Discussiones Mathematicae
General Algebra and Applications 45 (2025) 421–444
https://doi.org/10.7151/dmgaa.1483

RINGS SUCH THAT, FOR EACH UNIT u, $u^n - 1$ BELONGS TO THE $\Delta(R)$

Peter Danchev[1]

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

e-mail: danchev@math.bas.bg; pvdanchev@yahoo.com

Arash Javan, Omid Hasanzadeh, Mina Doostalizadeh

and

Ahmad Moussavi

Department of Mathematics
Tarbiat Modares University
14115-111 Tehran Jalal AleAhmad Nasr, Iran

e-mail: a.darajavan@modares.ac.ir; a.darajavan@gmail.com
o.hasanzade@modares.ac.ir; hasanzadeomiid@gmail.com
d_mina@modares.ac.ir; m.doostalizadeh@gmail.com
moussavi.a@modares.ac.ir; moussavi.a@gmail.com

Abstract

We study in-depth those rings R for which, there exists a fixed $n \geq 1$, such that $u^n - 1$ lies in the subring $\Delta(R)$ of R for every unit $u \in R$. We succeeded to describe for any $n \geq 1$ all reduced π-regular $(2n-1)$-ΔU rings by showing that they satisfy the equation $x^{2n} = x$ as well as to prove that the property of being exchange and clean are tantamount in the class of $(2n-1)$-ΔU rings. These achievements considerably extend results established by Danchev (Rend. Sem. Mat. Univ. Pol. Torino, 2019) and Koşan *et al.* (Hacettepe J. Math. & Stat., 2020). Some other closely related results of this branch are also established.

Keywords: n-ΔU ring, ΔU ring, n-JU ring, JU ring, (semi-)regular ring, clean ring.

2020 Mathematics Subject Classification: 16S34, 16U60.

[1]Corresponding author.

1. Introduction and motivation

In this paper, let R denote an associative ring with identity element, which is *not* necessarily commutative. For such a ring R, the sets $U(R)$, $Nil(R)$, $C(R)$ and $Id(R)$ represent the set of invertible elements, the set of nilpotent elements, the set of central elements, and the set of idempotent elements in R, respectively. Additionally, $J(R)$ denotes the Jacobson radical of R. The ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are denoted by $\mathrm{M}_n(R)$ and $\mathrm{T}_n(R)$, respectively. A ring is termed *abelian* if each its idempotent element is central.

The main instrument of the present article plays the set $\Delta(R)$, which was introduced by Lam in [15, Exercise 4.24] and recently studied by Leroy-Matczuk in [16]. As pointed out by the authors in [16, Theorems 3 and 6], $\Delta(R)$ is the largest Jacobson radical's subring of R which is closed with respect to multiplication by all units (quasi-invertible elements) of R. Also, $J(R) \subseteq \Delta(R)$. Moreover, $\Delta(R) = J(T)$, where T is the subring of R generated by units of R, and the equality $\Delta(R) = J(R)$ holds if, and only if, $\Delta(R)$ is an ideal of R. An element a in a ring R is from $\Delta(R)$ if $1 - ua$ is invertible for all invertible $u \in R$.

A ring R is said to be *n-UJ* provided $u - u^n \in J(R)$ for each unit u of R, where $n \geq 2$ is a fixed integer; that is, for any $u \in U(R)$, $u^{n-1} - 1 \in J(R)$. This notion was initially introduced by Danchev in [5] on 2019 and after that, hopefully independently, by Koşan *et al.* in [10] on 2020; note that these rings are a common generalization for $n = 1$ of the so-termed *JU* rings which were firstly defined by Danchev in [3] on 2016 and later redefined in [9] on 2018 under the name *UJ* rings. They showed that for $(2n)$-UJ rings the notions of semi-regular, exchange and clean rings are equivalent.

Likewise, letting $n \geq 2$ be fixed, a ring R is called *n-UU* if, for any $u \in U(R)$, $u^n - 1 \in Nil(R)$. This concept was introduced by Danchev (see [4]), and furthermore studied in more details in [6]. It is principally known that a ring R is said to be *strongly π-regular* provided that, for any $a \in R$, there exists an integer $n \geq 1$ depending on a such that $a^n \in a^{n+1}R$. In [6], the authors showed that a ring R is simultaneously $(n-1)$-UU and strongly π-regular if, and only if, R is strongly n-nil-clean (that is, the sum of an n-potent and a nilpotent which commute each other).

A ring R is said to be *regular* (resp., *unit-regular*) in the sense of von Neumann if, for every $a \in R$, there is $x \in R$ (resp., $x \in U(R)$) such that $axa = a$, and R is said to be *strongly regular* if, for every $a \in R$, $a \in a^2R$. Recall also that a ring R is *exchange* if, for each $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$, and a ring R is *clean* if every element of R is a sum of an idempotent and a unit (cf. [19]). Notice that every clean ring is exchange, but the converse is manifestly *not* true in general; however, it is true in the abelian case (see [19, Proposition

1.8]). In this aspect, a ring R is called *semi-regular* provided $R/J(R)$ is regular and idempotents lift modulo $J(R)$. It is well known that semi-regular rings are always exchange, but the opposite is generally untrue (see, for instance, [19]).

In 2019, Fatih Karabacak *et al.* introduced new rings that are a proper expansion of UJ rings. They called these rings ΔU in [8], namely a ring R is said to be ΔU if $1 + \Delta(R) = U(R)$.

So, as a possible non-trivial extension of ΔU rings, we introduce the concept of an n-ΔU ring. A ring R is called n-ΔU if, for each $u \in U(R)$, $u^n - 1 \in \Delta(R)$, where $n \geq 2$ is a fixed integer. Clearly, all ΔU rings and rings with only two units are n-ΔU. Also, every n-UJ ring is n-ΔU, but the reciprocal implication does *not* hold in all generality.

Our basic material is organized as follows: In the next section, we examine the behavior of n-ΔU rings comparing their crucial properties with these of the ΔU rings (see, for instance, Theorems 2.15, 2.23, 2.27 and 2.28, respectively). In the third section, we concentrate on the structure of some key extensions of n-ΔU ring demonstrating that there is an abundance of their critical properties (see, e.g., Propositions 3.1, 3.2, 3.4, 3.7, 3.8, 3.12, 3.14, etc. and Theorem 3.5). In closing, we pose a challenging question which, hopefully, will motivate a further research study of the explored subject.

2. n-ΔU RINGS

In this section, we begin by introducing the notion of n-ΔU rings and investigate its elementary properties. We now give our main tools.

Definition 2.1. A ring R is called n-ΔU if, for each $u \in U(R)$, $u^n - 1 \in \Delta(R)$, where $n \geq 2$ is a fixed integer.

Definition 2.2. A ring R is called π-ΔU if, for any $u \in U(R)$, there exists $i \geq 2$ depending on u such that $u^i - 1 \in \Delta(R)$.

According to the above two definitions, we observe that every ΔU ring is obviously an n-ΔU ring and that every n-ΔU ring is a π-ΔU ring. Besides, it is easy to see that if R is a finite π-ΔU ring, then one can find some number $m \in \mathbb{N}$ such that R is an m-ΔU ring.

We now arrive at the following construction.

Example 2.3. Once again, it is clear that n-UJ rings are always n-ΔU. However, the converse claim is generally invalid. For example, consider the ring $R = \mathbb{F}_2\langle x, y\rangle/\langle x^2\rangle$. Then, one calculates that $J(R) = \{0\}$, $\Delta(R) = \mathbb{F}_2x + xRx$ and $U(R) = 1 + \mathbb{F}_2x + xRx$. Thus, R is ΔU in view of [8, Example 2.2] and hence it is n-ΔU. But, evidently, R is *not* n-UJ.

