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ON INTERSECTION GRAPH OF IDEALS OF A NEAR-RING IN TERMS OF ITS ESSENTIAL IDEALS

Pavel Pal[1]

Department of Mathematics, Bankura University
Bankura-722155, India

e-mail: ju.pavel86@gmail.com

and

Jyotirmoy Jana

Department of Mathematics, Jadavpur University
Kolkata-700032, India

e-mail: jyotirmoyjana.1996@gmail.com

Abstract

Let N be a near-ring and $\mathrm{I}(N)$ denote the set of all non-trivial (i.e., non-zero proper) ideals of N. In this paper, the intersection graph of ideals of N has been introduced as a simple undirected graph denoted by $\mathrm{G}(N)$ whose vertex set is $\mathrm{I}(N)$ and two vertices I and J are adjacent if and only if $I \neq J$ and $I \cap J \neq \{0\}$. The graph $\mathrm{G}(N)$ has been studied mainly using the direct sum decomposition and the essential ideals of N. Here, some necessary and sufficient conditions for $\mathrm{G}(N)$ to be connected and complete have been obtained. Under some restrictions on N and $\mathrm{G}(N)$, some graph parameters such as chromatic number, independence number, domination number, hull number, geodetic number as well as some graphical properties of $\mathrm{G}(N)$ such as being chordal, star etc. have been obtained.

Keywords: intersection graph, near-ring, direct summand, essential ideal.

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[1]Corresponding author.

1. Introduction

One of the most important research areas of graph theory is algebraic graph theory where graphs are constructed depending on different algebraic structures. One of the main motivations to study such graphs is to find interplays between the graph theoretic properties and the properties of the algebraic structures under consideration.

Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The intersection graph $\mathrm{G}(F)$ is the graph whose vertices are $S_i, i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \phi$. Due to Marczewski [16, 17], we have the result for intersection graph which states that "Every simple graph is an intersection graph, i.e., for any simple graph G there exists a family F of sets $S_i, i \in I$, such that G is isomorphic to the intersection graph $\mathrm{G}(F)$". To make the study of the intersection graphs more interesting, Bosak [7] in 1964 defined intersection graph arising from a semigroup S. Then Csákány and Pollák [10] defined and studied the intersection graphs corresponding to finite groups. Zelinka [19] continued the work in the setting of finite Abelian groups. Later, Chakrabarty *et al.* [8] studied the intersection graph of ideals of a ring. For some other works of intersection graphs in the setting of groups and rings we may refer the readers to [1, 11, 14]. The theory of intersection graphs has a vast field of applications in different practical scenarios like secure sensor networks, wireless communication networks, epidemiology etc. (cf. [3, 5, 20]).

To continue the study of intersection graphs on different algebraic structures, in our present paper, we define the intersection graph $\mathrm{G}(N)$ on a near-ring N in terms of the non-zero proper ideals of N. The study of the intersection graphs in the setting of near-rings enriches the study of intersection graphs on various algebraic structures as the notion of near-rings indeed generalizes the notion of rings. In Section 3, $\mathrm{G}(N)$ is defined and studied mainly in terms of direct product of its subnear-rings. Among the other results, here we obtain a necessary and sufficient condition on a near-ring N so that $\mathrm{G}(N)$ becomes complete (cf. Theorem 3.4) and also we obtain that if $\mathrm{G}(N)$ is disconnected then the near-ring N is a direct product of two simple near-rings (cf. Theorem 3.10). We also obtain the degree of a maximal ideal of a near-ring N in $\mathrm{G}(N)$ (cf. Theorem 3.18). In Section 4, we obtain another necessary and sufficient condition for completeness of $\mathrm{G}(N)$ in terms of essential ideals of N (cf. Theorem 4.4). Here, we determine the domination number of $\mathrm{G}(N)$ corresponding to a near-ring N containing an essential ideal (cf. Theorem 4.7). We also discuss about the graph connectivity properties (viz. vertex connectivity, strong vertex connectivity, edge connectivity etc.) of $\mathrm{G}(N)$ (cf. Theorems 4.13, 4.29, respectively). Here, we find some conditions on N so that the corresponding graph $\mathrm{G}(N)$ becomes star, asteroidal triple-free, chordal or non-bipartite (cf. Theorems 4.14, 4.23, 4.27, Corollary 4.16, respectively). Also, we obtain the degree of vertices (cf. Theorem 4.17) and determine several graphical parameters such as chromatic number, independence number, geodetic number and hull number of a planar graph G(N) (cf. Theorems 4.22, 4.24, 4.25, 4.26, respectively).

2. PRELIMINARIES

We first recall some definitions of the theory of near-rings (cf. [13]) for their use in the sequel.

Definition 2.1. A *near-ring* is a set N together with two binary operations '+' and '·' such that

(a) $(N, +)$ is a group (not necessarily Abelian),

(b) $(N, \cdot)$ is a semi-group and

(c) for all $n_1, n_2, n_3 \in N : (n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ ("right distributive law").

N is called *zero-symmetric* if for all $n \in N, n0 = 0$. N is called an *integral* if N has no non-zero divisors of zero. N has the *DCCI* if N satisfy DCC (descending chain condition) on ideals. If $(N^* = N - \{0\}, \cdot)$ is a group, N is called a *near-field*.

Definition 2.2. Let N be a near-ring. A normal subgroup I of $(N, +)$ is called *ideal* of N if

(i) $IN \subseteq I$ and

(ii) $n(n' + i) - nn' \in I$, for all $n, n' \in N$ and for all $i \in I$.

An ideal I of a near-ring N is called a *direct-summand* of N if there exist an ideal J of N such that $N = I \bigoplus J$. J is then called a *direct complement* of I in N. A near-ring N is called *simple* if its ideals are $\{0\}$ and N only.

Definition 2.3. A subdirect product N of near-rings N_i $(i \in I)$ is called *trivial* if there exists $i \in I$ such that the projection map $\pi_i : N \to N_i$ is an isomorphism. A near-ring N is called *subdirectly irreducible* if N is not isomorphic to a non-trivial subdirect product of near-rings.

Definition 2.4. Let N be a near-ring. Then N is called *decomposable*, if it is the direct sum of non-trivial ideals (or, equivalently, it has a non-trivial direct summand), otherwise *indecomposable*.

Definition 2.5. Let N be a near-ring and $(\Gamma, +)$ be a group. Let $f : \underset{(n,\gamma)\to n\gamma}{N \times \Gamma \to \Gamma}$. Then (Γ, f) is called a N-*group* (denoted by N^Γ) if $(n + n')\gamma = n\gamma + n'\gamma$ and $(nn')\gamma = n(n'\gamma)$, for all $n, n' \in N$ and for all $\gamma \in \Gamma$. A subgroup Δ of N^Γ with $N\Delta \subseteq \Delta$ is said to be an N-*subgroup* of Γ. N is said to be *local* if N has a unique maximal N-subgroup.

Definition 2.6. Let N, N' be two near-rings. Then $h : N \to N'$ is called a *near-ring homomorphism* if $h(m+n) = h(m) + h(n)$ and $h(mn) = h(m)h(n)$, for all $m, n \in N$.

