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ON $L(2,1)$-ORDER SUM SIGNED GRAPH OF A FINITE GROUP

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Abstract

In this paper, we have constructed a color-induced signed graph of an algebraic graph, called the $L(2,1)$-order sum signed graph of a group. Based on the nature of the group, the $L(2,1)$-span of the order sum graph is obtained and the structural aspects of thus obtained $L(2,1)$-order sum signed graph such as planarity, chordality, etc. have been investigated. We have also defined an automorphism which turns out to be the only possible automorphism on the graph and have investigated the structural aspects of the graph such as edge transitivity and vertex transitivity. Further, a line-signed graph of $L(2,1)$-order sum signed graph, which is a line graph with a signing protocol defined for the edges, has also been introduced. We have also explored the regularity of the line-signed graph.

Keywords: $L(2,1)$-coloring, $L(2,1)$-order sum signed graph, signed graph homomorphism, pseudo-planarity, positive chordality, negative chordality.

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1. Introduction

The notion of the signed graph was introduced in [9] to model a social psychological problem. Given a room with a set of people, where the vertices of the graph represent people and if two persons share a healthy relation then the edge is positive and if the two persons are enemies then they share a negative edge. An undirected graph with signed edges is called a *signed graph*, denoted by S. We can obtain a signed graph from every undirected graph by defining a function

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called the *signature function* which does the job of assigning signs to the edges of the graph. There are structural properties of signed graphs that are of significance, some of them are; *balance*, a signed graph is said to be balanced if every cycle in the graph has an even number of negative edges; *clusterability*, a signed graph is said to be *k-clusterable* if we can partition its vertex set into k subsets such that every positive edge has both its end vertices in the same cluster and every negative edge has both its end vertices in two different clusters. A graph is said to be *marked graph* if its vertices are assigned signs. A signed graph that is also a marked graph is said to be *sign compatible* if every negative edge has both its end vertices marked as negative and no positive edge has both its end vertices marked as negative. The *switching* of a signed graph may be defined as a mapping $\sigma^{\mu}(e) : E(\Gamma) \to \{+,-\}$, defined by $\sigma^{\mu}(uv) = \mu(u)\sigma(e)\mu(v)$, for all $uv \in E(\Gamma)$, where μ is the marking and σ is the signature function on S. For detailed content on the signed graph concepts, see [16].

Graph coloring originated from the well-known four-color problem (see [6]). Due to its wide variety of applications, graph coloring has emerged as a separate field in graph theory, called the chromatic graph theory.

Chromatic graph theory is itself a vast field in which we come across numerous types of coloring a graph. The proper coloring of a graph is assigning colors to the vertices in such a way that no two adjacent vertices receive the same color. One example of the application of proper vertex coloring is the scheduling problem. The channel assignment problem is another such application of graph coloring. For more details on graph colorings, see [6].

The types of colorings, mostly used in channel assignment problems are distance-based colorings. In our study, we are interested in one such distance-based coloring, called the $L(h,k)$-coloring. An $L(h,k)$-coloring, for $h,k \in \mathbb{Z}^+ \cup \{0\}$, is a vertex coloring such that the adjacent vertices should have a color difference of at least h and the vertices at a distance 2 should have a color difference of at least k. In our paper, we restrict ourselves to $L(2,1)$-coloring which is a special case of $L(h,k)$-coloring. A graph Γ is said to admit an $L(2,1)$-coloring if the adjacent vertices receive colors in such a way that their color difference is at least 2 and the vertices at distance 2 receive colors in a way that their color difference is at least 1. It is interesting to note that $L(2,1)$-coloring admits proper coloring. A single graph can be colored in any number of ways to obtain an $L(2,1)$-coloring, but we are interested in an $L(2,1)$-coloring of a graph that uses minimum colors and thus the term $L(2,1)$-span, denoted by $\lambda(\Gamma)$, was introduced. Let c be one of the $L(2,1)$-colorings of the graph Γ. Then, the c-span, denoted by $\lambda(c)$ of the graph Γ is the difference between the maximum color and minimum color used in coloring the Γ. The $L(2,1)$-span of the graph Γ is the minimum value obtained out of all c-spans of Γ.