We continue with the following technicalities.

Proposition 2.4. *Let R be an n-ΔU ring, where n is an odd number. Then, $2 \in \Delta(R)$.*

Proof. Writing $-1 = (-1)^n \in 1 + \Delta(R)$ whence $-2 \in \Delta(R)$, we apply [16, Lemma 1(2)] to conclude that $2 \in \Delta(R)$, as formulated. ■

Remark 2.5. The condition "n is an odd number" in Proposition 2.4 is essential. For instance, $\mathbb{Z}_6$ is a 2-ΔU ring, but a simple computation shows that $2 \notin \Delta(\mathbb{Z}_6)$.

Proposition 2.6. *Let R be an n-ΔU ring and $k \in \mathbb{N}$ such that $n|k$. Then, R is a k-ΔU ring.*

Proof. Since R is an n-ΔU ring, for any $u \in U(R)$ we may write that $u^n = 1 + r$, where $r \in \Delta(R)$. Since $n|k$, there exists an integer t such that $k = tn$. Thus,

$$u^k = (u^n)^t = (1 + r)^t = 1 + r',$$

where $r' = (1 + r)^t - 1$, which is obviously in $\Delta(R)$ because it is a subring of R. Therefore, $u^k = 1 + r'$, where $r' \in \Delta(R)$. Hence, R is a k-ΔU ring, as stated. ■

Proposition 2.7. *A division ring D is n-ΔU if, and only if, $u^n = 1$ for every $u \in U(D)$.*

Proof. It is straightforward by noticing that for any division ring D we have $\Delta(D) = \{0\}$. ■

Lemma 2.8. *Suppose $\mathbb{F}$ is a field. Then, $\mathbb{F}$ is n-ΔU if, and only if, $\mathbb{F}$ is finite and $(|\mathbb{F}| - 1)|n$.*

Proof. Let $f(x) = 1 - x^n \in \mathbb{F}[x]$. Since $\mathbb{F}$ is a field, the polynomial $f(x)$ has at most n roots in $\mathbb{F}^*$. So, if we suppose A to be the set of all roots of f in $\mathbb{F}^*$, we will have $\mathbb{F}^* = A$. Consequently, $|\mathbb{F}^*| = |A| < n$.

On the other hand, as $\mathbb{F}^*$ is a cyclic group, there exists $a \in \mathbb{F}^*$ such that $\mathbb{F}^* = \langle a \rangle$. Since $a^n = 1$, we get $o(a)|n$, and hence $n = o(a)q = |\mathbb{F}^*|q$. Therefore, $|\mathbb{F}^*||n$ and, finally, $(|\mathbb{F}| - 1)|n$, as pursued.
The reverse implication is elementary. ■

Lemma 2.9. *Let D be a division ring and $n \geq 2$. If D is n-ΔU, then D is a finite field and $(|D| - 1)|n$.*

Proof. Certainly, $\Delta(D) = \{0\}$. So, for any $a \in D$, we have $a^n = 1$, whence $a = a^{n+1}$. Furthermore, appealing to the famous Jacobson's Theorem [14, 12.10], we detect that D must be commutative, and thus a field, as expected.

The second part follows at once from Lemma 2.8. ■

Corollary 2.10. *If D is a division ring which is π-ΔU, then D is a field.*

Example 2.11. Consider the ring $\mathbb{Z}$. Knowing that $U(\mathbb{Z}) = \{1, -1\}$, it is not too hard to see that $\Delta(\mathbb{Z}) = \{0\}$. Hence, $\mathbb{Z}$ is an n-ΔU. Nevertheless, for an arbitrary prime number p, the ring $\mathbb{Z}_p$ is *not* n-ΔU for every n unless $p-1$ divides n by Lemma 2.8.

Proposition 2.12. *A direct product $\prod_{i\in I} R_i$ of rings R_i is n-ΔU if, and only if, each direct component R_i is n-ΔU.*

Proof. As the equalities $\Delta(\prod_{i\in I} R_i) = \prod_{i\in I} \Delta(R_i)$ and $U(\prod_{i\in I} R_i) = \prod_{i\in I} U(R_i)$ are fulfilled, the result follows at once. ■

Proposition 2.13. *Let R be an n-ΔU ring. If T is an epimorphic image of R such that all units of T lift to units of R, then T is n-ΔU.*

Proof. Suppose that $f : R \to T$ is a ring epimorphism. Let $v \in U(T)$. Then, there exists $u \in U(R)$ such that $v = f(u)$ and $u^n = 1 + r \in 1 + \Delta(R)$. Thus, we have

$$v^n = (f(u))^n = f(u^n) = f(1 + r) = f(1) + f(r) = 1 + f(r) \in 1 + \Delta(T),$$

as asked for. ■

Proposition 2.14. *Let R be an n-ΔU. For any unital subring S of R, if $S \cap \Delta(R) \subseteq \Delta(S)$, then S is an n-ΔU ring. In particular, the center of R is an n-ΔU ring.*

Proof. Let $v \in U(S) \subseteq U(R)$. Since R is n-ΔU, we have $v^n - 1 \in \Delta(R) \cap S \subseteq \Delta(S)$. So, S is necessarily an n-ΔU ring. The rest of the statement follows directly from [16, Corollary 8]. ■

Our first major assertion is the following necessary and sufficient condition.

Theorem 2.15. *Let $I \subseteq J(R)$ be an ideal of a ring R. Then R is n-ΔU if, and only if, so is R/I.*

Proof. Let R be n-ΔU and $u + I \in U(R/I)$. Then, $u \in U(R)$ and thus $u^n = 1 + r$, where $r \in \Delta(R)$. Now, $(u + I)^n = u^n + I = (1 + I) + (r + I)$, where $r + I \in \Delta(R)/I = \Delta(R/I)$ in virtue of [16, Proposition 6].

Conversely, let R/I be n-ΔU and $u \in U(R)$. Then, $u + I \in U(R/I)$ whence $(u+I)^n = (1+I)+(r+I)$, where $r+I \in \Delta(R/I)$. Thus, $u^n+I = (1+r)+I$ and so $u^n - (1 + r) \in I \subseteq J(R) \subseteq \Delta(R)$. Therefore, $u^n = 1 + r'$, where $r' \in \Delta(R)$. Hence, R is n-ΔU, as required. ■

As an automatic consequence, we extract:

Corollary 2.16. *A ring R is n-ΔU if, and only if, $R/J(R)$ is n-ΔU.*

We next proceed by proving the following structural affirmations.

Proposition 2.17. *Let R be an n-ΔU (resp., a π-ΔU) ring and let e be an idempotent of R. Then, eRe is an n-ΔU (resp., a π-ΔU) ring.*

Proof. Let $u \in U(eRe)$. Thus, $u + (1 - e) \in U(R)$. By hypothesis,

$$(u + (1 - e))^n = u^n + (1 - e) = 1 + r \in 1 + \Delta(R).$$

So, we have $u^n - e \in \Delta(R)$. Now, we show that $u^n - e \in \Delta(eRe)$. Let v be an arbitrary unit of eRe. Apparently, $v + 1 - e \in U(R)$. Note that $u^n - e \in \Delta(R)$ gives us that $u^n - e + v + 1 - e \in U(R)$ utilizing the definition of $\Delta(R)$. Taking $u^n - e + v + 1 - e = t \in U(R)$, one checks that

$$et = te = ete = u^n - e + v,$$

and so $ete \in U(eRe)$. It now follows that $u^n - e + U(eRe) \subseteq U(Re)$. Then, we deduce $u^n - e \in \Delta(eRe)$ implying $u^n \in e + \Delta(eRe)$ which yields that the corner ring eRe is an n-ΔU ring, as wanted.