Definition 2.7. [15] A non-zero proper ideal I of a near-ring N is called an *essential ideal* of N if for any non-zero ideal J of N, $I \cap J \neq \{0\}$.

We recall the following notations of graph theory from [4, 6, 9, 12] and [18].

Definition 2.8. Let $G = (V(G), E(G))$ be a graph. A shortest path between two vertices v_1 and v_2 in G is called a $v_1 - v_2$ *geodesic*. The *distance* $d(x, y)$ between any two vertices x and y is the length of a shortest path from x to y. The *diameter* of G is $diam(G) = \max\{d(x, y) : x, y \in V(G)\}$ and the *girth* of G is the length of a smallest cycle in G. The *eccentricity* of a vertex x in G is defined as $e(x) = \max\{d(x, z) : z \in V(G)\}$. The *radius* of G is the minimum eccentricity among the vertices of G, which is denoted by $rad(G)$. A vertex $v \in V(G)$ is said to be *central* if $e(v) = rad(G)$. The set of all vertices in G lying on a $v_1 - v_2$ geodesic is denoted by $I[v_1, v_2]$. For any subset S of G, let $I[S] = \bigcup_{v_1, v_2 \in S} I[v_1, v_2]$. If $I[S] = V(G)$, then S is called a geodetic set of G. The minimum cardinality of a geodetic set of G is called the *geodetic number* $g(G)$ of G. A vertex v of a connected graph G is said to be a *cut vertex* if the deletion result of the vertex v is a disconnected graph. A vertex cut of G is a subset $S \subseteq V(G)$ such that the graph $G - S$ is disconnected. The *vertex connectivity* of G is defined by $\kappa(G) = \min\{n \geq 0 : \text{there exists a vertex cut } S \subseteq V(G) \text{ such that } |S| = n\}$. A graph whose edge set is empty is called a null graph or a totally disconnected graph. The *strong vertex connectivity* of G is defined by $K(G) = \min\{n \geq 0 :$ there exists a vertex subset $S \subseteq V(G)$ with $|S| = n$ such that $G - S$ is totally disconnected $\}$. An edge cut of G is a subset $T \subseteq E(G)$ such that the graph $G - T$, whose vertex set is $V(G)$ and edge set is $E(G) - T$, is disconnected. The *edge connectivity* of G is defined by $\lambda(G) = \min\{n \geq 0 : \text{there exists an edge cut } T \subseteq E(G) \text{ such that } |S| = n\}$.

Definition 2.9. A vertex set S of a graph G is a *dominating set* if each vertex of G either belongs to S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of S as S varies over all dominating sets of G.

Definition 2.10. An *independent vertex set* of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge in G. The *independence number* $\alpha(G)$ of G is the cardinality of the largest independent vertex set.

Definition 2.11. A subset S of $V(G)$ is *convex* if $I[S] = S$. If A is a subset of $V(G)$, then the *convex hull* of A (denoted by $[A]$) is the smallest convex set in

G containing A. If $[A] = V(G)$, then A is called a *hull set* of G. The smallest cardinality of a hull set of G is called the *hull number* of G and is denoted by $h(G)$.

Definition 2.12. Let G be a graph. Suppose C is a cycle G with more than three vertices. A *chord* of C is an edge of G, which is not an edge of C, but whose endpoints are vertices of C. G is called *chordal* if every n-cycle in G with $n > 3$ possesses a chord.

Definition 2.13. An *asteroidal triple* of a graph (with at least six vertices) is an independent set of three vertices such that for any pair of distinct vertices of the set, there is a path between the two that contains no vertex adjacent to the third.

Definition 2.14. Two graphs are *edge-disjoint* if they have no edge in common. A *decomposition* of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that $\bigcup_{F\in\mathcal{F}} E(F) = E(G)$. A separation of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. This common vertex is called a *separating vertex* of the graph G.

Definition 2.15. A circuit in a graph that includes all the edges of the graph is called an *Euler circuit*. A graph G is said to be *Eulerian* if either G is a trivial graph or G has an Euler circuit.

3. INTERPLAY BETWEEN THE INTERSECTION GRAPH OF IDEALS OF A NEAR-RING AND DIRECT PRODUCT OF ITS SUBNEAR-RINGS

Definition 3.1. Let $\mathrm{I}(N)$ be the set of all non-zero proper ideals of a near-ring N. The *intersection graph of ideals of* N, denoted by $\mathrm{G}(N)$, is a undirected simple graph (without loops and multiple edges) with vertex set $\mathrm{I}(N)$ and two vertices I and J are adjacent if and only if $I \neq J$ and $I \cap J \neq \{0\}$.

Example 3.2. Consider the near-ring $N = \{0, a, b, c\}$ together with two binary operations '+' and '$\cdot$' defined by

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

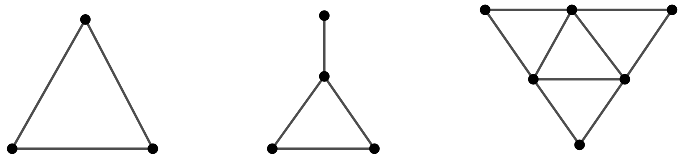
and

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	b	b	b	b
c	b	c	b	c

The non-zero proper ideals of N are $I = \{0, a\}$ and $J = \{0, b\}$. So, the intersection graph $\mathrm{G}(N)$ consists of two vertices I and J. The graph $\mathrm{G}(N)$ is:

Figure 1.

Example 3.3. $M_c(\mathbb{Z}_n)$, the set of all constant mappings from $\mathbb{Z}_n$ to $\mathbb{Z}_n$, forms a near-ring with pointwise addition and composition. The intersection graphs corresponding to $M_c(\mathbb{Z}_n)$ for $n = p^4,\ p^2q,\ pqr$, where p, q and r are distinct primes, are given below.



(a)$n = p^4$ (b)$n = p^2q$ (c)$n = pqr$

Figure 2.

Theorem 3.4. *Let N be a near-ring. Then* $\mathrm{G}(N)$ *is complete if and only if N is subdirectly irreducible.*

Proof. Let N be a subdirectly irreducible near-ring. Then $\bigcap_{I(\neq\{0\})\ \textit{ideal in } N} I \neq \{0\}$ (cf. Theorem 1.60, pp. 25 [13]). This implies that $I \cap J \neq \{0\}$, for any two non-trivial ideals I and J. Hence, $\mathrm{G}(N)$ is complete. Conversely, let $\mathrm{G}(N)$ be complete. To prove N is subdirectly irreducible, it is enough to prove that $\bigcap_{K(\neq\{0\})\ \textit{ideal in } N} K \neq \{0\}$ in view of Theorem 1.60[13]. If possible, let $\bigcap_{K(\neq\{0\})\ \textit{ideal in } N} K = \{0\}$. Then there exist at least two non-zero proper ideals $L,\ M$ such that $L \cap M = \{0\}$, which contradicts the fact that $\mathrm{G}(N)$ is complete. This proves the result. ■

Though the following results can be proved in a similar fashion to those in rings (cf. [2, 8]). We state them for their immediate use.