2. Results and discussions

Recall that a *clique* of a graph Γ is an induced subgraph of Γ which is a complete graph. Also, a vertex of a graph Γ is a *simplicial vertex* if its neighbours induce a clique.

Definition 2.1. A graph is called a $(k+1)$-*clique graph* if each simplicial vertex forms $(k+1)$-clique with the same set of vertices. It is a complete split graph.

2.1. L(2,1)-span of order sum graph of finite groups

In this paper, the group is denoted by G, $o(G)$ is the order of the group.

Definition 2.2 [2]**.** An *order sum graph* $\Gamma_{os}(G)$ of a finite group G is a graph with $V(\Gamma_{os}(G)) = G$ and $u, v \in V(\Gamma_{os}(G))$ are adjacent if $o(u) + o(v) > o(G)$.

A group is cyclic if it has at least one element whose order is equal to the order of the group. If the group is not cyclic then every element in the group has oder strictly less than n, the order of the group . Thus, we have the following proposition.

Proposition 2.3 [2]**.** The order sum graph $\Gamma_{os}(G)$ of order n is a null graph if and only if G is not a cyclic group.

Throughout the paper, the color set used is $\mathcal{S} = \{0, 1, 2, \ldots\}$. Using Proposition 2.3, we get the L-span of the order sum graph of an acyclic group as follows.

Proposition 2.4. The L-span of an order sum graph of an acyclic group is 0.

Proof. By Proposition 2.3, we have the order sum graph obtained to be a null graph for an acyclic group. Thus, all the vertices can be assigned the same color, the minimum color from the set $\mathcal{S}$ and hence L-span of such graphs is 0. ■

In a group of prime order except the identity element, every other element acts as a generator of the group. This leads to the following observation.

Proposition 2.5 [2]**.** The order sum graph associated with a group of prime order p is complete.

For a complete graph, the L-span is known. Hence, the L-span of the order sum graph of a group of prime order leads to the following proposition.

Proposition 2.6. The L-span of the order sum graph of the prime order group is $2(n-1)$.

Proof. By Proposition 2.5, the prime order group is a cyclic group with all its elements being generators except the identity. Thus, $\Gamma_{os}(G)$ induces a complete graph as an order sum graph. Hence, as we know for any complete graph of order n, the L-span is $2(n-1)$. The result follows. ■

A cyclic group will have at least one such element whose order is equal to the order of the group. This observation leads to the following proposition.

Proposition 2.7 [2]**.** If G is a cyclic group of finite order then $diam(\Gamma_{os}(G)) \leq 2$.

The L-span of an order sum graph of a finite cyclic group with k generators is obtained and discussed in the following proposition.

Proposition 2.8. The L-span of an order sum graph of a finite cyclic group of order n is $n+k-1$, where k is the number of generators in the group.

Proof. By Proposition 2.7, for an order sum graph of a finite cyclic group, the diameter is less than or equal to 2. Suppose the diameter is 1, then the graph is complete, and hence the L-span is $2(n-1)$. If the diameter is 2, then consider the group with k generators, this implies that the corresponding $L(2,1)$-order sum signed graph will be a $(k+1)$-clique graph with say, $v_1, v_2, \ldots, v_k$ vertices adjacent to all the vertices. As it is known that L-span of K_n is $2n-2$. Thus, our K_{k+1} will be colored with the span being $2k-2$. The remaining simplicial vertices will be colored with consecutive colors starting from $2k$ and so on. The remaining $n-k$ vertices are non-generators and exactly $n-k$ vertices are mutually non-adjacent, hence the L-span is $2k+(n-k-1) = n+k-1$. ■

3. L(2,1)-ORDER SUM SIGNED GRAPH OF A FINITE CYCLIC GROUP

Definition 3.1 [3]**.** An *induced signed graph* of a graph is a signed graph obtained from an ordinary graph Γ whose signature function is defined by $\sigma(uv) = (-1)^{|\phi(u)-\phi(v)|}$, where ϕ is a mapping defined by $\phi : V(\Gamma) \longrightarrow \mathbb{Z}$, set of weights (or labels), associated with each vertex in the signed graph.