The case of π-ΔU rings is quite similar, so we omit the arguments. ■

Proposition 2.18. *For any ring $R \neq \{0\}$ and any integer $n \geq 2$, the ring $M_n(R)$ is not a $(2k - 1)$-ΔU ring whenever $k \geq 1$.*

Proof. Since it is long known that $M_2(R)$ is isomorphic to a corner subring of $M_n(R)$ for $n \geq 2$, it suffices to show that $M_2(R)$ is not a $(2k-1)$-ΔU ring bearing in mind Proposition 2.17. To this goal, consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in U(M_2(R)).$$

Thus, $A^{2k-1} = A$ or $A^{2k-1} = -A$. Now, let $M_2(R)$ be $(2k - 1)$-ΔU. If firstly $A^{2k-1} = A$, then we conclude that

$$B := A - I = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \in \Delta(M_2(R)).$$

But, we know that B is a unit. So, utilizing [16, Lemma 1], we infer that $BB^{-1} \in \Delta(M_2(R))$ and hence $I \in \Delta(M_2(R))$. This, however, is an obvious contradiction.

If now $A^{2k-1} = -A$, it can be concluded that $I \in \Delta(M_2(R))$ and again this is a contraposition. So, $M_2(R)$ is really not a $(2k - 1)$-ΔU ring, as desired. ■

Example 2.19. Consider the matrix ring $R = M_2(\mathbb{Z}_2)$. We have

$$U(R) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

With a simple calculation at hand, we may derive that, for any $u \in U(R)$, $u^6 - 1 \in \Delta(R)$. So, R is a 6-ΔU ring. In general, $M_n(\mathbb{Z}_2)$ $(n \geq 2)$ is *not* n-ΔU if n is an even number. However, this observation does *not* hold in general for odd values of n.

Let us now recollect that a set $\{e_{ij} : 1 \leq i, j \leq n\}$ of non-zero elements of R is said to be a system of n^2 *matrix units* if $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{jj} = 1$ and $\delta_{js} = 0$ for $j \neq s$. In this case, $e := \sum_{i=1}^{n} e_{ii}$ is an idempotent of R and $eRe \cong M_n(S)$, where

$$S = \{r \in eRe : re_{ij} = e_{ij}r \ \text{ for all } \ i, j = 1, 2, \ldots, n\}.$$

Recall also that a ring R is said to be *Dedekind-finite* provided $ab = 1$ implies $ba = 1$ for any two $a, b \in R$. In other words, all one-sided inverse elements in the ring must be two-sided.

We are now prepared to establish the following.

Proposition 2.20. *Every $(2k-1)$-ΔU ring is Dedekind-finite, provided $k \geq 1$.*

Proof. If we assume the contrary that R is *not* a Dedekind-finite ring, then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Assuming $e_{ij} = a^i(1 - ba)b^j$ and $e = \sum_{i=1}^{n} e_{ii}$, there exists a non-zero ring S such that $eRe \cong M_n(S)$. However, owing to Proposition 2.17, eRe is a $(2k-1)$-ΔU ring, so $M_n(S)$ has to be a $(2k-1)$-ΔU ring too, which contradicts Proposition 2.18, as expected. ∎

Recall that a ring R is said to be *semi-local* if $R/J(R)$ is a left artinian ring or, equivalently, if $R/J(R)$ is a semi-simple ring.

Proposition 2.21. *Let R be a ring and $n \geq 1$. Then, the following two conditions are equivalent for a semi-local ring:*

(i) *R is a $(2n-1)$-ΔU ring.*

(ii) *$R/J(R) \cong \prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$, where $(p^{k_i} - 1) | n$ and $\mathbb{F}_{p^{k_i}}$ is a field with p^{k_i} elements.*

Proof. (i) $\Rightarrow$ (ii) Since R is semi-local, $R/J(R)$ is semi-simple, so we have

$$R/J(R) \cong \prod_{i=1}^{m} \mathrm{M}_{n_i}(D_i),$$

where each D_i is a division ring. Then, employing Corollary 2.16 and Proposition 2.18, we deduce that $R/J(R) \cong \prod_{i=1}^{m} D_i$. On the other hand, invoking Lemma 2.9, we derive that $D_i \cong \mathbb{F}_{p^{k_i}}$, where $p^{k_i} - 1$ divides n, as claimed.

(ii) $\Rightarrow$ (i) According to Lemma 2.8, we conclude that every $\mathbb{F}_{p^{k_i}}$ is $(2n-1)$-ΔU for all i. Then, taking into account Proposition 2.12, we receive that $\prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$ is $(2n-1)$-ΔU and hence $R/J(R)$ is $(2n-1)$-ΔU. Thus, R is a $(2n-1)$-ΔU ring in accordance with Corollary 2.16, as asserted. ■

Lemma 2.22. *Let R be a $(2n-1)$-ΔU ring for some $n \geq 1$. If $J(R) = \{0\}$ and every non-zero right ideal of R contains a non-zero idempotent, then R is reduced.*

Proof. Suppose the reverse that R is *not* reduced. Then, there exists a non-zero element $a \in R$ such that $a^2 = 0$. Referring to [17, Theorem 2.1], there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T. However, thanks to Proposition 2.17, eRe is a $(2n-1)$-ΔU ring and hence $M_2(T)$ is a $(2n-1)$-ΔU ring as well. This, in turn, contradicts Proposition 2.18, as expected. ■

It is well known that a ring R is called *π-regular* if, for each a in R, $a^n \in a^n R a^n$ for some integer n. So, regular rings are always π-regular. Also, strongly π-regular rings are themselves π-regular.

Our second main statement is the following.

Theorem 2.23. *Let R be a ring and $n \geq 1$. The following three items are equivalent:*

(i) *R is a regular $(2n-1)$-ΔU ring.*

(ii) *R is a π-regular reduced $(2n-1)$-ΔU ring.*

(iii) *R has the identity $x^{2n} = x$.*

Proof. (i) $\Rightarrow$ (ii) Since R is regular, $J(R) = \{0\}$ and thus every non-zero right ideal contains a non-zero idempotent. So, Lemma 2.22 applies to get that R is reduced. Moreover, every regular ring is known to be π-regular and so the implication follows immediately, as promised.

(ii) $\Rightarrow$ (iii) Notice that reduced rings are always abelian, so R is abelian regular by [1, Theorem 3] and hence it is strongly regular. Then, R is unit-regular and so $\Delta(R) = \{0\}$ by [16, Corollary 16]. Thus, we have $Nil(R) = J(R) = \Delta(R) = \{0\}$.

On the other hand, one observes that R is strongly π-regular. Let $x \in R$. In view of [7, Proposition 2.5], there is an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$, $ex = xe \in Nil(R) = \{0\}$. So, it must be that

$$x = x - xe = x(1-e) = u(1-e) = (1-e)u.$$

But, since R is a $(2n-1)$-ΔU ring, $u^{2n-1} = 1$. It follows now that

$$x^{2n-1} = ((1-e)u)^{2n-1} = u^{2n-1}(1-e)^{2n-1} = (1-e).$$

Hence, $x = x(1-e) = x.x^{2n-1} = x^{2n}$, and we are done.

(iii) $\Rightarrow$ (i) It is trivial that R is regular. Let $u \in U(R)$. Then, we have $u^{2n} = u$ forcing that $u^{2n-1} = 1$ and thus R is a $(2n-1)$-ΔU ring, as promised. ■

We now can record the following interesting consequence.

