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A NOTE ON TRI-CLEAN RINGS

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Abstract

We introduce a new class of rings, in which elements are sum of units and tripotents. This class of rings is called tri-clean (T-clean) rings which is a generalized structure of clean rings and invo tri-clean rings. We derive several properties of T-clean rings. We show that if an element a is T-clean in a corner ring eRe for some idempotent e then it is also a T-clean element in R. If 2 is an unit in R then R is a T-clean ring if and only if $\frac{R}{I}$ is a T-clean ring for every nil ideal I of R. We also prove that all the upper triangular matrix rings over T-clean ring is a T-clean ring.

Keywords: clean rings, T-clean rings, tripotents, lifting tripotents, ideal extension.

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1. Introduction

In this discussion R denotes a ring with 1 unless specified. An element $a \in R$ is called clean if $a = u + e$ for some unit u and idempotent e in R. The concept of clean elements were first introduced 47 years back by Nicholson [19]. Hongbo introduced the generalized clean rings ($G - clean$) and showed several relationships of clean rings and $(s, 2)$ rings in the paper [21]. In 2008 Chen [5] extended clean elements of commutative reduced rings to arbitrary abelian rings. In 2017 Danchev [6] defined a ring R to be invo clean if each element of R can be expressed as a sum of an involution element and idempotent element. In the paper the author completely described the algebraic structure of invo-clean rings. Later in 2018 Danchev [9] gave some important criterion when a commutative group ring is invo-clean ring. For further details on invo clean rings and related topics, we refer to [7, 8]. Following in 2022 Ahmad *et al.* [1] extended the concept to involution tri clean rings and showed many applications in graph theory.

Meanwhile several authors have studied various properties of rings widely where every element of R can be splitted into two parts. For better insight into these rings the readers are referred to [3, 12, 20, 13, 15] etc.

All clean elements in a ring R satisfies the property (P_1) : $a \in R$ is clean if and only if $a-1$ is clean. It is well known that the clean elements in R are not closed under additive inverse means it does not satisfy the property (P_2) : a is clean in R if and only if $-a$ is clean in R. Rings satisfying property P_2 is investigated by Grigore and Horia in the paper [4]. But the tri-clean rings or T-clean ring satisfies property P_2 and does not satisfy property P_1. In this article, we characterize the tri-clean rings which satisfy property P_2 and do not satisfy property P_1. Apart from this, the core objective of this article is to study several properties of T-clean rings.

We denote $idem(R), T(R), U(R)$ are set of all idempotents, tripotents and group of units in a ring R respectively. $UT_n(R)$ denotes the $n \times n$ upper triangular matrix ring over R with the usual addition and multiplication of the matrix. We also recall $J(R)$ the Jacobson radical, which is equal to the intersection of all maximal right/left ideals in a ring R. It is well known that $J(R) = \{x \in R : 1 + yxz \in U(R), \forall y, z \in R\}$ which is indeed an ideal of R. Interestingly $J(R) + U(R) = U(R)$. $P(R)$ denotes the set of all elements $x \in R$ such that $x^n = x$ for some positive integer n, called the potent elements. For any $a \in R$, $I(a) = \{r \in R : ara = a\}$ is called the inner inverses of a in R. We also noticed that semi-tripotent rings are T-clean but the converse is not true. We know that idempotents can be lifted modulo every nil ideal in a ring R ([14], Theorem 2). However, the same is not true for tripotent elements. For example, $\bar{2} \in \frac{\mathbb{Z}_8}{2\mathbb{Z}_8}$ is a tripotent but 2 is not a tripotent in $\mathbb{Z}_8$. Khurana [14] studied the generalize conditions on a ring R for which potent elements lifted modulo every nil ideals. With this motivation we are inspired to establish the notion of T-clean rings and deduce several noteworthy findings.

2. Preliminaries

In this section, we provide the fundamental definitions and findings that are necessary for our work. We start with the definition of tri-clean elements and tri-clean rings.

Definition. An element a of a ring R is called tri-clean or T-clean if a can be written as a sum of an unit and a tripotent. If every element of R is tri-clean then R is called a T-clean ring.

We denote $T(R) = \{x \in R : x^3 = x\}$ and $T - clean(R) = \{a \in R : a = u + t,$ for some $u \in U(R)$ and $t \in T(R)\}$. If $R = T - clean(R)$ then R is a T-clean ring. We will often refer the equation $a = u + t$ for some $u \in U(R)$ and $t \in T(R)$ as tri-clean or T-clean decomposition of a.

