$. Since $xx^+xx^+x \le xx^+x \le x$, $x^+ \le x^+xx^+$. For $x \in S$, we denote the element x^+xx^+ by $x^\diamond$,

$$x^\diamond = x^+xx^+.$$

Hence, the following result holds.

Theorem 4. *Let S be an m-ordered semigroup. Then $x^+ \le x^\diamond$ for every $x \in S$.*

Considering Theorems 1 and 4, we can now verify that x is an associate of $x^\diamond$.

Theorem 5. *Let S be an m-ordered semigroup. Then $x^\diamond = x^\diamond x x^\diamond$ for every $x \in S$.*

Proof. Applying Theorem 4, we have $x^\diamond = x^+xx^+ \le x^\diamond x x^\diamond$ and, by Theorem 1,

$$x^\diamond x x^\diamond = x^+xx^+xx^+xx^+ \le x^+xx^+ = x^\diamond.$$

Thus, $x^\diamond = x^\diamond x x^\diamond$. ∎

In the subsequent theorems, we present a set of properties that follows from the preceding results.

Theorem 6. *Let S be an m-ordered semigroup. Then, for every $x \in S$.*

(i) $x^+x = x^\diamond x$ *is idempotent.*

(ii) $xx^+ = xx^\diamond$ *is idempotent.*

(iii) $x^+ = x^+x^{++}x^+ = (xx^+x)^+$.

(iv) $x^{\diamond +} = x^{++}$.

(v) $x^{++} \le x$.

(vi) $x^{+++} = x^+$.

(vii) $x^{+\diamond} = x^{++}$.

(viii) $x^{++} \le x^{\diamond\diamond}$.

(ix) $x^{\diamond\diamond} \le x$.

(x) $x^\diamond = x^{\diamond\diamond\diamond}$.

(xi) $x^+ \le x'$ *for all* $x' \in V(x)$.

(xii) x^{++} *is the least inverse of* x^+.

Proof. (i) By Theorems 4 and 1 we have $x^+x \le x^\diamond x = x^+xx^+x \le x^+x$. Hence $x^+x = x^\diamond x$ and $x^+x = x^+xx^+x$. Then x^+x is an idempotent element.

(ii) The proof is similar to that of (i).

(iii) Considering Theorem 1, we have $x^+x^{++}x^+ \le x^+$. So, by Theorem 3, it follows that $x^+x^{++}x^+ = x^+$.

By (ii), we have $xx^+xx^+xx^+x = xx^+x$, whence $(xx^+x)^+ \leq x^+$. Consequently, by Theorem 3, we have $(xx^+x)^+ = x^+$.

(iv) By (i) and (iii), we have $x^+xx^+x^{++}x^+xx^+ = x^+xx^+$. So,

$$x^{\diamond +} = (x^+xx^+)^+ \leq x^{++}$$

and, considering Theorem 3, it follows that $x^{\diamond +} = x^{++}$.

(v) By Theorem 5, we have $x^\diamond x x^\diamond = x^\diamond$, whence $x^{\diamond +} \leq x$. Thus, by (iv), we obtain $x^{++} \leq x$.

(vi) By (v), we have $x^{+++} \leq x^+$. Hence, by Theorem 3, $x^{+++} = x^+$.

(vii) This follows immediately from (vi) and (iii), since

$$x^{+\diamond} = x^{++}x^+x^{++} = x^{++}x^{+++}x^{++} = x^{++}.$$

(viii) Using the previous properties we have

$$\begin{aligned} x^{++} &= x^{++}x^{+++}x^{++} && \text{[by (iii)]} \\ &= x^{++}x^+x^{++} && \text{[by (vi)]} \\ &= x^{\diamond +}x^+x^{\diamond +} && \text{[by (iv)]} \\ &\leq x^{\diamond +}x^\diamond x^{\diamond +} && \text{[by Theorem 4]} \\ &= x^{\diamond\diamond}. && \text{[by definition of } x^{\diamond\diamond}\text{].} \end{aligned}$$

(ix) We have

$$\begin{aligned} x^{\diamond\diamond} &= x^{\diamond +}x^\diamond x^{\diamond +} && \text{[by definition of } x^{\diamond\diamond}\text{]} \\ &= x^{++}x^\diamond x^{++} && \text{[by (iv)]} \\ &\leq xx^\diamond x && \text{[by (v)]} \\ &= xx^+xx^+x && \text{[by definition of } x^\diamond\text{]} \\ &\leq x && \text{[by definition of } x^+\text{].} \end{aligned}$$

