Discussiones Mathematicae
General Algebra and Applications 45 (2025) 295–309
https://doi.org/10.7151/dmgaa.1482

DERIVATIONS AND SEMIDERIVATIONS IN SEMIRINGS

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Abstract

In this paper, we prove that if a non-trivial derivation of a semiring commutes then the semiring is commutative. We define semiderivation in semirings and derive some of its fundamental results. We prove that a map associated with a semiderivation is always a homomorphism in an additively cancellative yoked prime semiring. We also prove that a semiderivation σ of a semiring R induces a corresponding semiderivation σ^{Δ} in R^{Δ}, the ring of differences of the semiring R. This serves as a passage between rings and semirings and enable us to establish other conditions for commutativity of semirings with semiderivations. Our results generalizes the classical results in prime rings.

Keywords: derivations, semiderivations, semirings and commutativity.

2020 Mathematics Subject Classification: 16N60, 16U80, 16W25, 16Y60.

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1. Introduction

Posner's theorems [21] served as a foundation for Algebraists to investigate derivations in prime rings, semiprime rings and other algebraic structures. Researchers have extended the concept of derivations in many directions by defining Lie derivations [18], Jordan derivations [12, 15], inner derivations [4], reverse derivations [13, 15], generalized derivations [6], generalized reverse derivations [1, 13], semiderivations [7, 19] and many more in various algebraic structures. For more details on derivations in rings and semirings one may read [2] and [10] respectively. A part of our research work is motivated by the classical results due to Herstein [16, 17]. He proved that if a derivation of a prime ring (characteristic not two) commutes, then the ring is a commutative integral domain [16]. Further, he proved that if a fixed element of a prime ring (characteristic not two) commutes with the derivation then it must be in the centre [17]. Inspired by these works, we prove the analogous results in additively cancellative prime semirings. In 1983, Bergen [7] introduced a class of functions, called as semiderivations, which are more general than derivations. Several articles explored the structure of semiderivations and established commutativity conditions in prime rings with semiderivations [3, 5, 8, 9] (and references within). In 2015, Sindhu, *et al.* [20] defined semiderivations in semirings and generalized some properties of prime rings with derivations to semiprime semirings with semiderivations. Recently, Kim & Lee [19] introduced the notion of orthogonal reverse semiderivations in prime semirings and investigated conditions for two reverse semiderivations to be orthogonal. In this article, we generalize the results obtained by Chang in [8] to the class of additively cancellative prime semirings by obtaining some commutativity conditions.

2. Definitions and examples

In this section we define the concept of semiderivation in semirings. We adopt the basic definitions in semirings from Golan [14]. In particular, the following definitions are useful throughout the paper.

Definition 2.1 [14]**.** Let R be an additively cancellative semiring and then the corresponding ring of difference, denoted by R^{Δ} is defined as follows

$$R^{\Delta} = \{a - b : a, b \in R\}.$$

In R^{Δ}, we have $a - b = c - d$ if and only if there exists $r, r' \in R$ such that $a + r = c + r'$ and $b + r = d + r'$. The zero element and multiplicative identity of R^{Δ} are $r - r$ and $1 - 0$ respectively. For $a - b, c - d \in R^{\Delta}$, addition and multiplication is given by

$$(a - b) + (c - d) = (a + c) - (b + d)$$
$$(a - b)(c - d) = (ac + bd) - (ad + bc).$$

We also note that the embedding of R to the ring of differences R^{Δ} is due to the map $r \mapsto r - 0$, for each $r \in R$.

Definition 2.2 [14]**.** A function d of R into R is called a derivation of a semiring R if it satisfies the following conditions.

(i) $d(r + s) = d(r) + d(s), \quad \forall\, r, s \in R$ and

(ii) $d(rs) = d(r)s + rd(s), \quad \forall\, r, s \in R$.

Ganesh and Selvan [11] extended the above definition by defining derivation d^{Δ} in the ring of differences (R^{Δ}) of a semiring (R) corresponding to the derivation d in R.

Definition 2.3 [11]**.** Let R be a semiring and d be a derivation from R into R. Let R^{Δ} be the ring of differences of the semiring R. Then, d^{Δ} is a function in R^{Δ} induced by d, defined as follows

$$d^{\Delta} : R^{\Delta} \to R^{\Delta}$$
$$d^{\Delta}(a - b) = d(a) - d(b), \quad \forall\, a, b \in R.$$

It is easy to see that d^{Δ} is indeed a derivation in R^{Δ} [11].

Definition 2.4. Let R be a semiring. An additive mapping σ from R into R is called a semiderivation if there exists a map f also from R into R such that it satisfies the following conditions.