Corollary 2.24. *Suppose $n \geq 1$. The following four conditions are equivalent for a ring R:*

(i) *R is a regular $(2n-1)$-ΔU ring.*

(ii) *R is a strongly regular $(2n-1)$-ΔU ring.*

(iii) *R is a unit-regular $(2n-1)$-ΔU ring.*

(iv) *R has the identity $x^{2n} = x$.*

Proof. (i) $\Rightarrow$ (ii) In virtue of Lemma 2.22, R is reduced and hence abelian. Then, R is strongly regular.

(ii) $\Rightarrow$ (iii) This is pretty obvious, so we leave out the argumentation.

(iii) $\Rightarrow$ (iv) Let $x \in R$. Then, $x = ue$ for some $u \in U(R)$ and $e \in Id(R)$. We know that every unit-regular ring is by definition regular, so R is regular $(2n-1)$-ΔU whence R is abelian. On the other hand, [16, Corollary 16] leads us to $\Delta(R) = \{0\}$. Therefore, for any $u \in U(R)$, we have $u^{2n-1} = 1$ which means that $x^{2n-1} = u^{2n-1}e^{2n-1} = e$. So, we detect that $x^{2n} = x$, as required.

(iv) $\Rightarrow$ (i) It is clear by a direct appeal to Theorem 2.23. ■

Let us recall that a ring R is called *semi-potent* if every one-sided ideal *not* contained in $J(R)$ contains a non-zero idempotent.

The next difficult question arises quite logical.

Problem 2.25. Characterize semi-potent n-ΔU rings for an arbitrary $n \geq 1$.

The following technical claim is useful.

Proposition 2.26. *Suppose $k \geq 1$. Then, a ring R is ΔU if, and only if,*

(i) *$2 \in \Delta(R)$,*

(ii) *R is a 2^k-ΔU ring.*

(iii) *If, for every $x \in R$, $x^{2^k} \in \Delta(R)$, then $x \in \Delta(R)$.*

Proof. "$\Rightarrow$" As R is a ΔU ring, then $-1 = 1+r$ for some $r \in \Delta(R)$. This implies that $-2 \in \Delta(R)$ and so $2 \in \Delta(R)$. Besides, every ΔU ring is 2^k-ΔU. Now, the asked result follows from [8, Proposition 2.4(3)].

"$\Leftarrow$" Let $u \in U(R)$. By (ii), we have $u^{2^k} \in 1+\Delta(R)$ and hence, combining [16, Theorem 3(2) and Lemma 1(3)] with (i), we conclude that $(u-1)^{2^k} = 1+u^{2k}+r$ for some $r \in \Delta(R)$. So, $(u-1)^{2^k} \in \Delta(R)$. Thus, with the help of (iii), we infer that $u-1 \in \Delta(R)$, which ensures that R is a ΔU-ring, as required. ■

We now come to the next two pivotal assertions.

Theorem 2.27. *Let R be a $(2n-1)$-ΔU ring. Then, the following two points are equivalent:*

(i) *R is an exchange ring.*

(ii) *R is a clean ring.*

Proof. (ii) $\Rightarrow$ (i) This is obvious, because each clean ring is always exchange.

(i) $\Rightarrow$ (ii) If R is simultaneously exchange and $(2n-1)$-ΔU, then R is reduced thanking to Lemma 2.22, and hence it is abelian. Therefore, R is abelian exchange, so it is clean. ■

Theorem 2.28. *Let R be a (2^k-1)-ΔU ring for some $k \geq 1$. Then, the following three statements are equivalent:*

(i) *R is a semi-regular ring.*

(ii) *R is an exchange ring.*

(iii) *R is a clean ring.*

Proof. Observe that (ii) and (iii) are equivalent employing Theorem 2.27.

(i) $\Rightarrow$ (ii) This is obvious, since every semi-regular ring is always exchange.

(iii) $\Rightarrow$ (i) First, we show that $2 \in J(R)$. To this end, Proposition 2.4 assures that $2 \in \Delta(R)$. Let $r \in R$ and $r = e + u$ be a clean decomposition for r. We know that $2e-1 \in U(R)$ and hence $(2e-1) = (2e-1)^{2^k-1} \in 1 + \Delta(R)$, so that $2e \in \Delta(R)$. Thus, $2r = 2e + 2u \in \Delta(R) + \Delta(R) \subseteq \Delta(R)$. So, $1 - 2r \in U(R)$ and hence $2 \in J(R)$, as claimed.

On the other hand, $r^{2^k} = e + 2f + u^{2^k}$, where $f \in R$. So,

$$\begin{aligned} r - r^{2^k} &= (e+u) - \left(e + 2f + u^{2^k}\right) = (e+u) - \left(e + 2f + u\left(u^{2^k-1}\right)\right) \\ &= (e+u) - (e + 2f + u + d), \end{aligned}$$

whence

$$r - r^{2^k} = -(2f + d) \in \Delta(R),$$

where $d \in \Delta(R)$. Consider now $\overline{R} = R/J(R)$, where $\overline{R}$ is reduced and so abelian enabled via Lemma 2.22.

Next, we prove that $\Delta(R) = J(R)$. Letting $d \in \Delta(R)$ and $e \in Id(R)$, we have $1 - ed = f + u$, where $f \in Id(R)$ and $u \in U(R)$. So, $\overline{1} - \overline{e}\overline{d} = \overline{f} + \overline{u}$ and multiplying by the expression $\overline{(1-e)}$ on the left the previous equality, we derive that $\overline{(1-e)} = \overline{(1-e)}\overline{f} + \overline{(1-e)}\overline{u}$. Then, one inspects that

$$\overline{(1-e)}\,\overline{(1-f)} = \overline{(1-e)}\overline{u} \in U(\overline{(1-e)}\,\overline{R}\,\overline{(1-e)}) \cap Id(\overline{(1-e)}\,\overline{R}\,\overline{(1-e)}).$$

Consequently, $\overline{(1-e)}\,\overline{(1-f)} = \overline{(1-e)}$, so again using this trick for the expression $\overline{f}$ on the right of the previous equality, we deduce that $\overline{(1-e)}\overline{f} = \overline{0}$, so that $\overline{f} = \overline{e}\overline{f} \in \overline{e}\overline{R}\overline{e}$.

Furthermore, if we multiply the equation $\overline{1} - \overline{e}\overline{d} = \overline{f} + \overline{u}$ by $\overline{e}$ on the left, we will have $\overline{e} - \overline{e}\overline{d} = \overline{e}\overline{f} + \overline{e}\overline{u} = \overline{f} + \overline{e}\overline{u}$. Hence,

$$\overline{e} - \overline{f} = \overline{e}(\overline{u} + \overline{d}) \in U(\overline{e}\overline{R}\overline{e}) \cap Id(\overline{e}\overline{R}\overline{e}),$$

and so $\overline{e} - \overline{f} = \overline{e}$ concluding that $\overline{f} = \overline{0}$. Then, $f \in J(R) \cap Id(R) = \{0\}$. Thus, $f = 0$ and hence $1 - ed \in U(R)$.

On the other side,

$$1 - rd = 1 - ed - ud \in U(R) + \Delta(R) \subseteq U(R),$$

and we conclude that $d \in J(R)$. Hence, $r - r^{2^k} \in J(R)$. Thus, the quotient $\frac{R}{J(R)}$ is regular and also idempotents lift modulo $J(R)$, because by hypothesis R is a clean ring, whence finally R is a semi-regular ring, as required. ■

3. Some extensions of n-ΔU rings

As usual, we say that B is a unital subring of a ring A if $\emptyset \neq B \subseteq A$ and, for any $x, y \in B$, the relations $x - y$, $xy \in B$ and $1_A \in B$ hold. Let A be a ring and let B a unital subring of A, we denote by $R[A,B]$ the set

$$\{(a_1, \ldots, a_n, b, b, \ldots) : a_i \in A, b \in B, 1 \leq i \leq n\}.$$

Then, a routine check establishes that $R[A,B]$ forms a ring under the usual component-wise addition and multiplication. The ring $R[A,B]$ is called the *tail ring extension.*

We start our considerations here with the following helpful statement.