Definition 3.2 [4]**.** Let Γ be a simple graph having an $L(2,1)$-coloring, $c : V(\Gamma) \to \mathbb{N} \cup \{0\}$, with $\lambda(c) = \lambda(\Gamma)$. Then, the $L(2,1)$-*color induced signed graph* or $L(2,1)$-*signed graph* denoted by $\sigma((\Gamma))$ of Γ is the signed graph, induced by Γ, whose signing function σ is given by $\sigma(uv) = (-1)^{|c(u)-c(v)|}$ for all $uv \in E(\Gamma)$.

Figure 1 is an example of an $L(2,1)$-order sum signed graph of $G = \mathbb{Z}_8$.

Note that we are using dashed lines to represent negative edges and thick lines to represent positive edges throughout the paper.

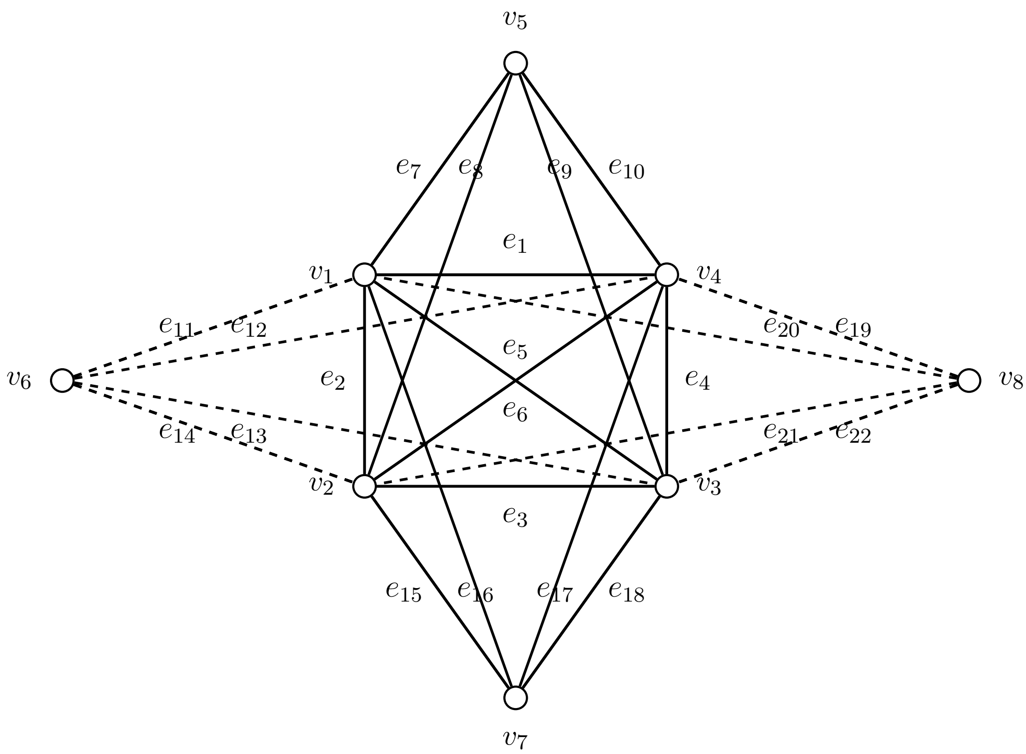

Figure 1. $L(2,1)$-order sum signed graph of G.

The $L(2,1)$-order sum signed graph denoted by $\sigma(\Gamma_{os}(G))$ is an $L(2,1)$-color induced signed graph of an order sum graph $\Gamma_{os}(G)$ of a finite group G. We will now see an interesting result on the number of positive and negative edges of an $L(2,1)$-order sum signed graph of a finite cyclic group.

Theorem 3.3. Let G be a cyclic group with k generators and $\Gamma_{os}(G)$ be its order sum graph. Then, $\Gamma_{os}(G)$ has

(i) $\frac{k(k-1)}{2} + pk$ positive edges if p simplicial vertices are colored with even numbers.