Example 1. $\mathbb{Z}_4$ is T-clean as well as clean. However, $\mathbb{Z}$ is not T-clean.

Example 2. If $a \in R$ is a T-clean then by simple computations we have $-a$ is also a T-clean.

It is clear that all clean rings are T-clean but the converse is not true. Here is an example.

Example 3. Let $A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ in the ring $UT_2(\mathbb{Z})$. Then A is T-clean by the decomposition given by

$$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

but A is not clean by Proposition 11 of [4].

There may be more than one T-clean decomposition of $a \in R$. If there is only one T-clean decomposition then it is called uniquely T-clean. Again, if the unit and the tripotent of the T-clean decomposition of a commute, then it is called strongly T-clean. However, we will not go into great detail about these properties of T-clean ring. In addition, determining all the T-clean or number of T-clean decompositions for a is a matter of another significant work. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & n \\ 0 & -1 \end{pmatrix}$ in the ring $UT_2(\mathbb{Z})$ is T-clean with infinitely many T-clean decomposition.

Definition [15]**.** A ring R is called *$semi - n - potent$* if each element of R can be written as a sum of an element from Jacobson's radical and an $n - potent$.

Example 4. Semi-tripotent (semi-3-potent) rings are T-clean ring. If R is a semi-tripotent ring then for any $a \in R$, $a + 1 = j + f$ for some $j \in J(R)$ and $f^3 = f$. Then $a = (j - 1) + f$ is a T-clean decomposition as $j - 1 \in U(R)$. However, the reverse implication does not hold. Here is an example.

Example 5. Let $R = M_2(\mathbb{Q})$. Then

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

is a T-clean in R but not semi-tripotent ([15], Remark 1).

Remark 6. ([16], Ex. 1.4) For any ring R. If $ab, ba \in U(R)$ then $a, b \in U(R)$. If R is a dedekind domain then only $ab \in U(R)$ is enough to conclude both $a, b \in U(R)$.

Let $a \in R$ be a tripotent (unit) element and I be an ideal of R. We say a lifts modulo I if there exists a tripotent (unit) element $t \in R$ ($u \in R$) such that $t - a \in I$ ($u - a \in I$).

The following lemmas are about the units and tripotents that can be lifted modulo every nil ideal of R.

Lemma 7. *For any ring R and I is a nil ideal of R. If $\bar{a} \in U(\frac{R}{I})$, then there exists $u \in U(R)$ such that $a - u \in I$.*

Proof. We consider $\bar{v} \in \frac{R}{I}$ such that $\bar{a}\bar{v} = \bar{1}$ in $\frac{R}{I}$. Thus, $av-1 \in I$ and $va-1 \in I$ and hence we get $av \in 1+I$ and $va \in 1+I$. But as I is nil ideal we have $1+I \subseteq U(R)$. So, $av, va \in U(R)$ and hence by Remark 6 $a, v \in U(R)$. Now we consider $u = v^{-1}$ then we get $a+I = (1+I)(v+I)^{-1} = (1+I)(v^{-1}+I) = (v^{-1}+I) = u+I$. Therefore, $a - u \in I$. ∎

Lemma 8. *For any ring R and Jacobson's radical $J(R)$ of R. If $\bar{a} \in U(\frac{R}{J(R)})$, then there exists $u \in U(R)$ such that $a - u \in J(R)$.*

Proof. The proof is similar to Lemma 7 because for any $x \in J(R)$, $1 + x$ is an unit. ∎

Lemma 9. *([14], Theorem 8) Let $I \subseteq R$ be an one sided nil ideal and $a \in R$ is such that $a^3 - a \in I$. If $2 \in R$ is an unit, then there exists $t \in R$ with $t^3 = t$ such that $a - t \in I$.*

The following lemma determines all the tripotent elements of $\mathbb{Z}_p[i]$, where p is a prime. If $p \equiv 3(mod\ 4)$ then $\mathbb{Z}_p[i]$ is a field and hence contains only the trivial tripotents $\{0, 1, p-1\}$.