(i) $\sigma(rs) = \sigma(r)f(s) + r\sigma(s), \quad \forall\, r, s \in R$,

(ii) $\sigma(rs) = \sigma(r)s + f(r)\sigma(s), \quad \forall\, r, s \in R$ and

(iii) $\sigma(f(r)) = f(\sigma(r)), \quad \forall\, r \in R$.

The following example demonstrate the existence of maps that are semiderivation but not derivation and serves as motivation to study such maps.

Example 2.5. Let S be a semiring and define $R := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in S \right\}$. It is easy to note that R is a semiring. Now we define maps σ and f both from R to R as follows

$$\sigma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}; \text{ and } f \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

Then, for $A, B \in R$, where $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in S$ and $B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $x, y \in S$ we have the following.

(i) $\sigma(AB) \neq \sigma(A)B + A\sigma(B)$, implying that σ is not a derivation.

(ii) $\sigma(AB) = \sigma(A)f(B) + A\sigma(B)$, satisfying condition (i) in Definition 2.4.

(iii) $\sigma(AB) = \sigma(A)B + f(A)\sigma(B)$, satisfying condition (ii) in Definition 2.4.

(iv) $\sigma(f(A)) = f(\sigma(A))$, satisfying condition (iii) in Definition 2.4.

Thus, σ is a semiderivation (associated with f) but not a derivation.

3. Some results on semiderivations in semirings

In this section, we introduce a map σ^{Δ} in R^{Δ} induced by a map σ in R. We first prove that σ^{Δ} is a semiderivation in the ring of differences (R^{Δ}) corresponding to a semiderivation σ of a semiring (R). We present some of the basic properties of semiderivations in semirings. In particular, we prove that sum of two semiderivations of a semiring is again a semiderivation of the semiring (if the underlying map remains same). We move on to prove the main result for this section: a map associated with the semiderivation is always a homomorphism. Throughout the section we assume the underlying semiring R to be additively cancellative (unless stated otherwise) whenever the ring R^{Δ} is being used.

Definition 3.1. Let R be a semiring and σ be a map from R into R. Let R^{Δ} be the corresponding ring of differences of the semiring R. Then, σ^{Δ} is a function in R^{Δ} induced by σ, defined as follows

$$\sigma^{\Delta} \; : \; R^{\Delta} \to R^{\Delta}$$
$$\sigma^{\Delta}(a-b) = \sigma(a) - \sigma(b), \quad \forall\, a, b \in R.$$

In the following lemma we prove that σ^{Δ} is indeed a semiderivation in R^{Δ}, if σ is a semiderivation in R.

Lemma 3.2. *Let R be a semiring and R^{Δ} be the corresponding ring of differences of R. Let σ be a semiderivation corresponding to a map f of R. Then, σ^{Δ} is a semiderivation of R^{Δ} (corresponding to σ) associated with the map f^{Δ} (corresponding to f).*

Proof. Let $x, y \in R^{\Delta}$ where $x = a - b$ and $y = c - d$ for $a, b, c, d \in R$. Then we have

$$\begin{aligned}
\sigma^{\Delta}(xy) &= \sigma^{\Delta}\left((a-b)(c-d)\right) \\
&= \sigma^{\Delta}\left((ac+bd)-(ad+bc)\right) \\
&= \sigma(ac+bd)-\sigma(ad+bc) \\
&= \sigma(ac)+\sigma(bd)-\sigma(ad)-\sigma(bc) \\
&= \sigma(a)f(c)+a\sigma(c)+\sigma(b)f(d)+b\sigma(d) \\
&\quad -\sigma(a)f(d)-a\sigma(d)-\sigma(b)f(c)-b\sigma(c) \\
&= \sigma(a)\left((f(c)-f(d)\right)-\sigma(b)\left((f(c)-f(d)\right) \\
&\quad +a(\sigma(c)-\sigma(d))-b(\sigma(c)-\sigma(d)) \\
&= \left((\sigma(a)-\sigma(b)\right)\left((f(c)-f(d)\right)+(a-b)\left((\sigma(c)-\sigma(d)\right) \\
&= \sigma^{\Delta}(x)f^{\Delta}(y)+x\sigma^{\Delta}(y).
\end{aligned}$$