Theorem 3.5. *The graph* $\mathrm{G}(N)$ *of a near-ring N is disconnected if and only if N contains at least two minimal ideals which are maximal, too and every non-trivial ideal of N is minimal as well as maximal.*

Corollary 3.6. *For any* $\mathrm{G}(N)$ *of a near-ring* N*, whenever* $\mathrm{G}(N)$ *is disconnected, it is a null-graph.*

Theorem 3.7. *Let* N *be a near-ring and* $\mathrm{G}(N)$ *be a connected graph. Then* $diam(\mathrm{G}(N)) \leq 2$.

Theorem 3.8. *The graph* $\mathrm{G}(N)$ *of a zero-symmetric integral near-ring* N *(but not a near-field) is complete.*

Theorem 3.9. *Let* N *be a near-ring which has the DCCI. Then* $\mathrm{G}(N)$ *is complete if and only if* N *has a unique minimal ideal.*

Theorem 3.10. *Let* N *be a near-ring. If* $\mathrm{G}(N)$ *is disconnected, then* N *is a direct product of two simple near-rings.*

Proof. Since $\mathrm{G}(N)$ is disconnected, Theorem 3.5 shows that there exist at least two maximal ideals I and J of N such that $I \cap J = \{0\}$ and $I + J = J + I = N$. Now, we are going to show that N is isomorphic to the direct product $N/I \times N/J$. Define $\phi : N \to N/I \times N/J$ by $\phi(n) = (n+I, n+J)$. Let $n_1, n_2 \in N$. Then $\phi(n_1) + \phi(n_2) = (n_1+I, n_1+J)+(n_2+I, n_2+J) = (n_1+n_2+I, n_1+n_2+J) = \phi(n_1+n_2)$ and $\phi(n_1)\phi(n_2) = (n_1 + I, n_1 + J)(n_2 + I, n_2 + J) = (n_1n_2 + I, n_1n_2 + J) = \phi(n_1n_2)$. Therefore, ϕ is a near-ring homomorphism. Now, $Ker\phi = \{n \in N | \phi(n) = (0 + I, 0 + J)\} = \{n \in N | (n + I, n + J) = (0 + I, 0 + J)\} = \{n \in N | n + I = 0 + I$ and $n+J = 0+J\} = \{n \in N | n \in I$ and $n \in J\} = \{n \in N | n \in I \cap J = \{0\}\} = \{0\}$. So, ϕ is one-one. Now, suppose that $(n_1+I, n_2+J) \in N/I \times N/J$. Since $N = I+J$, there exist $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $n_1 = i_1 + j_1, n_2 = i_2 + j_2$. Now, since I is normal in $(N, +), -j_1 + i_1 + j_1 \in I \Rightarrow -j_1 + i_1 + j_1 = i'_1$, for some $i'_1 \in I \Rightarrow i_1 + j_1 = j_1 + i'_1$. Also, $-j_1 + i_2 + j_1 \in I \Rightarrow -j_1 + i_2 + j_1 = i'_2$, for some $i'_2 \in I \Rightarrow i_2 = j_1 + i'_2 - j_1 \Rightarrow i_2 + j_1 = j_1 + i'_2$. We have, $\phi(j_1 + i'_2) = (j_1 + i'_2 + I, j_1 + i'_2 + J) = (j_1 + I, i_2 + j_1 + J) = (j_1 + i'_1 + I, i_2 + J) = (i_1 + j_1 + I, i_2 + j_2 + J) = (n_1 + I, n_2 + J)$. So, ϕ is onto. Hence, ϕ is an isomorphism. Also, since I and J are maximal in N, N/I and N/J are simple. ■

Theorem 3.11. *Let* N *be a near-ring containing a non-zero proper direct summand. Then* $\mathrm{G}(N)$ *is never a complete graph.*

Proof. Let I be a non-zero proper direct summand of N. By Definition 4.7, there exists a non-zero proper ideal J of N such that $I \bigoplus J = N$. This implies that $I \cap J = \{0\}$. So, the vertices I and J are not adjacent. Hence, $\mathrm{G}(N)$ is not complete. ■

Lemma 3.12. *Let* I *be a non-zero proper direct summand of a near-ring* N*. If* $deg(I)$ *is finite, then* N *has the DCCI.*

Proof. Suppose that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a chain of ideals of I. Then each I_i is an ideal of N (cf. Theorem 2.12, pp. 46 [13]). So, each I_i ($\neq \{0\}$) is a vertex in $\mathrm{G}(N)$ and is adjacent with I. If I does not have the DCCI, then $deg(I)$ is not finite, a contradiction. Thus, I has the DCCI. In a similar way, we can show that N/I has the DCCI. Hence, N has the DCCI (cf. Theorem 2.35, pp. 51 [13]). ∎

Theorem 3.13. *Let N be a near-ring containing a non-zero proper direct summand and $\mathrm{G}(N)$ be regular. Then $\mathrm{G}(N)$ is a null graph.*

Proof. Let N be a near-ring containing a non-zero proper direct summand I and $\mathrm{G}(N)$ be regular. If possible, suppose that $\mathrm{G}(N)$ is not null. By Corollary 3.6, $\mathrm{G}(N)$ is connected. Then $\mathrm{G}(N)$ is either complete or not. But Theorem 3.11 shows that $\mathrm{G}(N)$ is not complete. Clearly, $deg(I)$ is finite as $\mathrm{G}(N)$ is regular. So, from Lemma 3.12, N has the DCCI, which confirms the existence of a minimal ideal of N. Now, from Theorem 3.9, N has at least two minimal ideals I_1 and I_2 (say). Clearly, I_1 and I_2 are not adjacent and $diam(\mathrm{G}(N)) \leq 2$ (by Theorem 3.7). Therefore, there exists a non-trivial ideal J of N such that I_1-J-I_2 is a path. So, $I_1, I_2 \subsetneq J$. Thus, each vertex adjacent to I_1 is adjacent to J, too. Again, J is adjacent to I_2 but I_1 is not adjacent to I_2. This argument shows that $deg(J) > deg(I_1)$, which contradicts the fact that $\mathrm{G}(N)$ is regular. Hence, $\mathrm{G}(N)$ is a null graph. ∎

Theorem 3.14. *Let N be a finite near-ring such that $(N, +)$ is cyclic and let M be a maximal ideal of N. Then M is a non-zero proper direct summand of N if and only if $deg(M) = n - 2$, where $|V(\mathrm{G}(N)| = n$.*

Proof. Suppose that M is a non-zero proper direct summand of N. Then there exists a non-zero proper ideal I such that $M + I = N$ and $M \cap I = \{0\}$. This implies that $deg(M) \leq n - 2$. If possible, let $deg(M) \lneq n - 2$. Then there exists a non-zero proper ideal J ($\neq I$) such that $M \cap J = \{0\}$. Since M is maximal, $M + J = N$. Clearly, $|N| = |M + I| = |M||I|$ and $|N| = |M + J| = |M||J|$ as $M + I = M + J = N$ and $M \cap I = M \cap J = \{0\}$. So, $|I| = |J| = |N|/|M|$. Since $(N, +)$ is a finite cyclic group, $I = J$ which is a contradiction. Therefore, $deg(M) = n - 2$. Conversely, let $deg(M) = n - 2$, where $|V(\mathrm{G}(N)| = n$. Then there exists a non-zero proper ideal K of N such that $M \cap K = \{0\}$. Hence, $M + K = N$ as M is maximal. This implies that $M \bigoplus K = N$. Thus, M is a non-zero proper direct summand of N. ∎

Note 3.15. It is clear from the proof that N need not be finite and $(N, +)$ need not be cyclic for the converse part of the above theorem.