Proposition 3.1. *$R[A,B]$ is an n-ΔU ring if, and only if, both A and B are n-ΔU rings.*

Proof. Suppose $R[A,B]$ is an n-ΔU ring. Firstly, we prove that A is an n-ΔU ring. Let $u \in U(A)$. Then, $\bar{u} = (u, 1, 1, \ldots) \in U(R[A,B])$. By hypothesis, we have $(u^n - 1, 0, 0, \ldots) \in \Delta(R[A,B])$, so $(u^n - 1, 0, 0, \ldots) + U(R[A,B]) \subseteq U(R[A,B])$. Thus, for all $v \in U(A)$,

$$(u^n - 1 + v, 1, 1, \ldots) = (u^2 - 1, 0, 0, \ldots) + (v, 1, 1, \ldots) \in U(R[A,B]).$$

Hence, $u^n - 1 + v \in U(A)$, which insures that $u^n - 1 \in \Delta(A)$. Now, we show that B is an n-ΔU ring. To this target, choose $v \in U(B)$. Then, $(1, \ldots, 1, 1, v, v, \ldots) \in U(R[A,B])$. By hypothesis, $(0, \ldots, 0, v^n - 1, v^n - 1, \ldots) \in \Delta(R[A,B])$, so

$$(0, \ldots, 0, v^n - 1, v^n - 1, \ldots) + U(R[A,B]) \subseteq U(R[A,B]).$$

Thus, for all $u \in U(B)$,

$$(1, 1, \ldots, v^n - 1 + u, v^n - 1 + u, \ldots) \in U(R[A, B]).$$

We have $v^n - 1 + u \in U(B)$ and hence $v^n - 1 \in \Delta(B)$, as required.

Conversely, assume that A and B are both n-ΔU rings. Let

$$\bar{u} = (u_1, u_2, \ldots, u_t, v, v, \ldots) \in U(R[A, B]),$$

where $u_i \in U(A)$ and $v \in U(B) \subseteq U(A)$. We must show that $\bar{u}^n - 1 + U(R[A, B]) \subseteq U(R[A, B])$. In fact, for all $\bar{a} = (a_1, \ldots, a_m, b, b, \ldots) \in U(R[A, B])$ with $a_i \in U(A)$ and $b \in U(B) \subseteq U(A)$, take $z = \max\{m, t\}$. Then, we obtain

$$\bar{u}^n - 1 + \bar{a} = (u_1^n - 1 + a_1, \ldots, u_z^2 - 1 + a_z, v^n - 1 + b, v^n - 1 + b, \ldots).$$

Note that $u_i^n - 1 + a_i \in U(A)$ for all $1 \leq i \leq z$ and $v^n - 1 + b \in U(B) \subseteq U(A)$. We, thereby, deduce that $\bar{u}^n - 1 + \bar{a} \in U(R[A, B])$. Thus, $\bar{u}^n - 1 \in \Delta(R[A, B])$ and $\bar{u}^n \in 1 + \Delta(R[A, B])$. This unambiguously enables us that $R[A, B]$ is an n-ΔU ring, as asserted. ■

Let R be a ring and suppose that $\alpha : R \to R$ is a ring endomorphism. Traditionally, $R[[x; \alpha]]$ denotes the ring of *skew formal power series* over R; that is, all formal power series in x having coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x; 1_R]]$ is the ring of *formal power series* over R.

Proposition 3.2. *The ring $R[[x; \alpha]]$ is n-ΔU if, and only if, so is R.*

Proof. Consider $I = R[[x; \alpha]]x$. Then, a plain check gives that I is an ideal of $R[[x; \alpha]]$. Note that $J(R[[x; \alpha]]) = J(R) + I$, so $I \subseteq J(R[[x; \alpha]])$. Since $R[[x; \alpha]]/I \cong R$, the result follows at once exploiting Theorem 2.15. ■

As an automatic consequence, we yield.

Corollary 3.3. *The ring $R[[x]]$ is n-ΔU if, and only if, so is R.*

Let R be a ring and suppose that $\alpha : R \to R$ is a ring endomorphism. Standardly, $R[x; \alpha]$ denotes the ring of *skew polynomials* over R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[x] = R[x; 1_R]$ is the ring of *polynomials* over R. For an endomorphism α of a ring R, R is called *α-compatible* if, for any $a, b \in R$, $ab = 0 \Longleftrightarrow a\alpha(b) = 0$, as in this case α is evidently injective.

Let $Nil_*(R)$ denote the *prime radical* (or, in other terms, the *lower nil-radical*) of a ring R, i.e., the intersection of all prime ideals of R. We know that $Nil_*(R)$ is a nil-ideal of R. It is long known that a ring R is called 2-*primal* if its lower nil-radical $Nil_*(R)$ consists precisely of all the nilpotent elements of R. For instance, it is well known that both reduced and commutative rings are 2-primal.

Proposition 3.4. *Let R be a 2-primal and α-compatible ring. Then, the equality $\Delta(R[x,\alpha]) = \Delta(R) + Nil_*(R[x,\alpha])x$ is valid.*

Proof. Assuming $f = \sum_{i=0}^{n} a_i x^i \in \Delta(R[x,\alpha])$, then, for every $u \in U(R)$, we have that $1 - uf \in U(R[x,\alpha])$. Thus, [2, Corollary 2.14] employs to get that $1 - ua_0 \in U(R)$ and, for every $1 \le i \le n$, the relation $ua_i \in Nil_*(R)$ is true. Since $Nil_*(R)$ is an ideal, it must be that $a_0 \in \Delta(R)$ and, for every $1 \le i \le n$, the relation $a_i \in Nil_*(R)$ holds. But, as R is a 2-primal ring, [2, Lemma 2.2] is applicable to conclude that $Nil_*(R)[x,\alpha] = Nil_*(R[x,\alpha])$, as required.

Reciprocally, assume $f \in \Delta(R) + Nil_*(R[x,\alpha])x$ and $u \in U(R[x,\alpha])$. Then, owing to [2, Corollary 2.14], we have $u \in U(R) + Nil_*(R[x,\alpha])x$. Since R is a 2-primal ring, one has that

$$1 - uf \in U(R) + Nil_*(R[x,\alpha])x \subseteq U(R[x,\alpha]),$$

and thus $f \in \Delta(R[x,\alpha])$, as needed. ∎

We are now in a position to establish the following criterion.

Theorem 3.5. Let R be a 2-primal ring and α an endomorphism of R such that R is α-compatible. The following are equivalent:

(i) $R[x;\alpha]$ is an n-ΔU ring.

(ii) R is an n-ΔU ring.

Proof. (ii) $\Rightarrow$ (i) Let $f = \sum_{i=0}^{n} u_i x^i \in U(R[x,\alpha])$, so in view of [2, Corollary 2.14] one arrives at $u_0 \in U(R)$ and $u_i \in Nil(R)$ for each $i \ge 1$. Then, by hypothesis, $1 - u_0^n \in \Delta(R)$. Therefore, with [2, Corollary 2.14] at hand, there exists $g \in Nil_*(R)[x;\alpha]$ such that

$$f^n = u_0^n + gx \in 1 + \Delta(R) + Nil_*(R[x,\alpha])x,$$

and hence with the aid of Proposition 3.4 we obtain

$$f^n \in 1 + \Delta(R[x;\alpha]).$$

(i) $\Rightarrow$ (ii) Let $u \in U(R) \subseteq U(R[x;\alpha])$. Hence,

$$u^n \in 1 + \Delta(R[x;\alpha]) = 1 + \Delta(R) + Nil_*(R[x,\alpha])x.$$

Thus, we have $u^n \in 1 + \Delta(R)$ whence R is an n-ΔU ring, as wanted. ∎

As a valuable consequence, we arrive at the following.