(ii) qk negative edges if q simplicial vertices are colored with odd numbers.

Proof. Consider the k generators $v_1, v_2, \ldots, v_k$, in the finite cyclic group that are the vertices in the $L(2,1)$-order sum signed graph. These vertices induce a complete graph of order k; thus one may see that according to our signing protocol, all the edges in the clique receive a positive sign. Thus, $\frac{k(k-1)}{2}$ positive edges within the clique induced by generators. Now, suppose p simplicial vertices are colored with even numbers, then the edges between the vertices $v_1, v_2, \ldots, v_k$

and p simplicial vertices receive a positive sign. Thus, we have $\frac{k(k-1)}{2}+pk$ positive edges. Now, suppose q simplicial vertices are colored with odd numbers, then clearly the edges between $v_1, v_2, \ldots, v_k$, and each of q simplicial vertices receive a negative sign. Thus, we have qk negative edges in the graph. ■

The idea of clusterability helps us in partitioning the vertex set of a signed graph. The k-clusterability of a signed graph implies that we can partition its vertex set into k partitions or clusters. The theorem below gives us a result on the clusterability of an $L(2,1)$-order sum signed graph of a finite cyclic group.

Theorem 3.4. The $L(2,1)$-order sum signed graph of a finite cyclic group is at most $(q+1)$-clusterable, where q is the number of simplicial vertices that are colored with odd numbers.

Proof. The vertices in the clique share positively signed edges mutually. Further, the simplicial vertices that are assigned colors with even numbers, let them be p in number, also share positively signed edges with the vertices in the clique. Thus, all the vertices in the clique along with the p simplicial vertices can be put into one cluster. The remaining are the simplicial vertices that are colored with odd numbers, let them be q in number, implying that they share negatively signed edges with the vertices belonging to the clique. We can put each of these q vertices in one single cluster as they form an independent set or put each of q simplicial vertices into q different cluster, each cluster consisting of exactly one vertex. This is the maximum possible way of forming clusters of the vertices of our graph. Hence, the graph is at most $(q+1)$-clusterable. ■

A graph is said to be *planar* if we can draw the graph in a way that no two edges intersect each other. In the following definitions, we discuss about planarity of a signed graph restricting ourselves to the signs of the edges.

Definition 3.5. A signed graph S is said to be *pseudo-planar* if no two edges of the same parity cross each other.

There can be signed graphs where the edges crossing each other can be positive edges. Thus, we have the following definition.

Definition 3.6. A signed graph S is said to be *positive pseudo-planar* if no two positive edges cross each other.

There can be signed graphs where the edges crossing each other can be negative edges. Thus, we have the following definition.

Definition 3.7. A signed graph S is said to be *negative pseudo-planar* if no two negative edges cross each other.

The above definitions leads to the following results.

Theorem 3.8. An $L(2,1)$-order sum signed graph of a finite cyclic group with exactly two generators is pseudo-planar.

Proof. The $L(2,1)$-order sum signed graph of a group with two generators is a theta graph $\theta_{t,2}$, where $\theta_{t,2}$ is a graph such that two adjacent vertices have t internally disjoint paths of length 2. The graph has an embedding such that no two edges of any parity cross each other. Hence, the graph is pseudo-planar. ■

Theorem 3.9. An $L(2,1)$-order sum signed graph of a group with 3 generators is pseudo-planar if its independent set has not more than two vertices belonging to the same parity of colors.

Proof. Referring to the proof of Proposition 2.8, the $L(2,1)$-order sum signed graph of a group containing 3 generators is a 4-clique graph. The complete graph K_4 is a planar graph. But when considering more than two vertices belonging to the same parity of colors, we get an embedding of a complete bipartite graph $K_{3,3}$, giving rise to more than two chords of the same sign conflicting with each other. Hence, the graph in this case is not pseudo-planar. ■

Theorem 3.10. If the number of generators k of a finite cyclic group is at least 4, then the corresponding $L(2,1)$-order sum signed graph is not pseudo-planar.