(x) This follows from the observation that

$$\begin{aligned} x^\diamond = x^+xx^+ &= x^+x^{++}x^+xx^+x^{++}x^+ && \text{[by (iii)]} \\ &= x^+x^{++}x^\diamond x^{++}x^+ && \text{[by definition of } x^\diamond\text{]} \\ &= x^+x^{\diamond +}x^\diamond x^{\diamond +}x^+ && \text{[by (iv)]} \\ &= x^+x^{\diamond\diamond}x^+ && \text{[by definition of } x^{\diamond\diamond}\text{]} \\ &= x^{+++}x^{\diamond\diamond}x^{+++} && \text{[by (vi)]} \\ &= x^{\diamond\diamond +}x^{\diamond\diamond}x^{\diamond\diamond +} && \text{[by (iv)]} \\ &= x^{\diamond\diamond\diamond} && \text{[by definition of } x^{\diamond\diamond\diamond}\text{].} \end{aligned}$$

(xi) This is immediate since $xx'x = x$.

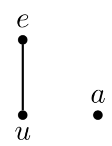
(xii) By (iii) and (vi) we have

$$x^+x^{++}x^+ = x^+ \text{ and } x^{++}x^+x^{++} = x^{++}x^{+++}x^{++} = x^{++}.$$

So, $x^{++} \in V(x^+)$ and, by (xi), x^{++} is the least inverse of x^+. ∎

The reverse of the inequality established at Theorem 4 and the reverse inequalities of Theorem 6(v), (viii), (ix) do not necessarily hold in an m-ordered semigroup. In Example 2, we have $u^{\diamond\diamond} = e \neq u$, which illustrates the last case. For the others, consider the following example.

Example 7. Let $S = (\{e, a, u\}, \cdot)$ be the group $(\{e, a\}, \cdot)$ with an additional identity u adjoined. Consider the order defined in S by the following Hasse diagram



Easily we verify that $e^+ = u$, $a^+ = a$ and $u^+ = u$ and so S is an m-ordered semigroup. Moreover, $e^\diamond = e \neq e^+$, $e^{++} = u \neq e$ and $e^{\diamond\diamond} = e \neq e^{++}$.

Let S be an m-ordered semigroup. By Theorem 3 we know that, for every $x \in S$, x^{++} is a minimal element of S and by Theorem 6(v) we also know that, for every $x \in S$, x is greater than or equal to the minimal element x^{++}. Furthermore, x^{++} is the only minimal element less than or equal to x. In fact, by the one hand, we know that if m is a minimal element of S, then $m \in S^+ = \{s^+ : s \in S\}$ (since $m^{++} \leq m$ and m is minimal, we have $m = m^{++}$). By the other hand, for every $x, y \in S$, if $y^+ \leq x$ then $x^\diamond y^+ x^\diamond \leq x^\diamond x x^\diamond = x^\diamond$ and so $(x^\diamond)^+ \leq y^+$. Thus, by Theorem 3 and Theorem 6(iv), we have $y^+ = x^{\diamond +} = x^{++}$.

Based on the previous observations, we derive the following result.

Theorem 8. *Let S be an m-ordered semigroup. Then for every $x, y \in S$,*

(i) *If $z^+ \leq x$ and $z^+ \leq y$, for some $z \in S$, then $x^+ = y^+$.*

(ii) *If $x \leq y$, then $x^+ = y^+$.*

(iii) *If $x \leq y$, then $x^\diamond \leq y^\diamond$.*

Proof. (i) If x, y are elements of S such that $z^+ \leq x$ and $z^+ \leq y$, for some $z \in S$, then

$$x^{++} = z^+ = y^{++},$$

considering the previous observation and that z^+ is a minimal element of S. From Theorem 6(vi) we get $x^+ = y^+$.

(ii) Follows from (i) and Theorem 6(v).

(iii) It follows from (ii). ∎

Theorem 9. *Let S be an m-ordered semigroup. Then*

$$S = \biguplus_{x \in S^+} (x)^\uparrow.$$

Proof. Considering Theorem 8(ii) and Theorem 6(vi), the union $\bigcup_{x\in S^+}(x)^\uparrow$ is a disjoint union. By Theorem 6(v), it is immediate that $S \subseteq \biguplus_{x\in S^+}(x)^\uparrow$. Since the converse inclusion is obvious, the equality follows. ∎

3. Idempotents in m-ordered semigroups

By Theorem 6(i),(ii) we know that the set of idempotents of an m-ordered semigroup S is not empty. Our purpose now is to study this set of elements.