Corollary 3.6. *Let R be a 2-primal ring. Then, the following are equivalent:*

(i) *$R[x]$ is an n-ΔU ring.*

(ii) *R is an n-ΔU ring.*

Let R be a ring and M a bi-module over R. The *trivial extension* of R and M is stated as

$$T(R, M) = \{(r, m) : r \in R \text{ and } m \in M\},$$

with addition defined component-wise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

One knows that the trivial extension $T(R, M)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M \right\}$$

of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. We also notice that the set of units of the trivial extension $T(R, M)$ is precisely

$$U(T(R, M)) = T(U(R), M).$$

Also, by [8], one may exactly write that

$$\Delta(T(R, M)) = T(\Delta(R), M).$$

We are now ready to prove the following.

Proposition 3.7. *Let R be a ring and M a bi-module over R. Then, the following hold:*

(i) *The trivial extension* $\mathrm{T}(R, M)$ *is an n-ΔU ring if, and only if, R is an n-ΔU ring.*

(ii) *The upper triangular matrix ring $T_n(R)$ is an n-ΔU if, and only if, R is an n-ΔU ring.*

Proof. (i) Set $A = \mathrm{T}(R, M)$ and consider the ideal $I := \mathrm{T}(0, M)$. Then, one finds that $I \subseteq J(A)$ such that $\frac{A}{I} \cong R$. So, the result follows directly from Theorem 2.15.

(ii) Let $I = \{(a_{ij}) \in T_n(R) \,|\, a_{ii} = 0\}$. Then, one establishes that $I \subseteq J(T_n(R))$ with $T_n(R)/I \cong R^n$. Therefore, the desired result follows from a plain combination of Theorem 2.15 and Proposition 2.12. ∎

Let α be an endomorphism of R and n a positive integer. It was defined by Nasr-Isfahani in [18] the *skew triangular matrix ring* like this

$$\mathrm{T}_n(R, \alpha) = \left\{ \left. \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \right| a_i \in R \right\}$$

with addition point-wise and multiplication given by:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ 0 & b_0 & b_1 & \cdots & b_{n-2} \\ 0 & 0 & b_0 & \cdots & b_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & b_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ 0 & c_0 & c_1 & \cdots & c_{n-2} \\ 0 & 0 & c_0 & \cdots & c_{n-3} \\ \ddots & \ddots & \ddots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix},$$

where

$$c_i = a_0\alpha^0(b_i) + a_1\alpha^1(b_{i-1}) + \cdots + a_i\alpha^i(b_0), \quad 1 \leq i \leq n-1.$$

We denote the elements of $\mathrm{T}_n(R,\alpha)$ by $(a_0, a_1, \ldots, a_{n-1})$. If α is the identity endomorphism, then one easily checks that $\mathrm{T}_n(R,\alpha)$ is a subring of the *upper triangular matrix ring* $\mathrm{T}_n(R)$.

All of the mentioned above guarantee the truthfulness of the following statement.

Proposition 3.8. *Let R be a ring and $k \geq 1$. Then, the following are equivalent:*

(i) $\mathrm{T}_n(R,\alpha)$ *is a k-ΔU ring.*

(ii) *R is a k-ΔU ring.*

Proof. Choose the set

$$I := \left\{ \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} \\ 0 & 0 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \middle| \, a_{ij} \in R \quad (i \leq j) \right\}.$$

Then, one easily verifies that $I \subseteq J(\mathrm{T}_n(R,\alpha))$ and $\frac{\mathrm{T}_n(R,\alpha)}{I} \cong R$. Consequently, Theorem 2.15 directly applies to get the expected result. ■

A simple manipulation with coefficients guarantees that there is a ring isomorphism

$$\varphi : \frac{R[x,\alpha]}{(x^n)} \to \mathrm{T}_n(R,\alpha),$$

given by

$$\varphi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + \langle x^n \rangle) = (a_0, a_1, \ldots, a_{n-1})$$

with $a_i \in R$, $0 \leq i \leq n-1$. So, one finds that $\mathrm{T}_n(R,\alpha) \cong \frac{R[x,\alpha]}{(x^n)}$, where (x^n) is the ideal generated by x^n.

We, thus, proceed by discovering the following two claims.

Corollary 3.9. *Let R be a ring and $k \geq 1$. Then, the following are equivalent:*

(i) *R is a k-ΔU ring.*

(ii) *For $n \geq 2$, the quotient-ring $\frac{R[x;\alpha]}{(x^n)}$ is a k-ΔU ring.*

(iii) *For $n \geq 2$, the quotient-ring $\frac{R[[x;\alpha]]}{(x^n)}$ is a k-ΔU ring.*

Corollary 3.10. *Let R be a ring. Then, the following are equivalent:*

(i) *R is a k-ΔU ring.*

(ii) *For $n \geq 2$, the quotient-ring $\frac{R[x]}{(x^n)}$ is a k-ΔU ring.*

(iii) *For $n \geq 2$, the quotient-ring $\frac{R[[x]]}{(x^n)}$ is a k-ΔU ring.*

Consider now R to be a ring and M to be a bi-module over R. Let

$$\mathrm{DT}(R,M) := \{(a,m,b,n) | a,b \in R, m,n \in M\}$$

with addition defined component-wise and multiplication defined by

$$\begin{aligned}&(a_1,m_1,b_1,n_1)(a_2,m_2,b_2,n_2)\\ &= (a_1a_2, a_1m_2+m_1a_2, a_1b_2+b_1a_2, a_1n_2+m_1b_2+b_1m_2+n_1a_2).\end{aligned}$$

Then, one claims that $\mathrm{DT}(R,M)$ is a ring which is isomorphic to $\mathrm{T}(\mathrm{T}(R,M), \mathrm{T}(R,M))$. Also, we have

$$\mathrm{DT}(R,M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} | a,b \in R, m,n \in M \right\}.$$

Likewise, one asserts that the following map is an isomorphism of rings: $\frac{R[x,y]}{\langle x^2,y^2\rangle} \to \mathrm{DT}(R,R)$, defined by

$$a+bx+cy+dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, detect the following.

Corollary 3.11. *Let R be a ring and M a bi-module over R. Then, the following statements are equivalent:*

(i) *R is an n-ΔU ring.*
(ii) *$\mathrm{DT}(R, M)$ is an n-ΔU ring.*
(iii) *$\mathrm{DT}(R, R)$ is an n-ΔU ring.*
(iv) *$\frac{R[x,y]}{\langle x^2,y^2\rangle}$ is an n-ΔU ring.*

Let A, B be two rings and M, N be (A, B)-bi-module and (B, A)-bi-module, respectively. Also, we consider the bi-linear maps $\phi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ that apply to the following properties:

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now, the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring with obvious matrix operations that is called a *Morita context* ring. Denote two-side ideals $Im\phi$ and $Im\psi$ to MN and NM, respectively, that are called the *trace ideals* of the *Morita context*.

We now have at our disposal all the ingredients necessary to establish the following.