Proof. If the number of generators is greater than or equal to 4, then we obtain a d-clique graph with $d \geq 5$. Thus, the embedding will have a complete graph K_5 induced subgraph in it having more than two chords of the same sign crossing each other. Hence, the graph is not pseudo-planar. ■

Definition 3.11 [14]**.** Let $C_k; k \geq 4$, be a cycle in a graph Γ. An edge in Γ which joins two non-consecutive vertices in Γ is called a *chord*. A graph Γ is said to be a chordal graph if every cycle $C_k; k \geq 4$, in Γ has a chord.

We have extended the idea of the chordality of a graph to a signed graph. Restricting the cycle to a positive homogeneous cycle in the signed graph we define positive chordality as follows

Definition 3.12. A signed graph is said to be *positive chordal* if it contains a positive homogeneous cycle of length greater than or equal to 4 with positive chords.

Further, restricting the cycle to a negative homogeneous cycle in the signed graph we define negative chordality as follows.

Definition 3.13. A signed graph is said to be *negative chordal* if it contains a negative homogeneous cycle of length greater than or equal to 4 with negative chords.

Using Definition 3.12, we have the following theorem on the positive chordality of $L(2,1)$-order sum signed graph.

Theorem 3.14. Every $L(2,1)$-order sum signed graph of a finite cyclic group is positive chordal.

Proof. The $L(2,1)$-order sum signed graph is a triangulated graph when the generators in the corresponding group are greater than or equal to 2. The smallest color used will be 0 and to obtain the minimum span we have considered coloring the non-simplicial vertices first. Thus, as the set of non-simplicial vertices forms a clique and by Definition 3.2 all of them will be assigned even colors. The edges between all the nonsimplicial vertices will be positive. Similarly, any simplicial vertex colored with an even number will have positive edges with their neighbours, thus making the graph positive chordal. ■

Using Definition 3.13 we have the following result on the negative chordality of $L(2,1)$-order sum signed graph.

Theorem 3.15. An $L(2,1)$-order sum signed graph of a finite cyclic group is negative chordal.

Proof. We know that every $L(2,1)$-order sum signed graph is a triangulated graph. Thus, any simplicial vertex that is colored with an odd number forms a triangle, the neighbours of whose vertices are already colored with even numbers. Hence, we never get a negative homogeneous induced C_3 as two negative edges are in each C_3 contributed by a simplicial vertex that is colored with an odd number. Thus, the graph is negative chordal. ■

Having made some structural observations in the previous results, we will now define an automorphism of an $L(2,1)$-order sum signed graph to itself as follows.

Theorem 3.16. The mapping $\gamma : V(\sigma(\Gamma_{os}(G)) \longrightarrow V(\sigma(\Gamma_{os}(G))$, where $\sigma(\Gamma_{os}(G))$ denotes the $L(2,1)$-order sum signed graph of a finite cyclic group G with k generators and $2 \leq p \leq q \leq n-k$, defined by
(1)

$$\gamma(v_i) = \begin{cases} v_j & \textit{whenever } \ v_i, v_j \in V(Q_k), \\ v_k & \textit{if } \ c(v_i) \textit{ and } c(v_k) \textit{ are of odd parity}, \ \textit{whenever } \ v_i, v_k \notin V(Q_k), \\ v_l & \textit{if } \ c(v_i) \textit{ and } c(v_l) \textit{ are of even parity}, \ \textit{whenever } \ v_i, v_l \notin V(Q_k). \end{cases}$$

where, $c(v_i)$ denotes the color received by vertex v_i in an $L(2,1)$-coloring c of the order sum graph and Q_k is the clique induced by the set of k generators in the group. Then, the mapping γ is an automorphism.

Proof. Consider an arbitrary vertex $v_i \in V(\sigma(\Gamma_{os}(G))$. If $v_i \in V(Q_k)$, then v_i will have both positive and negative edges incident on it. Hence, if we map v_i to any vertex that is not in Q_k, then some simplicial vertex will be mapped to a vertex in Q_k, which violates the nature of the edges incident on it. Thus, if $v_i \in V(Q_k)$, then it can be mapped to any vertex belonging to Q_k only. If $v_i \notin Q_k$, then v_i is a simplicial vertex. That is, all edges incident to it are all positive or all negative. Thus, if $c(v_i)$ is even, then v_i can only be mapped to a simplicial vertex that is colored with an even number. Similarly, if $c(v_i)$ is odd, then, v_i can be mapped to a simplicial vertex that is colored with an odd number. Thus, γ is an automorphism. ■

It is interesting to observe that γ is the only possible automorphism on $\sigma(\Gamma_{os}(G))$ and thus we can list out all possible ways of mapping the vertices, which is mentioned in the following corollary.