Theorem 10. *Let S be an m-ordered semigroup. Then for all e, f in $E(S)$,*

(i) $e^+ \leq e$.

(ii) $e^{++} = e^+$.

(iii) $e^+ \in E(S)$.

(iv) $e^+ = f^+$.

(v) $(ef)^+ \in E(S)$.

(vi) $e^\diamond \leq e$.

(vii) $e^{\diamond\diamond} = e^\diamond$.

(viii) $(ef)^\diamond \in E(S)$.

(ix) $e^\diamond \in E(S)$.

Proof. (i) Since $e \in E(S)$, we have $eee = e$. So $e^+ \leq e$.

(ii) This is an immediate consequence of (i) and Theorem 8.

(iii) Considering (i), we have $ee^+e^+e \leq eeee = e$ and so $e^+ \leq e^+e^+$. Thus $e^+e^+ \leq e^+e^+e^+$. Moreover, by (ii) and Theorem 6(iii) we have

$$e^+e^+e^+ = e^+e^{++}e^+ = e^+.$$

So $e^+e^+ \leq e^+$ and equality $e^+e^+ = e^+$ follows.

(iv) Since $e, f \in E(S)$, $e^+, f^+ \in E(S)$. Moreover,

$$\begin{aligned} f^+(e^+f^+)^\diamond e^+e^+f^+(e^+f^+)^\diamond e^+ &= f^+(e^+f^+)^\diamond e^+f^+(e^+f^+)^\diamond e^+ \\ &= f^+(e^+f^+)^\diamond e^+. \end{aligned}$$

Thus $(f^+(e^+f^+)^\diamond e^+)^+ \leq e^+$ and therefore $(f^+(e^+f^+)^\diamond e^+)^+ = e^+$. Similarly, we have $(f^+(e^+f^+)^\diamond e^+)^+ = f^+$. So $e^+ = f^+$.

(v) By (i) we have $e^+ \leq e$ and $f^+ \leq f$. So $e^+f^+ \leq ef$ and considering (iv) it follows that $e^+e^+ \leq ef$. Now, taking into account (iii), we have $e^+ \leq ef$ and, by Theorem 8 and by (ii), we obtain $e^+ = (ef)^+$. So, $(ef)^+$ is an idempotent.

(vi) By (i), we have $e^\diamond = e^+ee^+ \leq eee = e$.

(vii) We have

$$\begin{aligned}
e^{\diamond\diamond} &= (e^{\diamond})^{+}e^{\diamond}(e^{\diamond})^{+} && \text{[by definition of } e^{\diamond\diamond}\text{]}\\
&= e^{++}e^{\diamond}e^{++} && \text{[by Theorem 6(iv)]}\\
&= e^{+}e^{\diamond}e^{+} && \text{[by (ii)]}\\
&= e^{+}e^{+}ee^{+}e^{+} && \text{[by definition of } e^{\diamond}\text{]}\\
&= e^{+}ee^{+} && \text{[by (iii)]}\\
&= e^{\diamond} && \text{[by definition of } e^{\diamond}\text{]}
\end{aligned}$$

(viii) By definition of $(ef)^{\diamond}$, (v) and Theorem 6(i), we have

$$\begin{aligned}
(ef)^{\diamond}(ef)^{\diamond} &= (ef)^{+}ef(ef)^{+}(ef)^{+}ef(ef)^{+}\\
&= (e^{f})^{+}ef(ef)^{+}ef(ef)^{+}\\
&= (ef)^{+}ef(ef)^{+}\\
&= (ef)^{\diamond}.
\end{aligned}$$

So, $(ef)^{\diamond} \in E(S)$.

(ix) This follows from (viii). ∎

Now, we observe that in an m-ordered semigroup S, it is not always the case that the equality $e^{\diamond} = f^{\diamond}$ holds for idempotents e and f. For instance, in Example 7, $u, e \in E(S)$ and $u^{\diamond} \neq e^{\diamond}$.