We also have

$$\begin{aligned}
\sigma^{\Delta}(xy) &= \sigma^{\Delta}\left((a-b)(c-d)\right) \\
&= \sigma^{\Delta}\left((ac+bd)-(ad+bc)\right) \\
&= \sigma(a)c+f(a)\sigma(c)+\sigma(b)d+f(b)\sigma(d) \\
&\quad -\sigma(a)d-f(a)\sigma(d)-\sigma(b)c-f(b)\sigma(c) \\
&= \sigma(a)(c-d)-\sigma(b)(c-d)+f(a)\left(\sigma(c)-\sigma(d)\right)-f(b)\left(\sigma(c)-\sigma(d)\right) \\
&= \left(\sigma(a)-\sigma(b)\right)(c-d)+\left(f(a)-f(b)\right)\left(\sigma(c)-\sigma(d)\right) \\
&= \sigma^{\Delta}(x)y+f^{\Delta}(x)\sigma(y).
\end{aligned}$$

Thus conditions (i) and (ii) in the Definition 2.4 are true. Now let us prove the condition (iii) as follows.

$$\begin{aligned}
\sigma^{\Delta}(f^{\Delta}(x)) &= \sigma^{\Delta}(f(a)-f(b)) \\
&= \sigma(f(a))-\sigma(f(b)) \\
&= f(\sigma(a))-f(\sigma(b)) \\
&= f^{\Delta}(\sigma(a)-\sigma(b)) \\
&= f^{\Delta}(\sigma^{\Delta}(a-b)) \\
&= f^{\Delta}(\sigma^{\Delta}(x)). \qquad \blacksquare
\end{aligned}$$

Remark 3.3. We denote σ_f as the semiderivation associated with the map f of a semiring R and $\sigma^{\Delta}_{f^{\Delta}}$ be the semiderivation of R^{Δ} corresponding to σ, where f^{Δ} is a map of R^{Δ} corresponding to f of R.

We now present some of the basic properties of semiderivations in semirings and in ring of differences. Proof of these properties can be obtained through a routine procedure and hence not shown.

Lemma 3.4. *If σ_f and τ_f are semiderivations of R, then $(\sigma+\tau)_f$ is also a semiderivation of R.*

Remark 3.5. In general, if σ_f and τ_g are two semiderivations then one might expect $\sigma+\tau$ to be a semiderivation of $f+g$, which turns out to be false. That is, if the underlying map is different, then the sum of the semiderivation need not be a semiderivation.

Lemma 3.6. *If σ_f and τ_f are semiderivations of R and if $\sigma^{\Delta}_{f^{\Delta}}$ and $\tau^{\Delta}_{f^{\Delta}}$ are the corresponding semiderivations of R^{Δ} respectively, then we have*

$$(\sigma+\tau)^{\Delta}(a-b) = (\sigma^{\Delta}+\tau^{\Delta})(a-b), \quad \forall\ a-b \in R^{\Delta}, a, b \in R.$$

Remark 3.7. We note the following.

(i) Since every derivation is a semi-derivation, it is easy to note that the iterative of a semiderivation need not be a semiderivation again.

(ii) If the iterative $\sigma\tau$ of semiderivations is a semiderivation, then it is an easy routine to check that $(\sigma\tau)^{\Delta}$ is also a semiderivation of R^{Δ}. One may follow along the lines of proof in Lemma 2.8 [11] to verify the same.

Lemma 3.8. *Let σ_f and τ_f be semiderivations of R and $\sigma^{\Delta}_{f^{\Delta}}$ and $\tau^{\Delta}_{f^{\Delta}}$ be the corresponding semiderivations of R^{Δ} respectively. If the iterative of semiderivations $\sigma\tau$ is also a semiderivation, then we have*

$$(\sigma\tau)^{\Delta}(a-b) = \sigma^{\Delta}\tau^{\Delta}(a-b), \quad \forall\ a-b \in R^{\Delta}, a, b \in R.$$

Lemma 3.9. *Let R be an additively cancellative prime semiring and σ_f be a non-zero semiderivation of R. Let R^{Δ} be the corresponding ring of differences of R and $\sigma^{\Delta}_{f^{\Delta}}$ be the semiderivations of R^{Δ} corresponding to σ_f of R. If f is surjective then so is f^{Δ}.*

Now we prove that in an additively cancellative yoked prime semiring, the map f associated with the semiderivation σ is always a semiring homomorphism. This extends the result of Chang (Theorem 1) in [8]. The following lemma is crucial to prove this result.