Corollary 3.17. There are $k!p!q!$ permutations possible in the automorphism γ, where k denotes the number of generators in the group G, p and q denote the number of simplicial vertices in the graph $\sigma(\Gamma_{os}(G))$ that have only positive edges and negative edges incident to them respectively.

Proof. By Theorem 3.16, γ is an automorphism, which implies that γ is one-one and onto function. Given a set of k generators, they can be mapped in $k!$ ways amongst themselves. Similarly, p simplicial vertices can be mapped in $p!$ ways and q simplicial vertices can be mapped in $q!$ ways, amongst themselves. Thus. for each permutation out of $k!$ possible permutations, we have $p!$ possible combinations and thus $k!p!$ permutations possible. And for each permutation out of $k!p!$ permutations, there are $q!$ combinations possible. Thus, we have $k!p!q!$ permutations possible in the mapping defined by γ. ■

The idea of vertex transitivity and edge transitivity can be extended to signed graphs, but the only difference comes with signs of the edges being preserved when investigating the mapping. This leads us to the following observations.

Theorem 3.18. An $L(2,1)$-order sum signed graph $\sigma(\Gamma_{os}(G))$ of a finite cyclic group G with k generators and $2 \leq p \leq q \leq n-k$, is not vertex transitive.

Proof. The vertices contributing positive edges are the ones that belong to the clique Q_k or are the simplicial vertices that are colored with even numbers. Now, it is easy to observe that the edges incident on each vertex that is not part of the clique Q_k, are all positive or are all negative. Consider a vertex in the clique Q_k, say v_i, v_i has both positively signed and negatively signed edges incident on it. Thus, if we map any vertex in Q_k to any simplicial vertex that has positively signed edges incident on it, we can see that the negative edges that were adjacent to edges on v_i will now be adjacent to the positively signed edges of a simplicial vertex, thus violating the structure of our graph. Hence, we can only map any vertex in Q_k to a vertex in Q_k. Hence, the $L(2,1)$-order sum signed graph is not vertex-transitive. ■

Theorem 3.19. An $L(2,1)$-order sum signed graph $\sigma(\Gamma_{os}(G))$ of a finite cyclic group with k generators and $2 \leq p \leq q \leq n-k$, is not edge transitive.

Proof. Consider the positive edges in the graph. Let us assume that any two positive edges are mapped to each other. Let $e_1 = v_1v_2$ be a positive edge such that $v_1, v_2 \in V(Q_k)$ and $e_2 = v_1v_{k+1}$ be another positive edge where $v_{k+1} \notin V(Q_k)$. Let π be a permutation that maps e_1 to e_2; hence the possibility is that either v_1 is mapped to v_{k+1} and v_2 is mapped to v_1 or v_1 mapped to v_1 and v_2 mapped to v_{k+1}. If v_1 is mapped to v_k+1 we see that v_1 has both negative and positive edges incident to it, whereas, v_k+1 has only positive edges incident to it and hence we cannot map v_1 to v_k+1, the same argument holds if v_2 is mapped to v_k+1. Thus, the $L(2,1)$-order sum signed graph is not edge transitive. ■

4. The line-signed graph of L(2,1)-order sum signed graph of a finite group

Definition 4.1 [8]**.** For a signed graph $S = (\Gamma, \sigma(\Gamma))$, $L(S)$ is the line-signed graph of S with $V(L(S)) = E(S)$ and for any $u, v \in V(L(S))$, $uv \in E(L(S))$ if and only if u and v share a common vertex in S and $\sigma(uv) = \sigma(u)\sigma(v)$.

Figure 2 is an example of a line-signed graph of the graph in Figure 1.