Lemma 10. *([2], Proposition 1) Let p be a prime number such that $p \equiv 1(mod\ 4)$ and $x = a+ib \in \mathbb{Z}_p[i]$. Then x is a tripotent if and only if $a^2 = \frac{1-p}{4}$ and $b^2 = \frac{p-1}{4}$.*

Lemma 11. *Let $a \in R$ be such that $a^n = a$ for some $n \in \mathbb{N}$. Then a is T-clean.*

Proof. We consider $a^n = a$, $n \geq 2$ is a potent element in a ring R. Then by Lemma 2.1 of [15] we have $x = ue$ where $e^2 = e$, $u^{n-1} = 1$ and $ue = eu$. Now we suppose $v = ue - (1-e)$ which is an unit in R with $v^{-1} = u^{n-2}e - (1-e)$. Leaving $f = 1 - e$ we get $a = v + f$ is a clean decomposition. Therefore, a is clean and hence T-clean. ∎

Remark 12. Any potent ring is a T-clean ring.

Lemma 13. *([12], Theorem 1) The following statements are equivalent for any ring R.*

(1) *The ring R satisfies the identity $a^3 = a$.*

(2) *The ring R is commutative such that for every $a \in R$, $a = e + f$ for some $e, f \in idem(R)$.*

(3) *For any $a \in R$ we have $a = e + f$ for some $e, f \in idem(R)$ such that $ef = fe$.*

3. Main results

In this section, we present the main results. We start with a proposition that can be used for an alternative defintion of T-clean rings.

Proposition 14. *Let R be a ring. Then R is a T-clean ring if and only if for every $a \in R$ we get $a = t - u$ for some $t \in T(R)$ and $u \in U(R)$.*

Proof. We suppose R is a T-clean ring and $-a = t+u$ is a T-clean decomposition for some $t \in T(R)$ and $u \in U(R)$. Then we can write $a = (-t) - u$ where $(-t)^3 = -t^3 = -t \in T(R)$. Conversely, we suppose that for every $a \in R$, $-a = t - u$ for some $t \in T(R)$ and $u \in U(R)$. Then $a = (-t) + u$ where, $-t \in T(R)$ is a T-clean decomposition. So, R is a T-clean ring. ■

Proposition 15. *Let R be a ring with $char(R) = 3$. Then x is a T-clean if and only if $x - 1$ is a T-clean.*

Proof. (=>) We assume, $x = u + t$ for some $u \in U(R)$ and $t \in T(R)$. Then $x - 1 = u + (t - 1)$. As, $char(R) = 3$ we have $(t - 1)^3 = t^3 - 3t^2 + 3t - 1 = t^3 - 1 = t - 1 \in T(R)$.
(<=) we assume $x-1 = u+t$ for some $u \in U(R)$ and $t \in T(R)$. Then $x = u+(t+1)$. Again as, $char(R) = 3$ we obtain $(t+1)^3 = t^3 + 3t^2 + 3t + 1 = t + 1 \in T(R)$. Hence the result. ■

Proposition 16. *Any homomorphic image of a T-clean ring is a T-clean.*

Proof. We suppose $\phi : R \to S$ is an epimorphism and R is T-clean. Let $x \in S$ is any element. Then there exists $a \in R$ such that $\phi(a) = x$. So writing $a = u+t$ for some $u \in U(R)$ and $t \in T(R)$ we get $x = \phi(u + t) = \phi(u) + \phi(t)$. Since any ring homomorphism sends unit to unit we have $\phi(u) \in U(S)$ and $(\phi(t))^3 = \phi(t^3) = \phi(t) \in T(S)$. Therefore, S is a T-clean ring. ■

Remark 17. From Proposition 16 we have for any ring R and ideal I of R. If R is a T-clean ring then $\frac{R}{I}$ is T-clean too. But the reverse implication does not hold. For we take the following example.

Example 18. $\frac{\mathbb{Z}}{8\mathbb{Z}}$ is a T-clean ring but $\mathbb{Z}$ is not a T-clean ring.