The following corollary comes directly from the above theorem.

Corollary 3.16. *Let N be a near-ring. If M is a maximal ideal of N such that $deg(M) = n - 2$, where $|V(\mathrm{G}(N)| = n$, then M is not essential.*

Theorem 3.17. *Let N be a near-ring and M be a maximal ideal of N. If M is not a direct summand of N, then $G(N)$ is connected. Moreover, $deg(M) = n-1$, where $|V(G(N))| = n$.*

Proof. Suppose that $M \cap J = \{0\}$, where J is any ideal of N. If possible, let $J \neq \{0\}$. Then $M \subsetneq M + J$. We have, $M + J \subsetneq N$. Otherwise, if $M + J = N$, then M is a direct summand, a contradiction. Therefore, $M \subsetneq M+J \subsetneq N$, which contradicts the fact that M is maximal. Thus, $M \cap J = \{0\}$ implies $J = \{0\}$. So, we can conclude that M is adjacent to all other non-trivial ideals of N and hence, $G(N)$ is connected and $deg(M) = n - 1$. ■

Theorem 3.18. *Let N be a finite near-ring such that $(N, +)$ is a cyclic group. If M is a maximal ideal of N, then $deg(M)$ is either $(n - 1)$ or $(n - 2)$, where $|V(G(N))| = n$.*

Proof. Suppose that M is a non-zero proper direct summand of N. Then, from Theorem 3.14, we have $deg(M) = (n - 2)$. If M is not a direct summand of N, then, from Theorem 3.17, we have $deg(M) = (n - 1)$. This completes the proof. ■

Note that the above result is not true for a near-ring N such that $(N, +)$ is not cyclic, which is clear from the following example.

Example 3.19. Consider the near-ring $N = \{0, a, b, c\}$ together with two binary operations '+' and '·' defined by

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

and

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

Here, $(N, +)$ is not cyclic and the non-zero proper ideals of the near-ring N are $I = \{0, a\}$, $J = \{0, b\}$ and $K = \{0, c\}$. Clearly, I, J and K are non-zero maximal ideals of N. But $deg(I) = deg(J) = deg(K) = 0 = |V(G(N))| - 3$.

Theorem 3.20. *If N is an indecomposable near-ring, then $G(N)$ is connected.*

Proof. Let N be an indecomposable near-ring and let I, J be any two distinct non-zero proper ideals of N.

Case I. If $I \cap J \neq \{0\}$, then $G(N)$ becomes complete and hence, connected.

Case II. Suppose that $I \cap J = 0$. We have, $I + J \neq N$. Otherwise, if $I + J = N$, then N becomes decomposable, a contradiction. So, $I + J \in V(\mathrm{G}(N))$ and I-$(I+J)$-J is path between I and J. Thus, in any case, there exists a path between any two distinct vertices. Hence, $\mathrm{G}(N)$ is connected. ■

Remark 3.21. Note that the converse of the above result is not true in general. For example, if we consider the near-ring $M_c(\mathbb{Z}_{30})$ with pointwise addition and composition, where $M_c(\mathbb{Z}_{30})$ denotes the set of all constant mappings from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$, then $\mathrm{G}(M_c(\mathbb{Z}_{30}))$ is connected, but $M_c(\mathbb{Z}_{30})$ is decomposable.

Corollary 3.22. *For a local near-ring N, $\mathrm{G}(N)$ is connected. Moreover, $diam(\mathrm{G}(N)) \leq 2$.*

Proof. Let N be a local near-ring. Then N is indecomposable (cf. pp. 400 [13]). Thus, $\mathrm{G}(N)$ is connected. Hence, by Theorem 3.7, $diam(\mathrm{G}(N)) \leq 2$. ■

Theorem 3.23. *Let N be a near-ring. If I is a non-zero proper direct summand of N, then $\mathrm{G}(I)$ is an induced subgraph of $\mathrm{G}(N)$.*

Proof. Let I be a non-zero proper direct summand of N. Then each ideal of I is an ideal of N (cf. Theorem 2.12, pp. 46 [13]). So, $V(\mathrm{G}(I)) \subseteq V(\mathrm{G}(N))$. By Definition 3.1, it is clear that two vertices in $\mathrm{G}(I)$ are adjacent if and only if they are adjacent in $\mathrm{G}(N)$. Hence, $\mathrm{G}(I)$ is an induced subgraph of $\mathrm{G}(N)$. ■

Note that the converse is not true. As example: consider the near-ring $(\mathbb{Z}_8, +, \cdot)$. Here, $\mathrm{G}((\bar{2}))$ is an induced subgraph of $\mathrm{G}(\mathbb{Z}_8)$. But $(\bar{2})$ is not a direct summand of $\mathbb{Z}_8$.

4. Interplay between intersection graph and essential ideals of a near-ring

Theorem 4.1. *An ideal I of a near-ring N is essential if and only if $deg(I) = n - 1$, where $|V(\mathrm{G}(N)| = n$. Hence, the maximum degree of $\mathrm{G}(N)$, $\Delta(\mathrm{G}(N)) = deg(I)$.*

Proof. Let I be an essential ideal of a near-ring N with $|V(\mathrm{G}(N))| = n$. Then $I \cap J \neq \{0\}$ for any non-trivial ideal J, which implies that I is adjacent with all other vertices in $\mathrm{G}(N)$. Thus, $deg(I) = |V(\mathrm{G}(N))| - 1$. Conversely, let $deg(I) = |V(\mathrm{G}(N))| - 1$. Then I is adjacent with all other vertices in $\mathrm{G}(N)$. Therefore, $I \cap J \neq \{0\}$, for any non-trivial ideal J $(\neq I)$ of N. Hence, I is an essential ideal of N. ■

Theorem 4.2. *Let N be a near-ring. If N contains an essential ideal, then $\mathrm{G}(N)$ is connected.*

Proof. Let N be a near-ring containing an essential ideal I. Since I is an essential ideal of N, $I \cap J \neq \{0\}$ for any non-trivial ideal J $(\neq I)$ of N which implies that I is adjacent to all other vertices in $\mathrm{G}(N)$. Thus, for any two distinct vertices M and K, there exists a path M-I-K from M to K. Hence, $\mathrm{G}(N)$ is connected. ■

Note that the converse of this result is not true. For example: consider the near-ring $M_c(\mathbb{Z}_{30})$ with pointwise addition and composition, where $M_c(\mathbb{Z}_{30})$ denotes the set of all constant mappings from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$. Here, $\mathrm{G}(M_c(\mathbb{Z}_{30}))$ is connected (cf. Figure 2), but $M_c(\mathbb{Z}_{30})$ does not contain any essential ideal.