We also observe that, in the case of principally ordered semigroups, the corresponding conditions to Theorem 10 (ii), (iii) and (ix),

$$e^{*} \in E(S), e^{*} = e^{**}, e^{\circ} \in E(S) \qquad (e \in E(S)),$$

are not necessarily satisfied but are equivalent, as shown in [3, Theorem 2.2].

Based on the properties discussed earlier, we have the following result.

Theorem 11. *Let S be an m-ordered semigroup. Then $E(S)$ has a minimum element ω, and $\omega = e^{+}$, for every $e \in E(S)$.*

Proof. Since $E(S) \neq \emptyset$, let $e \in E(S)$. From Theorem 10(iii), we know that $e^{+} \in E(S)$ and, as a consequence of Theorem 10(iv), (i), it follows that, for all $f \in E(S)$, $e^{+} \leq f$. So, $E(S)$ has a minimum element ω, and $\omega = e^{+}$, for every $e \in E(S)$. ∎

Considering the existence of a minimum idempotent for every m-ordered semigroup, we have the following results.

Theorem 12. *Let S be an m-ordered semigroup and ω be the minimum element of $E(S)$. Then, for all $x \in S$, $x^{\diamond}\omega = x^{\diamond} = \omega x^{\diamond}$.*

Proof. Considering Theorem 6(ii) and Theorem 5, we have that, for every $x \in S$, $x^\diamond \omega \leq x^\diamond x x^\diamond = x^\diamond$ and $xx^\diamond \omega x \leq xx^\diamond xx^\diamond x = xx^\diamond x = xx^+x \leq x$, so $x^+ \leq x^\diamond \omega$. Then

$$x^\diamond = x^+ x x^+ \leq x^\diamond \omega x x^\diamond \omega \leq x^\diamond x x^\diamond x x^\diamond \omega = x^\diamond \omega.$$

Hence $x^\diamond = x^\diamond \omega$, and likewise $\omega x^\diamond = x^\diamond$. ■

We establish the following theorem, whose results are relevant to the next section.

Theorem 13. *Let S be an m-ordered semigroup and ω be the minimum element of $E(S)$. Then for all $x, y \in S$,*

(i) $(y(xy)^\diamond)^+ \leq x$.
(ii) $((xy)^\diamond x)^+ \leq y$.
(iii) $x^+ = (y(xy)^\diamond)^{++} \leq y(xy)^\diamond$.
(iv) $y^+ = ((xy)^\diamond x)^{++} \leq (xy)^\diamond x$.
(v) $(xy)^+ \leq y^+ x^+$.
(vi) $(xy)^+ = (y^+ x^+)^{++}$.
(vii) $(xx)^+ = x^+$ *if and only if* x^+ *is an idempotent.*
(viii) $xx^\diamond = xx^+ \leq xy(xy)^\diamond = xy(xy)^+$.
(ix) $y^\diamond y = y^+ y \leq (xy)^\diamond xy = (xy)^+ xy$.
(x) $x^{\diamond\diamond} = \omega x \omega$.
(xi) $(xx^\diamond)^\diamond = x^{\diamond\diamond} x^\diamond$.
(xii) $(x^\diamond x)^\diamond = x^\diamond x^{\diamond\diamond}$.

Proof. (i) We have $y(xy)^\diamond xy(xy)^\diamond = y(xy)^\diamond$ and so $(y(xy)^\diamond)^+ \leq x$.

(ii) The proof is similar to that of (i).

(iii) By (i) we have $(y(xy)^\diamond)^+ \leq x$ and by Theorem 8(ii) and Theorem 6(v) it follows that $x^+ = (y(xy)^\diamond)^{++} \leq y(xy)^\diamond$.

(iv) The proof is similar to that of (iii).

(v) From (iv) we have

$$xyy^+x^+xy \leq xy(xy)^\diamond xx^+xy \leq xy(xy)^\diamond xy \leq xy$$

and so $(xy)^+ \leq y^+x^+$.

(vi) By (v) and Theorem 8, we have $(xy)^{++} = (y^+x^+)^+$. Then by Theorem 6(vi), it follows that $(xy)^+ = (y^+x^+)^{++}$.