Lemma 3.10. *Let R be an additively cancellative yoked prime semiring. Let $a, b, c \in R$,*

(i) *if $a \neq 0$ and $arb = arc$ for all $r \in R$, then $b = c$*

(ii) *if $c \neq 0$ and $arc = brc$ for all $r \in R$, then $a = b$.*

Proof. We only prove (i) and the proof of (ii) can be obtained in a similar manner. Let $a \neq 0 \in R$. Since R is yoked, for any b, c in R we have either $b = c+s$ or $c = b + s$ for some $s \in R$. If $b = c + s$, then $arc = arb = ar(c + s) = arc + ars$ implies that $ars = 0$ for all $r \in R$ (since R is additively cancellative). Now, by the primeness of R we must have $s = 0$ (since $a \neq 0$) and thus we get $b = c$. On the other hand, if $c = b + s$, then $arb = arc = ar(b + s) = arb + ars$ and citing the same reasons mentioned earlier we get $s = 0$ implying $b = c$. ■

Theorem 3.11. *Let R be an additively cancellative yoked prime semiring and σ_f be a non-zero semiderivation of R, then f is a homomorphism of R.*

Proof. Let $r, s, t \in R$. Then we note the following

$$\begin{aligned}
\sigma(r(s+t)) &= \sigma(rs+rt)\\
\sigma(r)f(s+t) + r\sigma(s+t) &= \sigma(rs) + \sigma(rt) \text{ (by Definition 2.4(i))}\\
\sigma(r)f(s+t) + r\sigma(s) + r\sigma(t) &= \sigma(r)f(s) + r\sigma(s) + \sigma(r)f(t) + r\sigma(t)\\
\sigma(r)f(s+t) &= \sigma(r)f(s) + \sigma(r)f(t)\\
u\sigma(r)f(s+t) &= u\sigma(r)f(s) + \sigma(r)f(t), \text{ for } u \in R.
\end{aligned}$$

Now we apply Lemma 3.10(i) to see that $f(s + t) = f(s) + f(t), \forall s, t \in R$, proving additive nature of f. Next, to prove multiplicative property we again let $r, s, t \in R$. Then we have

$$\begin{aligned}
\sigma((rs)t) &= \sigma(r(st))\\
f(rs)\sigma(t) + \sigma(rs)t &= f(r)\sigma(st) + \sigma(r)st\\
f(rs)\sigma(t) + f(r)\sigma(s)t + \sigma(r)st &= f(r)f(s)\sigma(t) + f(r)\sigma(s)t + \sigma(r)st\\
&\qquad\text{(by Definition 2.4(ii))}\\
f(rs)\sigma(t) &= f(r)f(s)\sigma(t)\\
f(rs)\sigma(t)u &= f(r)f(s)\sigma(t)u, \text{ for } u \in R.
\end{aligned}$$

Now we apply Lemma 3.10(ii) to see that $f(rs) = f(r)f(s), \forall r, s \in R$. ■

Before we present the commutativity theorems, we shall state a basic result of Chang from [8] (Lemma 1).

Lemma 3.12 [8]**.** *Let R be a prime ring and σ be a non-zero semiderivation of R. If $r\sigma(s) = 0$ (or $\sigma(s)r = 0$) for all $s \in R$, then $r = 0$.*

We extend the above result to prime semirings as follows.

Lemma 3.13. *Let R be an additively cancellative prime semiring and σ be a non-zero semiderivation of R. If $r\sigma(s) = 0$ (or $\sigma(s)r = 0$) for all $s \in R$, then $r = 0$.*

Proof. Let $r \in R$. Then we have

$$\begin{aligned} r\sigma(s) &= 0 \\ r\sigma(st) &= 0, \forall t \in R \\ r\sigma(s)f(t) + rs\sigma(t) &= 0 \\ rs\sigma(t) &= 0, \forall s, t \in R \\ rR\sigma(R) &= 0. \end{aligned}$$

This imples $r = 0$ since R is prime and σ is non-zero. ■

4. Commutativity conditions in semirings

To begin with, we generalize two of the classical results of Herstein about derivations in prime rings [16, 17] to prime semirings. Then, we move on to generalize the results about semiderivations in prime rings by Chang [8] to additively cancellative prime semirings with semiderivations.

Throughout this section we denote R to be an additively cancellative prime semiring, R^Δ to be the corresponding prime ring of differences of the semiring R, $Z(R)$ to be the centre of R, $Z(R^\Delta)$ to be the centre of R^Δ and $[x, y] = xy - yx$, for $x, y \in R^\Delta$, the commutator in R^Δ. We also assume the map f corresponding to semiderivation σ as surjective throughout unless stated otherwise.

Remark 4.1. When R is additively cancellative semiring (which is not a ring), we observe that the characteristic of R^Δ must be zero. Otherwise, if characteristic of R^Δ is n, then $nx = 0$ for all $x \in R^\Delta$ and in particular $nr = 0$ for all $r \in R$. This will imply that R is a ring and so R will coincide with R^Δ.