Theorem 4.3. *Let N be a near-ring. If $\mathrm{G}(N)$ is complete, then N contains an essential ideal.*

Proof. Let $\mathrm{G}(N)$ be complete. Suppose that I is a vertex in $\mathrm{G}(N)$. Since $\mathrm{G}(N)$ is complete, $deg(I) = |V(\mathrm{G}(N))| - 1$. Hence, from Theorem 4.1, I is an essential ideal of N. ■

Note that the converse of the result is not true in general. As example: consider the near-ring $(\mathbb{Z}_{18}, +, \cdot)$. Here, $(\bar{3})$ is an essential ideal. But $\mathrm{G}(\mathbb{Z}_{18})$ is not complete (cf. Figure 3 [8]).

Theorem 4.4. *Let I be a minimal ideal of a near-ring N which has the DCCI. Then $\mathrm{G}(N)$ is complete if and only if I is essential.*

Proof. Let $\mathrm{G}(N)$ be complete. Then, from the proof of Theorem 4.3, it is clear that I is an essential ideal of N. Conversely, let I be a minimal as well as an essential ideal of N. Then I is the unique minimal ideal of N. Otherwise, if there exists another minimal ideal J $(\neq I)$, $I \cap J \neq \{0\}$, which contradicts the fact that I is an essential ideal of N. Hence, from Theorem 3.9, $\mathrm{G}(N)$ is complete. ■

Theorem 4.5. *Let N be a near-ring. If I is an essential ideal of N, then the eccentricity of I is* 1.

Proof. Let I be an essential ideal of N. Then, for any non-trivial ideal J of N, $I \cap J \neq \{0\}$ which implies that the distance between I and J, $d(I, J) = 1$. Thus, the eccentricity of I, $e(I) = \max\{d(I, J) : J \in V(\mathrm{G}(N))\} = 1$. ■

Theorem 4.6. *Let N be a near-ring. If I is an essential ideal of N, then I is central.*

Proof. Let I be an essential ideal of a near-ring N. Then, by Theorem 4.5, the eccentricity of I is 1 which implies that the radius of the graph $\mathrm{G}(N)$ is 1. Thus, $e(I) = rad(\mathrm{G}(N))$. Hence, I is central. ■

Theorem 4.7. *A near-ring N contains an essential ideal if and only if the domination number, $\gamma(\mathrm{G}(N)) = 1$.*

Proof. Let us consider that N contains an essential ideal I. Then I is adjacent to all other vertices in $\mathrm{G}(N)$. This implies that $\{I\}$ is a dominating set. Therefore, the minimum cardinality of any dominating set of $\mathrm{G}(N)$ is 1. Thus, the domination number, $\gamma(\mathrm{G}(N)) = 1$. Conversely, suppose that domination number, $\gamma(\mathrm{G}(N)) = 1$. Then there exists a dominating set $\{J\}$ in $\mathrm{G}(N)$ whence any non-trivial ideal K $(\neq \{0\})$ of N, K is adjacent to J. This implies that $J \cap K \neq \{0\}$ for any non-trivial ideal K of N. Therefore, J is an essential ideal of N. ■

Theorem 4.8. *Let N be a near-ring. If N contains an essential ideal, then the complement $\overline{\mathrm{G}(N)}$ of $\mathrm{G}(N)$ is disconnected.*

Proof. Let N be a near-ring containing an essential ideal I. Then, by Theorem 4.1, $deg(I) = |V(\mathrm{G}(N)| - 1$ in $\mathrm{G}(N)$ which implies that $deg(I) = 0$ in $\overline{\mathrm{G}(N)}$. So, I is an isolated vertex in $\overline{\mathrm{G}(N)}$. Hence $\overline{\mathrm{G}(N)}$ is disconnected. ■

Theorem 4.9. *Let N be a near-ring and I be the unique essential ideal of N. If $\mathrm{G}(N)$ contains at least one vertex of degree one and $|V(\mathrm{G}(N))| \geq 3$, then I is a cut vertex. Moreover, I is a separating vertex.*

Proof. Let J be a non-trivial ideal such that $deg(J) = 1$. Now, $|V(\mathrm{G}(N))| \geq 3$ implies that $I \neq J$. Theorem 4.2 shows that $\mathrm{G}(N)$ is connected. Clearly, I is the only vertex adjacent with J. So, J is an isolated vertex in the induced subgraph $\mathrm{G}'(N)$ with vertex set $V(\mathrm{G}(N)) - \{I\}$. Hence, I is a cut vertex in $\mathrm{G}(N)$. Again, since every cut vertex is a separating vertex, I is a separating vertex. ■

Theorem 4.10. *Let N be a near-ring containing a unique essential ideal I. If all non-trivial ideals except I are minimal, then $\mathrm{G}(N)$ is a star.*

Proof. Let $\{I_i : i \in H\}$, where H is an index set, be the set of all non-zero proper ideals (except I) of N. Since I is an essential ideal, I is adjacent to I_i for all $i \in H$. Also, since I_i's are minimal, $I_i \cap I_j = 0$ for all i, j $(i \neq j)$ implies I_i and I_j are not adjacent for all $i, j \in H$. So, the vertex set can be partitioned into two subsets $X = \{I\}$ and $Y = \{I_j : j \in H\}$ such that each edge has one end in X and one end in Y. Hence, $\mathrm{G}(N)$ is a star graph. ■

The following corollary comes directly from the above theorem.

Corollary 4.11. *Let N be a near-ring containing a unique essential ideal I. If all non-trivial ideals except I are minimal, then $\mathrm{G}(N)$ is a rooted tree with root vertex I.*

Theorem 4.12. *Let N be a near-ring containing at least one essential ideal. If $|V(\mathrm{G}(N))| = n$ (≥ 2), then $1 \leq \kappa(\mathrm{G}(N)) \leq \lambda(\mathrm{G}(N)) \leq \delta(\mathrm{G}(N)) \leq n - 1$, where $\kappa(\mathrm{G}(N))$, $\lambda(\mathrm{G}(N))$ and $\delta(\mathrm{G}(N))$ denote the vertex connectivity, the edge connectivity and the minimum degree of $\mathrm{G}(N)$ respectively.*

Proof. Let I an essential ideal of N. Theorem 4.2 shows that $\mathrm{G}(N)$ is connected, which implies that the minimum cardinality of a vertex cut is 1. Hence, $1 \leq \kappa(\mathrm{G}(N))$. Now, Theorem 4.1 shows that $\Delta(\mathrm{G}(N)) = deg(I)$, which implies that $\delta(\mathrm{G}(N)) \leq deg(I) = n-1$. Also, for any graph G, it is well known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$. Hence, the result is proved. ∎