Proposition 19. *Let R be a ring and $e \in idem(R)$. If $a \in eRe$ is a T-clean then a is T-clean in R.*

Proof. We consider $a = u' + t$ is a T-clean decomposition in eRe, where $t^3 = t \in eRe$ and $u' \in U(eRe)$ with inverse $u'w = e = wu'$ in eRe. Then we can easily prove that $u = u' - (1-e)$ is an unit with its inverse $u^{-1} = w - (1-e)$ in R. Now $a - u = (u'+t) - u = u' + t - u' + (1-e) = t + (1-e)$. Here,

$$\begin{aligned}(t+(1-e))^3 &= (t+(1-e))(t^2+(1-e)t+t(1-e)+(1-e))\\ &= (t+(1-e))(t^2+t-et+t-te+(1-e))\\ &= (t+(1-e))(t^2+(1-e))\ (as, t \in eRe => et = t = te)\\ &= (t+(1-e)) \in T(R).\end{aligned}$$

Therefore, $a = u + (t+(1-e))$ is a T-clean decomposition in R and hence a is a T-clean in R. ■

Theorem 20. *Let R be any ring and I be a nil ideal such that $2 \in U(R)$. Then R is a T-clean ring if and only if $\frac{R}{I}$ is a T-clean ring.*

Proof. (=>) It is clear from the proposition 16 using the canonical ring epimorphism from R to $\frac{R}{I}$.
(<=) We consider $\frac{R}{I}$ is a T-clean ring. Let $a \in R$, $\bar{a} \in \frac{R}{I}$. So there exists $\bar{u} \in U(\frac{R}{I})$ and $\bar{t} \in T(\frac{R}{I})$ such that $\bar{a} = \bar{u} + \bar{t}$. Since I is a nil ideal and by Lemma 7 we have units are lifted modulo I. Therefore, we may consider $u \in U(R)$ and $a - u$ is a tripotent modulo I. Since 2 is an unit in R, so by Lemma 9 we have that tripotents are lifted modulo I. Therefore, $a - u$ is a tripotent. So R is a T-clean ring. ■

We may wonder if 2 is not an unit, does the Theorem 20 still hold. This is unfavorable. Here is an example.

Example 21. Let $R = \mathbb{Z}[x]$ with $(x^2-1)^2 = 0$ and $I =< x^2 - 1 >$. Here, I is clearly a nil ideal in R also 2 is not an unit in R. We consider $\alpha + I = (1+I)+(x+I)$ in $\frac{R}{I}$. Then $(x+I)^3 = x^3+I = (x+I)(x^2+I) = (x+I)(1+I) = x+I$, is a tripotent. Hence, $\alpha + I$ is a T-clean element in $\frac{R}{I}$. Next we show that the tripotent $\alpha - 1 = x + I$ does not lift modulo I. If it is lifted modulo I then

$(x+(x^2-1)(ax+b))^3 = (x+(x^2-1)(ax+b)$ for some $a, b \in \mathbb{Z}$. This implies that

$$x^3 + 3x^2(x^2-1)(ax+b) = x + (x^2-1)(ax+b).$$

Simplifying we get

$$3ax^5 + 3bx^4 - 3ax^3 - 3bx^2 + x^3 = ax^3 + bx^2 - ax + x - b.$$

Again by replacing $x^4 = 2x^2 - 1$ from the given rule we get

$$(2a+1)x^3 + 2bx^2 - (2a+1)x - 2b = 0.$$

Comparing we get $2a+1=0$, which has no solution in $\mathbb{Z}$. Therefore, $\alpha - 1$ does not lift modulo I and consequently α is not a T-clean element in R.

We recall that the intersection of all prime ideals in a ring R is an ideal called prime radical of R which is denoted by $\mathcal{P}(R)$.

Corollary 22. *Let R be a ring and $2 \in U(R)$. Then R is a T-clean ring if and only if $\frac{R}{\mathcal{P}(R)}$ is a T-clean ring.*

Proof. It follows from the Theorem 20 as the prime radical $\mathcal{P}(R)$ is a nil ideal. ■

Theorem 23. *The usual product ring $R = \prod_{i=1}^{n} R_i$ is a T-clean if and only if R_i is a T-clean for each i.*

Proof. (=>) It follows from the Proposition 16 by considering the natural epimorphism $\phi_i : R \to R_i$ for each i.
(<=) We consider R_i is a T-clean ring for each i and $x = (x_1, x_2, ...x_n) \in R$. For each $x_i \in R_i$ we have the T-clean decomposition $x_i = u_i + t_i$ for some $u_i \in U(R_i)$ and $t_i \in T(R_i)$. Then clearly $u = (u_1, u_2, ...u_n) \in U(R)$ and $t = (t_1, t_2, ..., t_n) \in T(R)$ such that $x = u + t$. Therefore, R is a T-clean ring. ■

We consider the following example to show if one of R_1 or R_2 is not T-clean then $R = R_1 \times R_2$ may failed to be T-clean.