Theorem 4.2. *Let $d \neq 0$ be a derivation of R. If $d(r)d(s) = d(s)d(r)$ for all $r, s \in R$, then R is a commutative semi-integral domain (a semiring where the product of two non-zero elements is always non-zero).*

Proof. Let d^Δ be the derivation of R^Δ corresponding to the derivation d of R. We first prove that $d^\Delta(x)d^\Delta(y) = d^\Delta(y)d^\Delta(x)$ for all $x, y \in R^\Delta$. Let $x = a - b$ and $y = c - d$ be any two elements of R^Δ where $a, b, c, d \in R$. Then we have

$$\begin{aligned} d^\Delta(x)d^\Delta(y) &= d^\Delta(a - b)d^\Delta(c - d) \\ &= (d(a) - d(b))(d(c) - d(d)) \\ &= d(a)d(c) - d(a)d(d) - d(b)d(c) + d(b)d(d) \\ &= d(c)d(a) - d(d)d(a) - d(c)d(b) + d(d)d(b) \\ &= (d(c) - d(d))(d(a) - d(b)) \end{aligned}$$

$$= d^{\Delta}(c-d)d^{\Delta}(a-b)$$
$$= d^{\Delta}(y)d^{\Delta}(x).$$

Also, it is clear that $char(R^{\Delta}) \neq 2$ by Remark 4.1. Now we see that the hypotheses of Theorem 2 [16] are satisfied for R^{Δ} and thus R^{Δ} is a commutative integral domain. Hence we get the desired result since R is embedded in R^{Δ} and the restriction of d^{Δ} to R is simply the derivation d. ■

Theorem 4.3. *Let $d \neq 0$ be a derivation of R and suppose $a \in R$ such that $ad(r) = d(r)a$ for all $r \in R$. Then $a \in Z(R)$.*

Proof. Let d^{Δ} be the derivation of R^{Δ} corresponding to the derivation d of R. We note that by Remark 4.1 R^{Δ} must be of characteristic not two. Now let $a \in R$ and then we prove that $ad^{\Delta}(x) = d^{\Delta}(x)a$ for all $x \in R^{\Delta}$. Let $x = u - v$ for $u, v \in R$

$$\begin{aligned} ad^{\Delta}(x) &= ad^{\Delta}(u-v) \\ &= a(d(u) - d(v)) \\ &= ad(u) - ad(v) \\ &= d(u)a - d(v)a \\ &= d^{\Delta}(u-v)a \\ &= d^{\Delta}(x)a. \end{aligned}$$

Now by part (1) of Herstein's Theorem part in [17] we get that $a \in Z(R^{\Delta})$ and thus $a \in Z(R)$. ■

Lemma 4.4. *Let σ_f be a semiderivation of R and $\sigma^{\Delta}_{f^{\Delta}}$ be the corresponding semiderivation of R^{Δ}. Then, we have $(\sigma(R))^{\Delta} = \sigma^{\Delta}(R^{\Delta})$.*

Proof. Let $x \in (\sigma(R))^{\Delta}$ such that $x = a - b$ for $a, b \in \sigma(R)$. Since $a, b \in \sigma(R)$ there exists r, s such that $\sigma(r) = a$ and $\sigma(s) = b$. Then, $x = a - b = \sigma(r) - \sigma(s) = \sigma^{\Delta}(r - s)$ and hence $x \in \sigma^{\Delta}(R^{\Delta})$. Thus we have $(\sigma(R))^{\Delta} \subseteq \sigma^{\Delta}(R^{\Delta})$.

On the other had, let $x \in \sigma^{\Delta}(R^{\Delta})$. Then, $x = \sigma^{\Delta}(u - v)$ for some $u - v \in R^{\Delta}$ where $u, v \in R$. So, $x = \sigma(u) - \sigma(v) = a' - b'$, where $a', b' \in \sigma(R)$ and hence $x \in (\sigma(R))^{\Delta}$. Thus we have $\sigma^{\Delta}(R^{\Delta}) \subseteq (\sigma(R))^{\Delta}$ and so we are done. ■

Lemma 4.5. *Let $\sigma \neq 0$ be a semiderivation of R such that $\sigma(R) \subset Z(R)$, then R is a commutative semi-integral domain.*

Proof. Let $x \in \sigma^{\Delta}(R^{\Delta})$ where σ^{Δ} is the semiderivation of R^{Δ} corresponding to σ of R. Let $x = a - b$ where $a, b \in \sigma(R)$. If we let $y \in R^{\Delta}$, such that $y = c - d$ for $c, d \in R$, then we have $xy = (a-b)(c-d) = (ac+bd) - (ad+bc) =$