Theorem 4.13. *Let N be a near-ring containing exactly r $(r \geq 1)$ essential ideals. If $|V(\mathrm{G}(N))| = n$ $(\gneq r)$, then $r \leq \kappa(\mathrm{G}(N)) \leq \lambda(\mathrm{G}(N)) \leq \delta(\mathrm{G}(N)) \leq n-1$, where $\kappa(\mathrm{G}(N))$, $\lambda(\mathrm{G}(N))$ and $\delta(\mathrm{G}(N))$ denote the vertex connectivity, the edge connectivity and the minimum degree of $\mathrm{G}(N)$ respectively.*

Proof. Let $S = \{I_i : i = 1, 2, \ldots, r\}$ be the finite set of essential ideals of N. Theorem 4.12 shows that the result is true for $r = 1$. Suppose that $r \geq 2$. Let $S' \subseteq V(\mathrm{G}(N))$ such that $V(\mathrm{G}(N)) - S'$ contains at least one essential ideal. Then, by Theorem 4.2, it is clear that $\mathrm{G}(N) - S'$ is connected. So, S' can not be a vertex cut. Therefore, any vertex cut V' is a superset of S. Thus, the vertex connectivity $\kappa(\mathrm{G}(N)) \geq r$. Hence, by Theorem 4.12, we have $r \leq \kappa(\mathrm{G}(N)) \leq \lambda(\mathrm{G}(N)) \leq \delta(\mathrm{G}(N)) \leq n-1$. ∎

Theorem 4.14. *Let N be a near-ring containing a unique essential ideal I.*

(i) *If $\mathrm{G}(N)$ does not contain a cycle, then $\mathrm{G}(N)$ is a star.*

(ii) *If $\mathrm{G}(N)$ contains a cycle, then I is contained in a 3-cycle.*

Proof. (i) Suppose that $\mathrm{G}(N)$ does not contain a cycle. Then any two distinct vertices different from I are not adjacent. Since I is an essential ideal of N, I is adjacent with all other vertices in $\mathrm{G}(N)$. Thus, $\mathrm{G}(N)$ is a star.

(ii) Let us consider that $\mathrm{G}(N)$ contains a cycle $C_n = I_1\text{-}I_2 - \cdots - I_n\text{-}I_1$ $(n \geq 3)$. First consider that I is in C_n. If $n = 3$, then we are done. Suppose $n \geq 4$. Let $I = I_k$ $(k \notin \{1, n\})$. Then $I_1\text{-}I_k\text{-}I_n\text{-}I_1$ forms a 3-cycle. If $I = I_1$, then $I_1\text{-}I_2\text{-}I_3\text{-}I_1$ forms a 3-cycle. If $I = I_n$, then $I_n\text{-}I_1\text{-}I_2\text{-}I_n$ forms a 3-cycle. Now, suppose that I is not in C_n. Since I is adjacent with I_1 and I_2, $I\text{-}I_1\text{-}I_2\text{-}I$ forms a 3-cycle. Hence, the result is proved. ∎

Theorem 4.15. *Let N be a near-ring containing at least two essential ideals. If $|V(\mathrm{G}(N))| \geq 3$, then $\mathrm{G}(N)$ contains a 3-cycle.*

Proof. Let I, J be two essential ideals and K $(\neq I, J)$ be any non-trivial ideal of N. Clearly, I and J are adjacent and K is adjacent with I, J. Thus, $I\text{-}J\text{-}K\text{-}I$ forms a 3-cycle. ∎

The following corollary comes directly from the above theorem.

Corollary 4.16. *Let N be a near-ring containing at least two essential ideals. If $|V(\mathrm{G}(N))| \geq 3$, then*

(i) $\mathrm{G}(N)$ *cannot be a tree,*

(ii) *the girth of* $\mathrm{G}(N)$ *is* 3, *and*

(iii) $\mathrm{G}(N)$ *is never bipartite.*

Note that the converse of Corollary 4.16(iii) is not true in general. For example: consider the near-ring $M_c(\mathbb{Z}_{12})$ with pointwise addition and composition, where $M_c(\mathbb{Z}_{12})$ denotes the set of all constant mappings from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$. Here, $\mathrm{G}(\mathbb{Z}_{12})$ is not bipartite (cf. Figure 2), but $M_c(\mathbb{Z}_{12})$ contains exactly one essential ideal.

Theorem 4.17. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar, then $deg(I) = 3$, for any non-essential ideal I of N.*

Proof. Let J, K, L be three essential ideals and I be any non-trivial non-essential ideal of N. Clearly, $deg(I) \geq 3$. If possible, let $deg(I) \gneq 3$. Then there exists at least one non-trivial ideal P $(\neq I, J, K, L)$ of N such that I is adjacent to P. Since J, K, L are essential ideals, they are adjacent to P also. Thus, the graph induced by the following set of vertices $\{I, J, K, L, P\}$ is isomorphic to K_5, which contradicts the fact that $\mathrm{G}(N)$ is planar. Hence, $deg(I) = 3$. ■

Example 4.18. If N is a near-ring containing no essential ideal, then the above result in Theorem 4.17 is not true in general. Consider the near-ring $M_c(\mathbb{Z}_{30})$, the set of all constant mappings from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$, with pointwise addition and composition. Here, $M_c(\mathbb{Z}_{30})$ contains no essential ideal and $\mathrm{G}(M_c(\mathbb{Z}_{30}))$ is planar (cf. Figure 2). But there is no non-essential ideal of degree 3.

Example 4.19. If N is a near-ring containing exactly one essential ideal, then the above result in Theorem 4.17 is not true in general. Because, the near-ring $M_c(\mathbb{Z}_{12})$ contains exactly one essential ideal and $\mathrm{G}(M_c(\mathbb{Z}_{12}))$ is planar (cf. Figure 2.), but there is no non-essential ideal of degree 3 in $\mathrm{G}(M_c(\mathbb{Z}_{12}))$.

Example 4.20. Let N be a near-ring containing exactly two essential ideals. If $\mathrm{G}(N)$ is non-planar, then the above result in Theorem 4.17 is not true in general. Because, the near-ring $M_c(\mathbb{Z}_{24})$ contains exactly two essential ideals and $\mathrm{G}(M_c(\mathbb{Z}_{24}))$ is non-planar, but there is no non-essential ideal of degree 3 in $\mathrm{G}(M_c(\mathbb{Z}_{24}))$.

Example 4.21. Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is non-planar, then the above result in Theorem 4.17 is not true in general. Because, the near-ring $M_c(\mathbb{Z}_{36})$ contains exactly three essential ideals and $\mathrm{G}(M_c(\mathbb{Z}_{36}))$ is non-planar, but there is no ideal of degree 3 in $\mathrm{G}(M_c(\mathbb{Z}_{36}))$.