Example 24. Let $R = \mathbb{Z}_4 \times \mathbb{Z}$. We show that $(0,3)$ does not have any T clean decomposition in R. We suppose $(0,3) = (u_1, u_2) + (t_1, t_2)$ is a T-clean decomposition in R, where $(u_1, u_2) \in U(R)$ and $(t_1, t_2) \in T(R)$. Then $3 = u_2 + t_2 \implies t_2 = 3 - u_2$ in $\mathbb{Z}$. But units in $\mathbb{Z}$ are $\{-1, 1\}$. So, the tripotent t_2 is either 2 or 4. This is impossible as the tripotent elements of $\mathbb{Z}$ are $\{-1, 0, 1\}$.

The following Theorem characterizes the T-clean elements in an abelian ring.

Theorem 25. *Let R be ring which is abelian and $e \in idem(R)$. If $a \in R$ is a T-clean then the following holds*

1. *ae is a T-clean.*
2. *If $1+a$ is also T-clean, then $a+e$ is a T-clean in R.*
3. *If $b+e$ is clean, then b is a T-clean for any $b \in R$.*

Proof. 1. We consider $a = u + t$ is a T-clean decomposition in R such that $u \in U(R)$ and $t \in T(R)$. Now right multiplying both side by e we have $ae = ue + te$. Then clearly $ue \in U(eRe)$ and as R is abelian we have $(te)^3 = tetete = tet^2e = t^3e = te \in T(eRe)$. Thus ae is T-clean in eRe. Therefore, by Proposition 19 ae is T-clean in R.

2. We consider $a = u_1 + t_1$ and $1 + a = u_2 + t_2$ for some $u_1, u_2 \in U(R)$ and $t_1, t_2 \in T(R)$. Now $a + e = ae - ae + a + e = a(1-e) + (1+a)e = (u_1+t_1)(1-e) + (u_2+t_2)e = u_1(1-e) + u_2e + t_1(1-e) + t_2e$. Since R is an abelian ring we can clearly see $u_1(1-e) + u_2e \in U(R)$ as, $(u_1(1-e) + u_2)(u_1^{-1}(1-e) + u_2^{-1}e) = e + 1 - e = 1$. Again as R being abelian we have $(t_1(1-e) + t_2e)^3 = (t_1(1-e) + t_2e)(t_1^2(1-e) + t_2^2e) = t_1^3(1-e) + t_2^3e = t_1(1-e) + t_2e \in T(R)$. Therefore, $a+e$ is T-clean in R.

3. We consider $b + e = u + f$ for some $u \in U(R)$ and $f \in idem(R)$. Then $b = u + (f-e)$. As, R is an abelian ring e and f are commuting idempotent. Thus have $(f-e)^3 = f - e$. Therefore, b is T-clean in R.

∎

In support of the Theorem 25 we present the following example.

Example 26. Let $R = \mathbb{Z}_6 \times \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $e = (3, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) \in idem(R)$. We consider $a = (2, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) = (1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) + (1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ is a T-clean element in R. Then

1. $ae = (2, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})(3, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) = (0, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix})$ is a T-clean element in R. Because we have the T-clean decomposition as,
$$(0, \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}) = (1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) + (5, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}).$$

2. We have $1 + a = (3, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix})$ is a T-clean element in R with a T-clean decomposition as,
$$(3, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}) = (3, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}) + (0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}).$$

So, $a+e=(5,\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix})$ is a T-clean element in R with a T-clean decomposition as,

$$(5,\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}) = (5,\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}) + (0,\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}).$$

3. We take $b=(4,\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix})$ such that $b+e=(1,\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix})$ is a clean element in R with a clean decomposition as,

$$(1,\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}) = (1,\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}) + (0,\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}).$$

Then it is not difficult to show that b is a T-clean element in R.

$End(R)$ denotes the endomorphism ring of R consisting all the endomorphisms on R. Let $\sigma \in End(R)$ and define the rule $xa = \sigma(a)x$ for all $a \in R$ and some indeterminate x. Then $R[x,\sigma]$ consisting all polynomials of finite degree with the multiplication rule of a and x is a ring called the skew or twisted polynomial ring under polynomial addition and multiplication. The skew upper triangular matrix ring is denoted by $ST_n(R,\sigma)$ according to A.Isfahani [18]. For the multiplication rule of any two matrix in the ring $ST_n(R,\sigma)$ we refer to [18].