(vii) If $(xx)^+ = x^+$ then by (vi) and Theorem 6(vi) it follows that

$$(x^{++}x^{++})^+ = x^+.$$

Thus $(x^{++}x^{++})^{++} = x^{++}$ and so $x^{++} \leq x^{++}x^{++}$ by Theorem 6(v). Therefore

$$x^+ = x^+x^{++}x^+ \leq x^+x^{++}x^{++}x^+$$

and by Theorem (8) it follows that $x^{++} = (x^+x^{++}x^{++}x^+)^+$; so,

$$x^+ = (x^+x^{++}x^{++}x^+)^{++}.$$

Hence, since x^+x^{++} and $x^{++}x^+$ are idempotents, $(x^+x^{++}x^{++}x^+)^{++}$ is also idempotent by Theorem 10(v); thus x^+ is idempotent. Conversely, if x^+ is idempotent, by (v) we have

$$x(xx)^+x \leq xx^+x^+x = xx^+x \leq x$$

whence $x^+ \leq (xx)^+$. Thus, by Theorem 3 it follows that $x^+ = (xx)^+$.

(viii) This follows immediately from Theorem 6(ii) and by (iii).

(ix) Using Theorem 6(i) and considering (iv), the proof is similar to that of (viii).

(x) On the one hand, $\omega x\omega \leq x^{\diamond\diamond}x^\diamond xx^\diamond x^{\diamond\diamond} = x^{\diamond\diamond}x^\diamond x^{\diamond\diamond} = x^{\diamond\diamond}$. On the other hand,

$$\begin{aligned}
\omega x\omega &= (xx^+)^+x(x^+x)^+ && \text{[by Theorem 11]}\\
&\geq (xx^+)^+xx^+x(x^+x)^+ && \text{[by Theorem 1]}\\
&= (xx^\diamond)^\diamond xx^\diamond x(x^\diamond x)^\diamond && \text{[by Theorem 6(i),(ii)]}\\
&\geq (xx^\diamond)^\diamond xx^\diamond x^{\diamond\diamond} && \text{[by (viii)]}\\
&\geq x^{\diamond\diamond}x^\diamond x^{\diamond\diamond} && \text{[by (ix)]}\\
&= x^{\diamond\diamond} && \text{[by Theorem 5].}
\end{aligned}$$

Thus $x^{\diamond\diamond} = \omega x\omega$.

(xi) Considering Theorem 6(ii), Theorems 11 and 12 and by (x), we have

$$(xx^\diamond)^\diamond = (xx^\diamond)^+xx^\diamond(xx^\diamond)^+ = \omega xx^\diamond\omega = \omega x\omega x^\diamond = x^{\diamond\diamond}x^\diamond.$$

(xii) The proof is similar to (xi). ■

4. Subsemigroups of m-ordered semigroups

In this section, we focus on the subsemigroups of m-ordered semigroups. Specifically, we aim to determine whether for an m-ordered semigroup S, the sets $S^+ = \{x^+|x \in S\}$ and $S^\diamond = \{x^\diamond|x \in S\}$ are subsemigroups of S. We also explore the relationship between the class of m-ordered semigroups and the well-known class of inverse semigroups ordered by natural order. In particular, we provide a characterization of inverse semigroups that, under their natural order, are also m-ordered semigroups.

We already know that the set of idempotents of an m-ordered semigroup S is not empty. Moreover, given $e \in E(S)$, we have that, for all $f \in E(S)$, $f \in (e^+)^\uparrow$ and $(f^+)^\uparrow = (e^+)^\uparrow$ and we can state that, for every $e \in E(S)$, $(e^+)^\uparrow$ is a subsemigroup of S.

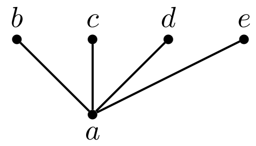
Theorem 14. *Let S be an m-ordered semigroup. For every $e \in E(S)$, $(e^+)^\uparrow$ is a subsemigroup of S.*

Proof. If $a, b \in (e^+)^\uparrow$, then $e^+ \leq a, b$. So, $e^+ = e^+e^+ \leq ab$ and therefore $ab \in (e^+)^\uparrow$. ∎

It was observed that, for every $e \in E(S)$, $E(S) \subseteq (e^+)^\uparrow$, but, as can be seen in the next example, the semigroup $(e^+)^\uparrow$ contains other elements than idempotents.

Example 15. Consider the ordered semigroup S described by the following Cayley table and Hasse diagram

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	e
c	c	c	c	c	c
d	a	d	a	e	b
e	a	e	a	b	d



It is readily seen that, for every $x \in S$, there exists $x^+ = \min\{y \in S : xyx \leq x\}$. Moreover, we have $a \in E(S)$, $a^+ = a$ and $(a^+)^\uparrow = S$ and it is evident that not all elements of $(a^+)^\uparrow$ are idempotents.