Proposition 3.12. *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring. Then, R is a $(2n-1)$-ΔU ring if, and only if, both A, B are $(2n-1)$-ΔU and $MN \subseteq J(A)$, $NM \subseteq J(B)$.*

Proof. Let R be a $(2n-1)$-ΔU ring. Consider $e := \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$. Then, one says that $eRe \cong A$ and $(1-e)R(1-e) \cong B$. So, thankfully to Proposition 2.17, we get that A, B are both $(2n-1)$-ΔU. Obviously, $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in U(R)$. Therefore,

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}^{2n-1} = \begin{pmatrix} 1 & (2n-1)m \\ 0 & 1 \end{pmatrix} \in 1 + \Delta(R)$$

and hence $\begin{pmatrix} 0 & (2n-1)m \\ 0 & 0 \end{pmatrix} \in \Delta(R)$. Similarly, we obtain $\begin{pmatrix} 0 & 0 \\ (2n-1)m' & 0 \end{pmatrix} \in \Delta(R)$, where $m' \in N$. Since $2 \in \Delta(R), 2n-1 \in U(A)$, for any $m \in M$ and $m' \in N$ we receive that

$$\begin{pmatrix} (2n-1)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & (2n-1)m \\ 0 & 0 \end{pmatrix} \in \Delta(R).$$

Then, it must be that $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \Delta(R)$. Also,

$$\begin{pmatrix} 0 & 0 \\ (2n-1)m' & 0 \end{pmatrix} \begin{pmatrix} (2n-1)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \Delta(R).$$

Thus, $\begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} \in \Delta(R)$. Since $\Delta(R)$ is a subring, we have $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix} \in \Delta(R)$. Then, for any $m \in M$ and $m' \in N$, we have

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} \in \Delta(R) \Rightarrow \begin{pmatrix} MN & 0 \\ 0 & 0 \end{pmatrix} \in \Delta(R),$$

$$\begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in \Delta(R) \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & NM \end{pmatrix} \in \Delta(R).$$

Since $\Delta(R)$ is a subring, we can verify that $I := \begin{pmatrix} MN & M \\ N & NM \end{pmatrix} \subseteq \Delta(R)$ and I is an ideal, whence $I \subseteq J(R)$. Consequently, $MN \subseteq J(A)$ and $NM \subseteq J(B)$ invoking [20, Theorem 2.5], as required.

Reciprocally, let A, B be $(2n-1)$-ΔU, where $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Then, utilizing [20, Lemma 3.1], we derive that $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$. Thus, the isomorphism $\frac{R}{J(R)} \cong \frac{A}{J(A)} \times \frac{B}{J(B)}$ is fulfilled. Finally, that R is $(2n-1)$-ΔU is guaranteed by virtue of Proposition 2.12 and Corollary 2.16, as needed. ■

The next comments are worthwhile.

Remark 3.13. Exploiting Proposition 3.12, we have that if R is $(2n)$-ΔU, then both A, B are $(2n)$-ΔU and the containments $(2n)MN \subseteq J(A)$, $(2n)NM \subseteq J(B)$ hold. Now, a quite logical question arises that, if A, B are $(2n)$-ΔU, where $(2n)MN \subseteq J(A)$ and $(2n)NM \subseteq J(B)$, can it be deduced that R is a $(2n)$-ΔU ring?

However, the answer is negative as the following construction illustrates: letting $R := \mathbb{F}_2\langle x, y | x^2 = 0\rangle$, then it can be checked that R is 2-ΔU and $2R = \{0\}$, but $M_2(R)$ is not 2-ΔU.

Moreover, an other natural question arises, namely that if R is a $(2n)$-ΔU ring, whether it could be derived that $MN \subseteq J(A)$ and $NM \subseteq J(B)$?
Again, the answer is contrapositive, because we know that $M_2(\mathbb{Z}_2)$ is 6-ΔU; in fact, supposing $A = B = M = N = \mathbb{Z}_2$, then $R = M_2(\mathbb{Z}_2)$ is 6-ΔU, but $MN \nsubseteq J(A)$ and $NM \nsubseteq J(B)$, as it can be verified without any difficulty.

The following result could also be of some helpfulness and importance.

Proposition 3.14. *Let* $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *be a Morita context ring such that* $MN \subseteq J(A)$ *and* $NM \subseteq J(B)$. *Then,* R *is an* n-ΔU *ring if, and only if, both* A *and* B *are* n-ΔU.

Proof. In view of [20, Lemma 3.1], we argue that

$$J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$$

and hence the isomorphism $\frac{R}{J(R)} \cong \frac{A}{J(A)} \times \frac{B}{J(B)}$ holds. Then, the result follows immediately from Corollary 2.16 and Proposition 2.12. ∎

Now, let R, S be two rings, and let M be an (R, S)-bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-module M, we can set

$$\mathrm{T}(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it forms a ring with the usual matrix operations. The so-stated formal matrix $\mathrm{T}(R, S, M)$ is called a *formal triangular matrix ring.* In Proposition 3.14, if we set $N = \{0\}$, then we will obtain the following.

Corollary 3.15. *Let* R, S *be rings and let* M *be an* (R, S)*-bi-module. Then, the formal triangular matrix ring* $\mathrm{T}(R, S, M)$ *is an* n-ΔU *ring if, and only if, both* R *and* S *are* n-ΔU.

Given a ring R and a central element s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition component-wise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$. A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring* over R. It was observed by Krylov in [13] that a ring S is a generalized matrix ring over R if, and only if, $S = K_s(R)$ for some $s \in C(R)$. Here, $MN = NM = sR$, so $MN \subseteq J(A) \iff s \in J(R)$, $NM \subseteq J(B) \iff s \in J(R)$.

We, thus, have all the instruments to state the following.

Corollary 3.16. *Let* R *be a ring and* $s \in C(R) \cap J(R)$. *Then,* $K_s(R)$ *is an* n-ΔU *ring if, and only if,* R *is* n-ΔU.

Following Tang and Zhou (cf. [21]), for $n \geq 2$ and for $s \in C(R)$, the $n \times n$ *formal matrix ring* over R, defined with the usage of s and denoted by $M_n(R; s)$, is the set of all $n \times n$ matrices over R with the usual addition of matrices and with the multiplication defined below:

For (a_{ij}) and (b_{ij}) in $M_n(R; s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where} \ \ (c_{ij}) = \sum s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, where δ_{jk}, δ_{ij}, δ_{ik} are the standard *Kroncker* delta symbols.

We now offer the validity of the following.

Corollary 3.17. *Let R be a ring and $s \in C(R) \cap J(R)$. Then, for any $k \geq 1$, $M_n(R; s)$ is a k-ΔU ring if, and only if, R is k-ΔU.*

Proof. If $n = 1$, then $M_n(R; s) = R$. So, in this case, there is nothing to prove. Let $n = 2$. By the definition of $M_n(R; s)$, we have $M_2(R; s) \cong K_{s^2}(R)$. Apparently, $s^2 \in J(R) \cap C(R)$, so the claim holds for $n = 2$ with the help of Corollary 3.16.