$(ca + db) - (da + cb) = (c - d)(a - b) = yx$. Thus we have $x \in Z(R^{\Delta})$ implying that $\sigma^{\Delta}(R^{\Delta}) \subset Z(R^{\Delta})$. Hence, by Lemma 2 in [8] R^{Δ} is a commutative integral domain. Therefore R must be semi-integral domain since R is embedded in R^{Δ}. ■

Theorem 4.6. *Let $\sigma \neq 0$ be a semiderivation of R such that $\sigma(r)\sigma(s) = \sigma(s)\sigma(r)$ for all $r, s \in R$. Then R is a commutative semi-integral domain.*

Proof. Let σ^{Δ} be the semiderivation of R^{Δ} corresponding to the semiderivation σ of R. Note that by Remark 4.1 we have $char(R^{\Delta}) \neq 2$. Next, let $x, y \in R^{\Delta}$, we prove $\sigma^{\Delta}(x)\sigma^{\Delta}(y) = \sigma^{\Delta}(y)\sigma^{\Delta}(x)$. Consider $x = a - b$ and $y = c - d$ for $a, b, c, d \in R$, then we have

$$\begin{aligned}
\sigma^{\Delta}(x)\sigma^{\Delta}(y) &= \sigma^{\Delta}(a-b)\sigma^{\Delta}(c-d) \\
&= (\sigma(a) - \sigma(b))(\sigma(c) - \sigma(d)) \\
&= \sigma(a)\sigma(c) - \sigma(a)\sigma(d) - \sigma(b)\sigma(c) + \sigma(b)\sigma(d) \\
&= \sigma(c)\sigma(a) - \sigma(d)\sigma(a) - \sigma(c)\sigma(b) + \sigma(d)\sigma(b) \\
&= (\sigma(c) - \sigma(d))(\sigma(a) - \sigma(b)) \\
&= \sigma^{\Delta}(c-d)\sigma^{\Delta}(a-b) \\
&= \sigma^{\Delta}(y)\sigma^{\Delta}(x).
\end{aligned}$$

Then by Theorem 2 in [8], we must have R^{Δ} to be a commutative integral domain and thus R must be semi-integral domain since R is embedded in R^{Δ}. ■

Theorem 4.7. *Let $\sigma \neq 0$ be a semiderivation of R and $r \neq 0 \in R$. If $r\sigma(R) \subset Z(R)$ then R is commutative.*

Proof. Let σ^{Δ} be the semiderivation of R^{Δ} corresponding to the semiderivation σ of R. Let $r \neq 0 \in R$, $x \in \sigma^{\Delta}(R^{\Delta})$ and $y \in R^{\Delta}$. Consider $x = a - b$ and $y = c - d$ where $a, b \in \sigma(R)$ and $c, d \in R$. Then we have

$$\begin{aligned}
(rx)y &= r(a-b)(c-d) \\
&= r(ac - ad - bc + db) \\
&= rac - rad - rbc + rbd \\
&= cra - dra - crb + drb \\
&= (c-d)(r(a-b)) \\
&= y(rx).
\end{aligned}$$

Thus, $r\sigma^{\Delta}(R^{\Delta}) \subset Z(R^{\Delta})$. Now, by Theorem 3 of Chang [8] we get R^{Δ} is a commutative ring. Since R is embedded in R^{Δ}, R must be commutative as well. ■

Theorem 4.8. *Let $\sigma \neq 0$ be a semiderivation of R and $r \in R$. If $r\sigma(R) = \sigma(R)r$ then $r \in Z(R)$.*

Proof. Let σ^{Δ} be the semiderivation of R^{Δ} corresponding to the semiderivation σ of R. Let $r \in R$ and let $x \in \sigma^{\Delta}(R^{\Delta})$. Consider $x = a - b$ where $a, b \in \sigma(R)$. Then we have $rx = r(a-b) = ra - rb = ar - br = (a-b)r = xr$. Hence we get $[r, \sigma^{\Delta}(R^{\Delta})] = 0$. So, by Theorem 4 in [8] we have $r \in Z(R^{\Delta})$ and hence $r \in Z(R)$. ∎

Theorem 4.9. *Let $\sigma \neq 0$ be a semiderivation of R and $r \in R$. For all $s, u \in R$, if $r\sigma(s)u + u\sigma(s)r = ur\sigma(s) + \sigma(s)ru$ then $r \in Z(R)$.*