Theorem 16. *Let S be an m-ordered semigroup. If U is a subsemigroup of S such that $U \subseteq (x^+)^\uparrow$, for some $x \in S$, then $U \subseteq (e^+)^\uparrow$ for every $e \in E(S)$.*

Proof. If U is a subsemigroup of S such that $U \subseteq (x^+)^\uparrow$ then there exists some $a \in U$ such that $x^+ \leq a$ and $x^+ \leq aa$. Then, by Theorem 8, we have $x^{++} = a^+$ and $x^{++} = (aa)^+$. So $a^+ = (aa)^+$ and, by Theorem 13(vii), it follows that a^+ is an idempotent of S. Hence, by Theorem 10(iii) and Theorem 6(vi), the element $x^+ = a^{++}$ is also an idempotent of S. Thus, considering Theorem 10(iv),(ii) and Theorem 6(vi), for every $e \in E(S)$, we have $e^+ = x^+$. ∎

Despite the previous result, an m-ordered semigroup S can have other subsemigroups that are not necessarily contained in subsemigroups $(e^+)^\uparrow$, with $e \in E(S)$.

Example 17. Consider the m-ordered semigroup S presented in Example 2. We have $e \in E(S)$, $(e^+)^\uparrow = \{e, u\}$, $S^\diamond = \{e, a, b, c\}$ is a subsemigroup of S and $S^\diamond \nsubseteq (e^+)^\uparrow$.

From Example 7, we know that the subset S^+ of an m-ordered semigroup S is not necessarily a subsemigroup of S: clearly $S^+ = \{a, u\}$ is not a subsemigroup of S. So, our aim now is to establish under which conditions the subset S^+ of an m-ordered semigroup S is a subsemigroup of S.

Theorem 18. *Let S be an m-ordered semigroup. Then*

(i) *S^+ is a subsemigroup of S if and only if $x^+y^+ = (yx)^+$, for all $x, y \in S$.*

(ii) *If S^+ is a subsemigroup of S, then S^+ is a regular semigroup.*

Proof. (i) Let $x, y \in S$. If S^+ is a subsemigroup of S, we have $x^+y^+ = s^+$ for some $s \in S$. Then, by Theorem 13(v), $(yx)^+ \leq s^+$ and from Theorem 3 it follows that $s^+ = (yx)^+$. The converse implication is obvious.

(ii) This follows immediately from Theorem 6(iii). ∎

Although S^+ is not necessarily a subsemigroup of an m-ordered semigroup S, we can prove that $S^\diamond$ is always a subsemigroup of S.

Theorem 19. *Let S be an m-ordered semigroup and ω be the minimum element of $E(S)$. Then $S^\diamond = \omega S \omega$ and is an inverse submonoid of S with identity ω. Moreover, $S^\diamond$ is dually naturally ordered.*

Proof. Considering that, for all $x \in S^\diamond$, $x = x^{\diamond\diamond}$ and, by Theorem 13(x), we have $x^{\diamond\diamond} = \omega x \omega$, it follows that $S^\diamond = \omega S \omega$. Clearly, $S^\diamond$ is a subsemigroup of S. The minimum element of $E(S)$ is an element of $S^\diamond$, since $\omega^\diamond = \omega^+\omega\omega^+ = \omega\omega\omega = \omega$, and obviously is the identity of $S^\diamond$. So, it remains to prove that $S^\diamond$ is an inverse semigroup. To do that, it is sufficient to show that $S^\diamond$ is a regular semigroup and that its idempotents commute. By Theorem 5, for all $x \in S^\diamond$, $xx^\diamond x = x^{\diamond\diamond}x^\diamond x^{\diamond\diamond} = x^{\diamond\diamond} = x$; so $S^\diamond$ is a regular semigroup. Now, considering that $S^\diamond$ is a subsemigroup of S and by Theorems 13(x) and 10(viii), for all $e, f \in E(S^\diamond)$, we have $ef = (ef)^{\diamond\diamond} = \omega ef \omega \leq fefe = (fe)^{\diamond\diamond}(fe)^{\diamond\diamond} = (fe)^{\diamond\diamond} = fe$. Similarly, $fe \leq ef$. Hence the idempotents of $S^\diamond$ commute. Thus $S^\diamond$ is an inverse semigroup.