To proceed by induction, assume now that $n > 2$ and that the claim holds for $M_{n-1}(R; s)$. Set $A := M_{n-1}(R; s)$. Then, $M_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a *Morita context*, where

$$M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \quad \text{and} \quad N = (M_{n1} \dots M_{n,n-1})$$

with $M_{in} = M_{ni} = R$ for all $i = 1, \dots, n-1$, and

$$\begin{aligned} \psi &: N \otimes M \to N, \quad n \otimes m \mapsto snm \\ \phi &: M \otimes N \to M, \quad m \otimes n \mapsto smn. \end{aligned}$$

Besides, for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = (y_{n1} \dots y_{n,n-1}) \in N$, we write

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \dots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \dots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \dots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in sA$$

and

$$yx = s^2 y_{n1}x_{1n} + s^2 y_{n2}x_{2n} + \cdots + s^2 y_{n,n-1}x_{n-1,n} \in s^2R.$$

Since $s \in J(R)$, we see that $MN \subseteq J(A)$ and $NM \subseteq J(A)$. Thus, we obtain that

$$\frac{M_n(R;s)}{J(M_n(R;s))} \cong \frac{A}{J(A)} \times \frac{R}{J(R)}.$$

Finally, the induction hypothesis and Proposition 3.14 yield the claim after all. ■

A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called *trivial* if the context products are trivial, i.e., $MN = (0)$ and $NM = (0)$. Consulting with [11], we now are able to establish that

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context. We, therefore, begin the proof-check of the following.

Corollary 3.18. *The trivial Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is an* n-ΔU *ring if, and only if, both* A *and* B *are* n-ΔU.

Proof. It is apparent to see that the two isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong \mathrm{T}(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}$$

are true. Then, the rest of the proof follows by combining Propositions 3.7(i) and 2.12, as needed. ■

As usual, for an arbitrary ring R and an arbitrary group G, the symbol RG stands for the *group ring* of G over R. Standardly, $\varepsilon(RG)$ designates the kernel of the classical *augmentation map* $\varepsilon : RG \to R$, defined by

$$\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g,$$

and this ideal is traditionally called the *augmentation ideal* of RG.

Here we will explore group rings that are n-ΔU, as for the case of JU group rings we refer the interested reader to [12]. Specifically, we continue by establishing the next three technicalities.

Lemma 3.19. *If* RG *is an* n-ΔU *ring, then* R *is too* n-ΔU.

Proof. Choosing $u \in U(R)$, then $u \in U(RG)$. Thus, $u^n = 1 + r$, where $r \in \Delta(RG)$. Since $-r = 1 - u^n \in R$, it suffices to show that $r \in \Delta(R)$, which is obviously true, because, for any $v \in U(R) \subseteq U(RG)$, we have $v - r \in U(RG) \cap R \subseteq U(R)$. Therefore, $r \in \Delta(R)$, as required. ■

We say that a group G is a *p-group* if the order of every element of G is a power of the prime number p. Besides, a group G is said to be *locally finite* if every finitely generated subgroup of G is finite.

In this light, the following two statements hold.

Lemma 3.20 [22, Lemma 2]**.** *Let p be a prime with $p \in J(R)$. If G is a locally finite p-group, then $\varepsilon(RG) \subseteq J(RG)$.*

Lemma 3.21. *If R is an n-ΔU ring and G is a locally finite p-group, where p is a prime number such that $p \in J(R)$, then RG is an n-ΔU ring.*

Proof. One looks that Lemma 3.20 tells us that $\varepsilon(RG) \subseteq J(RG)$. On the other hand, since the isomorphism $RG/\varepsilon(RG) \cong R$ holds, Theorem 2.15 is a guarantor that RG is an n-ΔU ring, as stated. ■

We close our work with the following intriguing problem.

Problem. Describe the structure of those rings R whose elements are a sum of a tripotent (or even of a potent) and an element from $\Delta(R)$ which commute each other.

Acknowledgement

The authors express their sincere gratitude to the expert referee for the numerous competent suggestions made which lead to a substantial improvement of the exposition.

Funding. The scientific work of the first-named author, P.V. Danchev, is partially supported by the project Junta de Andalucía under Grant FQM 264. All other four authors are supported by the Bonyad-Meli-Nokhbegan and receive funds from this foundation.

References

[1] A. Badawi, *On abelian π-regular rings*, Commun. Algebra **25(4)** (1997) 1009–1021. https://doi.org/10.1080/00927879708825906

[2] W. Chen, *On constant products of elements in skew polynomial rings*, Bull. Iran. Math. Soc. **41(2)** (2015) 453–462.

[3] P.V. Danchev, *Rings with Jacobson units*, Toyama Math. J. **38(1)** (2016) 61–74.

[4] P. V. Danchev, *On exchange π-UU unital rings*, Toyama Math. J. **39**(1) (2017) 1–7.

[5] P.V. Danchev, *On exchange π-JU unital rings*, Rend. Sem. Mat. Univ. Pol. Torino **77(1)** (2019) 13–23.

[6] P.V. Danchev, A. Javan and A. Moussavi, *Rings with $u^n - 1$ nilpotent for each unit u*, J. Algebra Appl. **25** (2026).
https://doi.org/10.1142/S0219498826500295

[7] A.J. Diesl, *Nil clean rings*, J. Algebra **383** (2013) 197–211.
https://doi.org/10.1016/j.jalgebra.2013.02.020

[8] F. Karabacak, M.T. Koşan, T. Quynh and D. Tai, *A generalization of UJ-rings*, J. Algebra Appl. **20** (2021).
https://doi.org/10.1142/S0219498821502170

[9] M.T. Koşan, A. Leroy and J. Matczuk, *On UJ-rings*, Commun. Algebra **46(5)** (2018) 2297–2303. https://doi.org/10.1080/00927872.2017.1388814

[10] M.T. Koşan, T.C. Quynh, T. Yildirim and J. Žemlička, *Rings such that, for each unit u, $u - u^n$ belongs to the Jacobson radical*, Hacettepe J. Math. Stat. **49(4)** (2020) 1397–1404.
https://doi.org/10.15672/hujms.542574

[11] M.T. Koşan, T.C. Quynh and J. Žemlička, *UNJ-Rings*, J. Algebra Appl. **19** (2020).
https://doi.org/10.1142/S0219498820501704

[12] M.T. Koşan and J. Žemlička, *Group rings that are UJ rings*, Commun. Algebra **49(6)** (2021) 2370–2377.
https://doi.org/10.1080/00927872.2020.1871000

[13] P.A. Krylov, *Isomorphism of generalized matrix rings*, Algebra Logic **47(4)** (2008) 258–262.
https://doi.org/10.1007/s10469-008-9016-y

[14] T.Y. Lam, A First Course in Noncommutative Rings, Second Edition (Springer Verlag, New York, 2001).

[15] T.Y. Lam, Exercises in Classical Ring Theory, Second Edition (Springer Verlag, New York, 2003).

[16] A. Leroy and J. Matczuk, *Remarks on the Jacobson radical*, Rings, Modules and Codes, Contemp. Math. **727**, Am. Math. Soc. (Providence, RI, 2019) 269–276.
https://doi.org/10.1090/conm/727/14640

[17] J. Levitzki, *On the structure of algebraic algebras and related rings*, Trans. Am. Math. Soc. **74** (1953) 384–409.

[18] A.R. Nasr-Isfahani, *On skew triangular matrix rings*, Commun. Algebra **39(11)** (2011) 4461–4469.
https://doi.org/10.1080/00927872.2010.520177

[19] W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Am. Math. Soc. **229** (1977) 269–278.
https://doi.org/10.2307/1998510

[20] G. Tang, C. Li and Y. Zhou, *Study of Morita contexts*, Commun. Algebra **42(4)** (2014) 1668–1681.
https://doi.org/10.1080/00927872.2012.748327

[21] G. Tang and Y. Zhou, *A class of formal matrix rings*, Linear Algebra Appl. **438** (2013) 4672–4688.
https://doi.org/10.1016/j.laa.2013.02.019

[22] Y. Zhou, *On clean group rings*, Advances in Ring Theory, Treads in Mathematics, Birkhauser (Verlag Basel/Switzerland, 2010) 335–345.
https://doi.org/10.1007/978-3-0346-0286-0-22

Received 1 November 2024
Revised 10 December 2024
Accepted 10 December 2024