Proof. Let σ^{Δ} be the semiderivation of R^{Δ} corresponding to the semiderivation σ of R. Let $r \in R$ and let $x, y \in R^{\Delta}$. We now prove that $[r, \sigma^{\Delta}(R^{\Delta})] \subset Z(R^{\Delta})$. That is, we need to prove the following equation

$$(r\sigma^{\Delta}(x) - \sigma^{\Delta}(x)r)y = y(r\sigma^{\Delta}(x) - \sigma^{\Delta}(x)r), \qquad \text{for all } x, y \in R^{\Delta}.$$

That is, to prove

$$r\sigma^{\Delta}(x)y + y\sigma^{\Delta}(x)r = yr\sigma^{\Delta}(x) + \sigma^{\Delta}(x)ry, \qquad \text{for all } x, y \in R^{\Delta}.$$

Assume $x = a - b, y = c - d$ where $a, b, c, d \in R$. Then we have

$$\begin{aligned}
r\sigma^{\Delta}(x)y + y\sigma^{\Delta}(x)r &= r(\sigma^{\Delta}(a-b))(c-d) + (c-d)(\sigma^{\Delta}(a-b))r \\
&= r(\sigma(a) - \sigma(b))(c-d) + (c-d)(\sigma(a) - \sigma(b))r \\
&= (r\sigma(a) - r\sigma(b))(c-d) + (c-d)(\sigma(a)r - \sigma(b)r) \\
&= r\sigma(a)c - r\sigma(a)d - r\sigma(b)c + r\sigma(b)d \\
&\quad + c\sigma(a)r - c\sigma(b)r - d\sigma(a)r + d\sigma(b)r \\
&= (cr\sigma(a) + \sigma(a)rc) + (dr\sigma(b) + \sigma(b)rd) \\
&\quad - (dr\sigma(a) + \sigma(a)rd) - (cr\sigma(b) + \sigma(b)rc) \\
&= (c-d)(r\sigma(a) - r\sigma(b)) + (\sigma(a)r - \sigma(b)r)(c-d) \\
&= yr\sigma^{\Delta}(x) + \sigma^{\Delta}(x)ry.
\end{aligned}$$

Thus by Theorem 5 in [8], we have $r \in Z(R^{\Delta})$ which implies that $r \in R$. ∎

Theorem 4.10. *Let $\sigma \neq 0$ be a semiderivation of R such that $\sigma(r)\sigma(s)u + u\sigma(s)\sigma(r) = \sigma(s)\sigma(r)u + u\sigma(r)\sigma(s)$ for all $r, s, u \in R$. Then R is commutative.*

Proof. Applying Theorem 4.9 we see that $\sigma(r) \subset Z(R)$ and then Lemma 4.5 implies that R must be commutative. ∎

Theorem 4.11. *Let $\sigma \neq 0$ be a semiderivation of R. If $\sigma^2(R) \subset Z(R)$, then R is commutative.*

Proof. Let $g = \sigma^{\Delta}$ be the semiderivation of R^{Δ} corresponding to the semiderivation σ of R. Let $r \in R$ and by hypothesis we have $\sigma^2(r)s = s\sigma^2(r)$ for all $r, s \in R$. Now, let $x, y \in R^{\Delta}$ and $x = a - b, y = c - d$ for $a, b, c, d \in R$. We prove $g^2(R^{\Delta}) \subset Z(R^{\Delta})$ as follows

$$\begin{aligned}
g^2(x)y &= (g^2(a-b))(c-d) \\
&= (\sigma^2(a) - \sigma^2(b))(c-d) \\
&= \sigma^2(a)c - \sigma^2(a)d - \sigma^2(b)c - \sigma^2(b)d \\
&= c\sigma^2(a) - d\sigma^2(a) - c\sigma^2(b) - d\sigma^2(b) \\
&= (c-d)(\sigma^2(a)\sigma^2(b)) \\
&= (c-d)(g^2(a-b)) \\
&= yg^2(x).
\end{aligned}$$

Hence by Theorem 7 in [8], we must have R^{Δ} to be commutative and so is R. ■

Theorem 4.12. *Let $\sigma_f, \tau_f \neq 0$ be two semiderivation of R. If $\sigma\tau(R) \subset Z(R)$ and if the iterative $(\sigma\tau)_f$ is also a semiderivation of R, then R is commutative.*