If now $e, f \in E(S^\diamond)$ with $e \leq_n f$ then $e = ef = fe$ gives $e = efe \geq \omega f \omega = f$. Thus $S^\diamond$ is dually naturally ordered. ∎

Corollary 20. *Let S be an m-ordered semigroup. Then $E(S^\diamond)$ is a subsemigroup of $S^\diamond$.*

We now characterize when a naturally ordered inverse semigroup is m-ordered.

Theorem 21. *An inverse semigroup S, under its natural order $\leq_n$, is m-ordered if and only if it has a smallest idempotent. In this case, $S^\diamond = S^+$ and is a subgroup of S.*

Proof. Suppose that S is an inverse semigroup with a smallest idempotent ω under $\leq_n$. If $x, y \in S$ and $xyx \leq_n x$, then there exists $e \in E(S)$ such that $xyx = ex$. Consequently, $x^{-1}xyx = x^{-1}ex \in E(S)$. Then $\omega \leq_n x^{-1}xyx \leq_n yx$ whence $\omega x^{-1} \leq_n yxx^{-1} \leq_n y$. Since also $x\omega x^{-1}x = xx^{-1}x\omega = x\omega \leq_n x$ it follows that x^+ exists and is ωx^{-1}. Likewise, $x^+ = x^{-1}\omega$ and so $x^+ = \omega x^{-1}\omega$. The reverse implication is clear.

Finally, if S is an inverse and an m-ordered semigroup, we have $x^\diamond = x^+xx^+ = \omega x^{-1}xx^{-1}\omega = \omega x^{-1}\omega = x^+$, for all $x \in S$. Then $S^\diamond = S^+$ and is a subgroup since is an inverse submonoid of S and, for every $e \in E(S^\diamond)$, $e = e^{\diamond\diamond} = e^\diamond = e^+ = \omega$. ■

Example 22. Let $S = \mathbb{Z} \times \mathbb{N}_0$. The algebra $\mathcal{S} = (S, *)$, where $*$ is the binary operation defined by

$$(a_1, b_1) * (a_2, b_2) = (a_1 + a_2, \min(b_1, b_2)), \text{ for all } (a_1, b_1), (a_2, b_2) \in S,$$

is an inverse semigroup. For all $(a, b) \in S$, $(-a, b)$ is the unique inverse of (a, b). Moreover, $E(S) = \{0\} \times \mathbb{N}_0$ and, under the natural order defined on S, $(0, 0)$ is the smallest idempotent. Hence, by the previous result, S is an m-ordered semigroup.

Consider now the subset

$$T = \{x \in S \mid x^\diamond \in V(x)\}$$

of an m-ordered semigroup S.

Theorem 23. *Let S be an m-ordered semigroup. Then T is an ideal of S.*

Proof. If $x \in T$, then, by Theorem 12, for every $y \in S$,

$$xy(xy)^\diamond xy = xy(xy)^+xy \leq xy$$

and

$$xy(xy)^\diamond xy = xx^\diamond xy(xy)^\diamond xy \geq xx^\diamond wxy = xx^\diamond xy = xy.$$

Consequently, by Theorem 5, $xy \in T$ and so T is a left ideal of S. Similarly, T is a right ideal. Thus T is a subsemigroup of S which is clearly regular. ■

Theorem 24. *Let S be an m-ordered semigroup. Then for all $x \in T$,*

$$x^\diamond = \min V(x).$$

Proof. Let $x \in T$ and $y \in V(x)$. Then $xyx = x$ gives $x^+ \leq y$ whence

$$x^\diamond = x^+xx^+ \leq yxy = y.$$

Hence $x^\diamond$ is the least inverse of x. ■

Theorem 25. *Let S be an m-ordered semigroup and ω be the minimum element of $E(S)$. Then $S^\diamond = \omega T\omega$ and is an inverse transversal of T.*

Proof. Clearly, $S^\diamond = \omega T \omega$ and, for every $x \in T$, we have $x^\diamond \in S^\diamond \cap V(x)$. If now $y, z \in S^\diamond \cap V(x)$, then $y = yxy = yxzxy \geq \omega z \omega = z$, and similarly $z \geq y$. Hence $y = z$ and consequently $S^\diamond \cap V(x) = \{x^\diamond\}$. ■

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