Theorem 27. *Let R be a ring with $2 \in U(R)$ and $\sigma \in End(R)$. Then the followings are equivalent for any $n \in \mathbb{N}$.*

1. *R is a T-clean ring.*
2. *$\frac{R[x,\sigma]}{<x^n>}$ is T-clean.*
3. *$ST_n(R,\sigma)$ is T-clean.*

Proof. $(a) <=> (b)$: Define a ring homomorphism $\phi : \frac{R[x,\delta]}{<x^n>} \to R$ by $\phi(a_0 + a_1x + ... + a_{n-1}x^{n-1} + <x^n>) = a_0$. Then clearly ϕ is an epimorphism. We consider $I = ker(\phi) =< x+ <x^n>>$ which is a nil ideal of $\frac{R[x,\delta]}{<x^n>}$. Therefore, by Theorem 20 we have the required result.
$(b) <=> (c)$: From [18] we have $ST_n(R,\delta) \cong \frac{R[x,\delta]}{<x^n>}$. Hence the result. ∎

Theorem 28. *The upper triangular matrix ring $R = UT_n(R)$ is T-clean if and only if R is T-clean.*

Proof. $(=>)$ It is clear from the Proposition 16 with the epimorphism $\phi : UT_n(R) \to R$ by $\phi((a_{ij})_n) = a_{11}$.

(<=) For the converse it is enough to prove for $n = 2$. We consider R is a T-clean ring and $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in UT_2(R)$. Then we have $a = u_1 + t_1$ and $c = u_2 + t_2$ for some $u_1, u_2 \in U(R)$ and $t_1, t_2 \in T(R)$. Now rewriting,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} u_1 & b \\ 0 & u_2 \end{pmatrix} + \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

we get a T-clean decompositions in $UT_2(R)$. Therefore, the result is true for $n = 2$. Following this process we can easily prove the result for any n. ■

Example 29. The upper triangular matrix ring over $\mathbb{Z}_4$ is a T-clean ring, but over $\mathbb{Z}$ is not a T-clean ring. Because $\mathbb{Z}_4$ is a T-clean ring but $\mathbb{Z}$ is not a T-clean ring.

Theorem 30. *Let A, B are any ring and M be left$-A$ right$-B$ bimodule. Then the general upper triangular matrix ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is T-clean if and only if A and B are T-clean.*

Proof. (=>) It is clear from the Proposition 16 by defining natural epimorphisms $R \to A$ and $R \to B$.
(<=) We consider $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in R$ be any element. As A, B are T-clean there exist $u_1 \in U(A), u_2 \in U(B), t_1 \in T(A)$ and $t_2 \in T(B)$ such that

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} u_1 + t_1 & m \\ 0 & u_2 + t_2 \end{pmatrix} = \begin{pmatrix} u_1 & m \\ 0 & u_2 \end{pmatrix} + \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

which is a T-clean decomposition in R. Therefore, R is a T-clean ring. ■

Corollary 31. *If R be an abelian ring and $t \in T(R)$. Let $\bar{t} = 1 - t^2$. Then tR and $\bar{t}R$ are T-clean ring if and only if R is a T-clean ring.*

Proof. Since R is an abelian ring by the pierce decomposition of R we have $R \cong tR \oplus \bar{t}R \cong \begin{pmatrix} tR & 0 \\ 0 & \bar{t}R \end{pmatrix}$. Now from Theorem 30 we get the result. ■

Definition [11]**.** For any ring R. Then $a \in R$ is called an $(s, 2)$ element if a is a unit or sum of two units in R. A ring R is called an $(s, 2)$ ring if every element of R is an unit or sum of two units in R.

Theorem 32. *Let R be a ring such that it contains no non-trivial idempotents. If R is a T-clean ring then R is an $(s, 2)$ ring.*

Proof. Let R be a T-clean ring and $a = u + t$. Then by [15] Lemma 2.1 there exist $e \in idem(R)$ and $v \in U(R)$ with $v^2 = 1$ and $ev = ve$ such that $t = ev$. If $e = 0$ we have $t = 0$ and if $e = 1$ we have $t = v$. Therefore, in both the cases we have $a = u$ or $a = u + v$. Therefore, R is an $(s, 2)$ ring. ∎

Example 33. $\mathbb{Z}_8$ is a T-clean ring with only trivial idempotents. We notice that $1, 3, 5, 7$ are units also we write all non units as $0 = 1 + 7, 2 = 1 + 1, 4 = 1 + 3, 6 = 1 + 5$. Therefore, $\mathbb{Z}_8$ is an $(s, 2)$ ring.