Theorem 4.22. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| \geq 4$, then the chromatic number of $\mathrm{G}(N)$ is 4.*

Proof. Let J, K, L be three essential ideals and $\{I_i : i \in H\}$, where H is an index set, be the set of non-zero proper non-essential ideals of N. Clearly, J, K, L are adjacent to each other. This shows that we need 3 different colours to colour J, K, L. Moreover, Theorem 4.17 shows that each I_i is adjacent to J, K, L and I_i, I_j are not adjacent for all i, j $(i \neq j)$. So, all I_i's can be coloured by only one colour different from the colours of J, K, L. Thus, the chromatic number of $\mathrm{G}(N)$ is 4. ■

Theorem 4.23. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$ $(n \geq 3)$, then $\mathrm{G}(N)$ contains no asteroidal triple.*

Proof. Suppose that $\{I_i : i = 1, 2, \ldots, n\}$ is the set of all non-zero proper non-essential ideals of N. Then $V' = \{I_1, I_2, \ldots, I_n\}$ is the largest independent vertex subset of $V(\mathrm{G}(N))$. We take a vertex subset $\bar{V} = \{u, v, w\}$ of V'. Note that any two among these three vertices are connected by an essential ideal which is adjacent with the third one. Thus, any vertex subset $\bar{V}$ of V' is not an asteroidal triple. Hence, the theorem is proved. ■

Theorem 4.24. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$ $(n \in \mathbb{N})$, then the independence number $\alpha(\mathrm{G}(N)) = n$.*

Proof. Let I, J, K be three essential ideals and $\{I_i : i = 1, 2, \ldots, n\}$ be the set of non-zero proper non-essential ideals of N. Since $\mathrm{G}(N)$ is planar, Theorem 4.17 shows that $deg(I_i) = 3$ for all $i = 1, 2, \ldots, n$. So, each I_i is adjacent with I, J, K only. Thus, $\{I_i : i = 1, 2, \ldots, n\}$ is the largest independent vertex set in $\mathrm{G}(N)$. Hence, the independence number $\alpha(G) = |\{I_i : i = 1, 2, \ldots, n\}| = n$. ■

Theorem 4.25. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n + 3$ $(n \geq 3)$, then the geodetic number $g(\mathrm{G}(N)) = n$.*

Proof. Let I, J, K be three essential ideals and $\{I_i : i = 1, 2, \ldots, n\}$ be the set non-trivial non-essential ideals of N, where $n \geq 3$. We are going to show that $S = \{I_i : i = 1, 2, \ldots, n\}$ is a geodetic set of $\mathrm{G}(N)$. Since $\mathrm{G}(N)$ is planar, Theorem 4.17 shows that $deg(I_i) = 3$ i.e., I_i is adjacent with I, J, K only, for all $i = 1, 2, \ldots, n$. We take the vertices $I_1, I_2, I_3 \in S$. Clearly, the paths I_1-I-I_2, I_1-J-I_3, I_2-K-I_3 are I_1-I_2, I_1-I_2, I_2-I_3 geodesics (respectively). This implies that $I, J, K \in I[S]$. So, $I[S] = V(G)$ i.e., S is a geodetic set of G. To prove the theorem it is enough to show that S is a geodetic set with minimum cardinality. If possible, let there exist $S' \subsetneq S$ such that S' is a geodetic set of $\mathrm{G}(N)$. Without loss of generality, suppose that $S' = S - \{I_r\}$ for some $r \in \{1, 2, \ldots, n\}$. Since each I_i is adjacent with I, J, K only, there is no shortest path from I_i to I_j $(i, j \neq r)$ through I_r. This implies that I_r does not belong to $I[S']$. Therefore, $I[S'] \neq V(\mathrm{G}(N))$, which contradicts the fact that S' is a geodetic set of $\mathrm{G}(N)$. Hence, S is a geodetic set of $\mathrm{G}(N)$ with minimum cardinality. Thus, the geodetic number $g(\mathrm{G}(N)) = |S| = n$. ■

Theorem 4.26. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$ $(n \geq 3)$, then the hull number $h(\mathrm{G}(N))$ is 2.*

Proof. Let I, J, K be three essential ideals and $\{I_i : i = 1, 2, \dots, n\}$ be the set non-trivial non-essential ideals of N, where $n \geq 3$. Theorem 4.17 shows that each vertex I_i of $\bar{V} = \{I_1, I_2, \dots, I_n\}$ is adjacent with I, J, K only. Clearly, the convex hull $[v]$ of any single vertex v is $\{v\}$. Now, we take the vertex subset $S = \{I_i, I_j\}$ for some $i, j \in \{1, 2, \dots, n\}$. We are to show that S is a hull set with minimum cardinality. Now, for any vertex subset S' of $V(\mathrm{G}(N))$ such that $S \subseteq S' \subseteq \bar{V}$, $I[S'] \neq S'$ as at least one of I, J, or K belongs to $I[V']$, where $V' \subseteq \bar{V}$ such that $|V'| = k$ $(2 \leq k \leq n)$. Now, $\bar{V} \cup \{I\} \cup \{J\} \subseteq I[\bar{V} \cup \{I\}]$ and $I[\bar{V} \cup \{I\} \cup \{J\}] = V(\mathrm{G}(N))$. Therefore, $V(\mathrm{G}(N))$ is the smallest convex set containing S. So, S is a hull set with minimum cardinality. Hence, the hull number $h(\mathrm{G}(N)) = |S| = 2$. ■

Theorem 4.27. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$ $(n \geq 2)$, then $\mathrm{G}(N)$ is chordal.*

Proof. Let I, J, K be three essential ideals and $\{I_i : i = 1, 2, \dots, n\}$ be the finite set of non-trivial non-essential ideals of N, where $n \geq 1$. Theorem 4.17 shows that no two vertices of $\{I_i : i = 1, 2, \dots, n\}$ are adjacent. So, any $n-$cycles C_n $(n \geq 4)$ contains at least two vertices of $\{I, J, K\}$ which are not adjacent in C_n but adjacent in $\mathrm{G}(N)$ because I, J, K are adjacent with each other. This implies that every n-cycle $(n \geq 4)$ possesses a chord. Hence, $\mathrm{G}(N)$ is chordal. ■

Theorem 4.28. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$ $(n \geq 2)$, then $\mathrm{G}(N)$ is never Eulerian.*

Proof. Let $\mathrm{G}(N)$ be planar and $|V(\mathrm{G}(N))| = n+3$ $(n \geq 2)$. Theorem 4.17 shows that $deg(I) = 3$ for any non-essential ideal I. Clearly, $\mathrm{G}(N)$ is connected and degree of each vertex is not even. Hence, $\mathrm{G}(N)$ is not Eulerian (cf. Theorem 3.1.1 [12]). ■

Theorem 4.29. *Let N be a near-ring containing exactly three essential ideals. If $\mathrm{G}(N)$ is planar and $|V(\mathrm{G}(N))| = n+3$, then the vertex connectivity $\kappa(\mathrm{G}(N))$ and the edge connectivity $\lambda(\mathrm{G}(N))$ are 3. Moreover, the strong vertex connectivity $K(\mathrm{G}(N)) = \kappa(\mathrm{G}(N)) = 3$.*