Proof. Let σ^{Δ} and τ^{Δ} be the semiderivation of R^{Δ} corresponding to the semiderivation σ and τ respectively of R. Let $r, s \in R$ and by the assumption of the theorem we have $(\sigma\tau(r))s = s(\sigma\tau(r))$ for all $r, s \in R$. Now, let $x, y \in R^{\Delta}$ and $x = a - b, y = c - d$ for $a, b, c, d \in R$. We prove $(\sigma\tau)^{\Delta}(R^{\Delta}) \subset Z(R^{\Delta})$ as follows

$$\begin{aligned}
((\sigma\tau)^{\Delta}(x))y &= ((\sigma\tau)^{\Delta}(a-b))(c-d) \\
&= (\sigma\tau(a) - \sigma\tau(b))(c-d) \\
&= \sigma\tau(a)c - \sigma\tau(a)d - \sigma\tau(b)c - \sigma\tau(b)d \\
&= c\sigma\tau(a) - d\sigma\tau(a) - c\sigma\tau(b) - d\sigma\tau(b) \\
&= (c-d)(\sigma\tau(a) - \sigma\tau(b)) \\
&= (c-d)((\sigma\tau)^{\Delta}(a-b)) \\
&= y((\sigma\tau)^{\Delta}(x)).
\end{aligned}$$

Hence by Theorem 8 in [8], we must have R^{Δ} to be commutative and so is R. ■

Lemma 4.13. *Let R be an additively cancellative yoked semiring. Let σ_f be a semiderivation of R such that $r\sigma(r) = \sigma(r)r$, for all $r \in R$, then*

(i) $s\sigma(u) + u\sigma(s) = \sigma(s)u + \sigma(u)s, \quad$ *for some* $u \in R$

(ii) $r\sigma(s) + s\sigma(r) = \sigma(r)s + \sigma(s)r, \quad \forall r, s \in R$

Proof. Follow along the lines of the proof in Lemma 2.11 in [11]. ■

Remark 4.14. In fact, the above result is true for any additive map of an additively cancellative yoked semiring R.

We conclude this paper with a theorem that extends the results due to Posner [21] and Chang [8].

Theorem 4.15. *Let R be an additively cancellative yoked prime semiring. Let $\sigma_f \neq 0$ be a semiderivation of R. If $r\sigma(r) = \sigma(r)r$, then R is commutative.*

Proof. Let σ^Δ be the semiderivation of R^Δ corresponding to the semiderivation σ of R. Let $r \in R$ and let $x, y \in R^\Delta$. We first note that $char(R^\Delta) \neq 2$ since R is additively cancellative (Remark 4.1). Next, we notice that R^Δ is a prime ring since R is a prime semiring (Lemma 2.6 in [11]). Finally, we prove that $[x, \sigma^\Delta(x)] \in Z(R^\Delta)$. That is, we need to prove the following equation

$$(x\sigma^\Delta(x) - \sigma^\Delta(x)x)y = y(x\sigma^\Delta(x) - \sigma^\Delta(x)x), \qquad \text{for all } x, y \in R^\Delta.$$

That is, to prove

$$x\sigma^\Delta(x)y + y\sigma^\Delta(x)x = yx\sigma^\Delta(x) + \sigma^\Delta(x)xy, \qquad \text{for all } x, y \in R^\Delta.$$

Assume $x = a - b, y = c - d$ where $a, b, c, d \in R$. Then we have

$$\begin{aligned}
x\sigma^\Delta(x)y + y\sigma^\Delta(x)x &= (a-b)(\sigma^\Delta(a-b))(c-d) + (c-d)(\sigma^\Delta(a-b))(a-b) \\
&= (a-b)(\sigma(a) - \sigma(b))(c-d) + (c-d)(\sigma(a) - \sigma(b))(a-b) \\
&= (a\sigma(a) - a\sigma(b) - b\sigma(a) + b\sigma(b))(c-d) \\
&\quad + (c-d)(\sigma(a)a - \sigma(a)b - \sigma(b)a + \sigma(b)b) \\
&= (\sigma(a)a - \sigma(a)b - \sigma(b)a + \sigma(b)b)(c-d) \\
&\quad + (c-d)(a\sigma(a) - a\sigma(b) - b\sigma(a) + b\sigma(b)) \\
&\qquad\qquad\qquad\qquad \text{[by hypothesis and by Lemma 4.13]} \\
&= (\sigma(a) - \sigma(b))(a-b)(c-d) + (c-d)(a-b)(\sigma(a) - \sigma(b)) \\
&= (\sigma^\Delta(a-b))(a-b)(c-d) + (c-d)(a-b)(\sigma^\Delta(a-b)) \\
&= (\sigma^\Delta(x))xy + yx(\sigma^\Delta(x)) \\
&= yx\sigma^\Delta(x) + \sigma^\Delta(x)xy.
\end{aligned}$$

Now, we apply Theorem 9 of Chang [8] to see that R^Δ is commutative and thus we obtain R to be commutative (since the R is embedded into R^Δ). ∎

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Received 22 July 2024
Revised 2 September 2024
Accepted 2 September 2024