Proposition 34. *Let p be a prime with $p \equiv 1(mod\ 4)$ and $R = \mathbb{Z}_p[i]$. Then the T-clean elements of R are of the form $(a+c)+i(b+d)$ such that $a^2+b^2 \in U(\mathbb{Z}_p)$, $c^2 = \frac{1-p}{4}$ and $d^2 = \frac{p-1}{4}$.*

Proof. Rewriting $x = (a + c) + i(b + d) = (a + ib) + (c + id)$. Then clearly $a + ib \in U(R)$ since $(a + ib)^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = (a^2 + b^2)^{-1}(a - ib) \in R$. Again by Lemma 10 we have $c + id \in T(R)$. Therefore, x is a T-clean. ∎

In support of Proposition 34 we consider the following example.

Example 35. Let $p = 13$ and $R = \mathbb{Z}_{13}[i]$. Then $9 + 10i$ is a T-clean element as $9 + 10i = (2 + 6i) + (7 + 4i)$. Here, $2 + 6i \in U(R)$ and $7 + 4i \in T(R)$.

Definition [17]**.** For any ring R an idempotent $e \in R$ is called $q - central$ if $eR(1 - e)Re = 0$ equivalently $erse = erese$ for all $r, s \in R$. If each idempotents in R are $q - central$ then R is called a $q - abelian$ ring.

Proposition 36. *Let R be a T-clean ring and $e \in idem(R)$ be a q-central. Then eRe is also a T-clean ring.*

Proof. Let $eae \in eRe$ be any element. As R is T-clean there exist $u \in U(R)$ with $u^{-1} = v$ and $t \in T(R)$ such that $a = u+t$. Then $eae = e(u+t)e = eue+ete$. Since e is q-central by applying Definition 3 we have $(eue)(eve) = eueve = euve = e = (eve)(eue)$ also $(ete)^3 = (ete)(ete)(ete) = (etete)(ete) = et^2ete = et^3e = ete$. Therefore, $eue \in U(eRe)$ and $ete \in T(eRe)$ and hence eRe is a T-clean ring. ∎

Theorem 37. *Let R be a ring which is abelian and $e \in idem(R)$. Then R is a T-clean ring if and only if eRe is a T-clean ring.*

Proof. (=>) We consider $eae \in eRe$. Replacing $a = u + t$ for some $u \in U(R)$ with $u^{-1} = v$ and $t \in T(R)$, we have $eae = eue + ete$. As, R is an abelian ring we have $eue.eve = e$ and $(ete)^3 = etetete = ete \in T(eRe)$. Therefore, eRe is a T-clean ring.
(<=) Proposition 19. ∎

Corollary 38. *Let R be a ring which is abelian. Then R is a clean ring if and only if R is T-clean.*

Proposition 39. *Let $a = u + t$ be a T-clean decomposition. Then for any unit $w \in U(R)$; aw, wa is T-clean if and only if $w \in I(t)$.*

Proof. We have $aw = uw + tw$. Since product of two unit is an unit we got uw is an unit. Again we have $(tw)^3 = twtwtw = tw \iff twt = t$. Thus $tw \in T(R) \iff w \in I(t)$. Similar proof for wa. Hence the result. ∎

Proposition 40. *Let R be a T-clean ring. Then any element of R can be written as sum of unit, idempotent, and square root of 1.*

Proof. We consider $x = w + t$ be a T-clean decomposition. As, $t \in T(R)$ by Lemma 11 we can express $t = f + v$ for some $f \in idem(R)$ and $v \in U(R)$. Applying a similar computation as in the proof of Lemma 11 we have $v = ue - (1 - e)$. Now $v^2 = (ue - (1 - e))^2 = ueue - (1 - e)ue - ue(1 - e) + (1 - e) = 1$. Therefore, $x = w + f + v$ is the required decomposition. ∎

Proposition 41. *If R is an uniquely T-clean ring. Then $2 = 0$ and $2t^2 = 0$ in R for any $t \in T(R)$. In other words uniquely T-clean ring has characteristics 2.*

Proof. Let $t \in T(R)$. Writing the T-clean decompositions of 0 in R we have.