Proof. Let $S = \{I, J, K\}$ be the set of essential ideals and $\{I_i : i = 1, 2, \ldots, n\}$ be the finite set of non-trivial non-essential ideals of N, where $n \geq 2$. Since $\mathrm{G}(N)$ is planar, Theorem 4.17 shows that $deg(I_i) = 3$ for all $i = 1, 2, \ldots, n$. So, the minimum degree $\delta(\mathrm{G}(N)) = 3$. Then, by Theorem 4.13, we have $3 \leq \kappa(\mathrm{G}(N)) \leq \lambda(\mathrm{G}(N)) \leq 3$ which implies that $\kappa(\mathrm{G}(N)) = \lambda(\mathrm{G}(N)) = 3$. Moreover, since $\mathrm{G}(N) - S$ is a null set i.e., $\mathrm{G}(N) - S$ is totally disconnected, S is a strong vertex cut. Clearly, S is a minimum strong vertex cut. So, the strong vertex connectivity $K(\mathrm{G}(N)$ is 3. Hence, the result is proved. ■

Theorem 4.30. *Let N be a near-ring containing at least four essential ideals. If $|V(\mathrm{G}(N))| \geq 5$, then $\mathrm{G}(N)$ is non-planar.*

Proof. Let I_1, I_2, I_3, I_4 be essential ideals and J be an non-trivial ideal of N different from I_i $(i = 1, 2, 3, 4)$. Then the induced subgraph of $\mathrm{G}(N)$ with vertex set $\{I_1, I_2, I_3, I_4, J\}$ is K_5. Hence, $\mathrm{G}(N)$ is non-planar. ■

Note that the converse of the above theorem is not true in general. For example: consider the near-ring $M_c(\mathbb{Z}_{56})$ with pointwise addition and composition, where $M_c(\mathbb{Z}_{56})$ denotes the set of all constant mappings from $\mathbb{Z}_{56}$ to $\mathbb{Z}_{56}$. Here, $\mathrm{G}(M_c(\mathbb{Z}_{56}))$ is non-planar, but $M_c(\mathbb{Z}_{56})$ contains exactly two essential ideals.

5. Concluding remarks

In this paper, we have studied several graphical properties of the intersection graph of ideals of a near-ring N in terms of direct summand and essential ideal of N. Based on our work done here, we mention below some possible future scope for further study.

1. One can determine the chromatic number, the independence number, the geodetic number and the hull number of a non-planar intersection graph.
2. One can study the graphical properties of the intersection graph of ideals of the direct product of near-rings.
3. It will be nice if one can exhibit a near-ring containing a unique essential ideal I such that I is a cut vertex as well as a separating vertex but the corresponding intersection graph has no vertex of degree one to counter the converse part of Theorem 4.9.
4. $M(\mathbb{Z}_n)$ denotes the set of all mappings from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ and $M_c(\mathbb{Z}_n)$ denotes the set of all constant mappings from $\mathbb{Z}_n$ to $\mathbb{Z}_n$. Here, we observed that the intersection graph of $M_c(\mathbb{Z}_n)$ and the intersection graph of $\mathbb{Z}_n$ are isomorphic (cf. Figure 2 and Figure 3 [8]). One can establish the relation between the intersection graphs corresponding to $\mathbb{Z}_n$ and $M(\mathbb{Z}_n)$ and hence one can generalize that result for an arbitrary additively written group Γ.

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References

[1] S. Akbari, F. Heydari and M. Maghasedi, *The intersection graph of a group*, J. Algebra Appl. **14(5)** 2015. https://doi.org/10.1142/S0219498815500656

[2] S. Akbari, R. Nikandish and M.J. Nikmehr, *Some results on the intersection graphs of ideals of rings*, J. Algebra Appl. **12(4)** 2013. https://doi.org/10.1142/S0219498812502003

[3] F.G. Ball, D.J. Sirl and P.S. Trapman, *Epidemics on random intersection graphs*, Ann. Appl. Probab. **24(3)** (2014) 1081–1128. https://doi.org/10.1214/13-AAP942

[4] S. Bhavanari, S.P. Kuncham and B.S. Kedukodi, *Graph of a nearring with respect to an ideal*, Commun. Algebra **38(5)** (2010) 1957–1967. https://doi.org/10.1080/00927870903069645

[5] M. Bloznelis, J. Jaworski and K. Rybarczyk, *Component evolution in a secure wireless sensor network*, Networks **53(1)** (2009) 19-26. https://doi.org/10.1002/net.20256

[6] J.A. Bondy, and U.S.R. Murty, Graduate Texts in Mathematics, Graph Theory (Springer, 2008).

[7] J. Bosák, *The graphs of semigroups*, Theory of Graphs and Application (M. Fielder, ed.), (Academic Press, New York, 1964) 119–125.

[8] I. Chakrabarty, S. Ghosh, T.K. Mukherjee and M.K. Sen. *Intersection graphs of ideals of rings*, Discrete Math. **309(17)** (2009) 5381–5392. https://doi.org/10.1016/j.disc.2008.11.034

[9] I. Chakrabarty and J. V. Kureethara, *A survey on the intersection graphs of ideals of rings*, Commun. Comb. Optim. **7(2)** (2022) 121–167. https://doi.org/10.22049/cco.2021.26990.1176

[10] B. Csákány and G. Pollák, *The graph of subgroups of a finite group*, Czech. Math. J. **19(2)** (1969) 241–247. https://doi.org/10.21136/CMJ.1969.100891

[11] A. Das, *On perfectness of intersection graph of ideals of* $\mathbb{Z}_n$, Discuss. Math. Gen. Algebra Appl. **37(2)** (2017) 119–126. arXiv:1611.01153v1

[12] D.S. Malik, M.K. Sen and S. Ghosh, Introduction to Graph Theory (Cengage Learning, Singapore, 2014).

[13] G. Pilz, Near-Rings, Revised edition (North-Holland, Amsterdam, 1983).

[14] N.J. Rad, S.H. Jafari and S. Ghosh, *On the intersection graphs of ideals of direct product of rings*, Discuss. Math. Gen. Algebra Appl. **34(2)** (2014) 191–201. https://doi.org/10.7151/dmgaa.1224

[15] Y.V. Reddy and B. Satyanarayana, *A Note on N-Groups*, Indian J. Pure Appl. Math. **19(9)** (1988) 842–845.

[16] E. Szpilrajn-Marczewski, *Sur deux proprieéteś des classes d'ensembles* (*in french*), Fund. Math. **33(1)** (1945) 303–307.

[17] E. Szpilrajn-Marczewski, *Sur deux propriétés des classes d'ensembles*, Fund. Math. **33(1)** (1945) 303–307 (Trans. Bronwyn Burlingham. "A Translation of Sur deux propriétés des classes d'ensembles", University of Alberta, 2009).

[18] D.B. West, Introduction to Graph Theory (Prentice Hall of India, New Delhi, 2003).

[19] B. Zelinka, *Intersection graphs of finite abelian groups*, Czech. Math. J. **25(2)** (1975) 171–174.

[20] J. Zhao, O. Yağan and V. Gligor, *Secure k-connectivity in wireless sensor networks under an on/off channel model*, 2013, IEEE International Symposium on Information Theory (2013) 2790–2794.