$$0 = (-1) + 1 \tag{1}$$

$$0 = 1 + (-1) \tag{2}$$

$$0 = (2t^2 - 1) + (1 - 2t^2) \tag{3}$$

From equations 1 and 2 we have $2 = 0$ and from equations 1 and 3 we have $2t^2 = 0$. ∎

Proposition 42. *Let R be a ring and $J(R)$ be its Jacobson's radical. If $S = \frac{R}{J(R)}$ has identity $a^3 = a$ and idempotents are lifted modulo $J(R)$ then R is a T-clean ring.*

Proof. Let $x \in R$ and $S = \frac{R}{J(R)}$ has identity $a^3 = a$. Then by Lemma 13 we have two commuting idempotents $\bar{e}, \bar{f} \in S$ such that $\bar{x} = \bar{e} + \bar{f}$. As idempotents are lifted modulo $J(R)$ we may consider $e, f \in idem(R)$ which also commutes such that $a = e + f + b$ for some $b \in J(R)$. Rewriting $a = e - (1 - f) + (1 + b)$. Here $1 + b \in U(R)$ and as difference of two commuting idempotent is a tripotent, we get $e - (1 - f) \in T(R)$. Therefore, a is a T-clean and hence R is T-clean ring.
∎

Let M be an $(R-R)$ bimodule. The trivial ideal extension is defined as $\mathcal{I}(R, M) = R \oplus M$, which is clearly an abelian group and it is a ring if we define the multiplication as $(r, m_1)(s, m_2) = (rs, rm_2 + m_1 s)$ for any $r, s \in R$ and $m_1, m_2 \in M$. $(1, 0)$ is the multiplicative identity of the ring $\mathcal{I}(R, M)$. In the same abelian group structure $R \oplus M$, if we define the multiplication as $(r, m_1)(s, m_2) = (rs, rm_2 + m_1 s + m_1 m_2)$. Then we get the general ideal extension ring we denote it by $\mathbf{I}(R, M)$. For more details of ideal extension we refer to [10].

Proposition 43. *Let M be an $(R-R)$ bimodule. If the trivial ideal extension $\mathcal{I}(R, M)$ is T-clean, then R is T-clean. The reverse implication holds if for every $m \in R$ and $u \in U(R)$, $mu = um$.*

Proof. We assume that $\mathcal{I}(R, M)$ is a T-clean ring. Define $\phi : \mathcal{I}(R, M) \to R$ by $\phi(r, m) = r$. Then clearly ϕ is an epimorphism. Therefore, by Proposition 19, R is a T-clean ring. For the converse, we assume R is T-clean. Let $(r, m) \in \mathcal{I}(R, M)$. There exist $u \in U(R)$ and $t \in T(R)$ such that $r = u + t$. Thus $(r, m) = (u, m) + (t, 0)$. Clearly $(t, 0) \in T(\mathcal{I}(R, M))$. Now Let $x = -(u^{-1})^2 m$. Then $(u, m)(u^{-1}, x) = (1, ux + mu^{-1}) = (1, mu^{-1} - u^{-1}m) = (1, 0)$. As by assumption $mu = um \implies u^{-1}m = mu^{-1}$. similarly we can show $(u^{-1}, x)(u, m) = (1, 0)$. Therefore, $(u, m) \in \mathcal{I}(R, M)$. Therefore, $\mathcal{I}(R, M)$ is a T-clean ring. ∎

Proposition 44. *Let M be an $(R-R)$ bimodule. If the general ideal extension $\mathbf{I}(R, M)$ is T-clean, then R is T-clean. The everse implication holds if for every $m \in R$ and $u \in U(R)$, $m + mu + m^2 u = 0$ with $mu = um$.*

Proof. Proof is similar literature to the Proposition 44. For the converse part we take $x = u^{-1}m$. ∎

4. Conclusions and scope of further study

In the past few years, researchers have contributed several results and characterized rings where every elements can be written as sum of two other elements. We have discussed several properties and established certain characterizations of T-clean rings. This class of rings can be considered to be a more generalized version of the algebraic structure of clean rings and invo tri-clean rings. About the uniquely T-clean ring structure, Proposition 41 tells us that it can have characteristic 2. In future we can study more about uniquely T-clean rings and its characterization. From Proposition 36 we can further characterize this class of rings using the concept of quarter central elements, which is a fairly new concept given by T.Y. Lam. We can also study the graph theoretic representations of T-clean elements in a ring R. Moreover, we can extend the characterizations of T-clean rings using the notion of additive commutators.

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