In a signed graph, $d^-(u)$ refers to the total number of negative edges incident to $u \in V(S)$, and $d^+(u)$ refers to the total number of positive edges incident to $u \in V(S)$. The following proposition gives the degree of a vertex in the line-signed graph. The term *tri-regular* in our context implies that $(d^-(u), d^+(u))$ takes three fixed pairs of values.

Proposition 4.2. The line-signed graph $L(S)$ of an $L(2,1)$-order sum signed graph of a finite cyclic group with k generators and $2 \leq p \leq q \leq n-k$, is a tri-regular signed graph.

Proof. By a known theorem, we have that for an edge $uv \in E(\Gamma)$ of a graph Γ, its degree $d(uv) = d(u)+d(v)-2$, where $u, v \in V(\Gamma)$. Since in the line graph, this edge acts as a vertex, the equation should hold. Consider $e \in V(L(S))$ such that $e = uv$ in $\sigma(\Gamma_{os}(G))$. For $u, v \in V(\sigma(\Gamma_{os}(G)))$, let $V(Q_k)$ denote the vertex set of the clique Q_k, $V(V_p)$ denote the set of p simplicial vertices and $V(V_q)$ denote the set of q simplicial vertices. We consider the following cases.

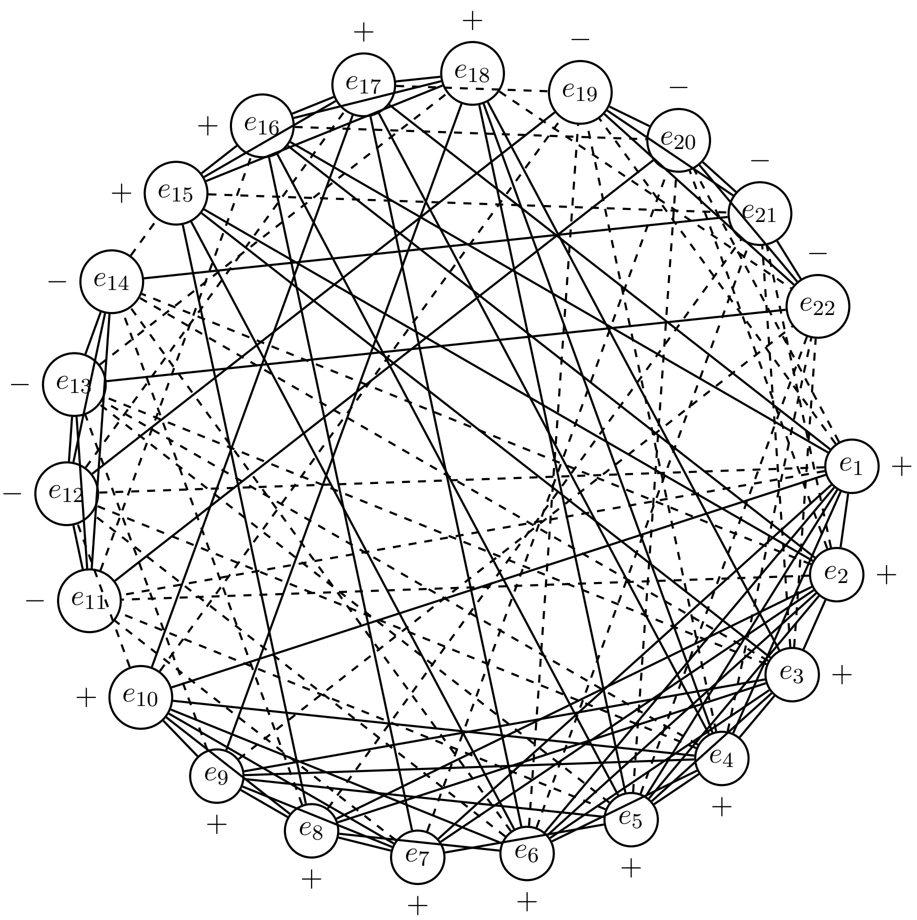

Figure 2. Line-signed graph of an $L(2,1)$-order sum signed graph from Figure 1.

Case 1. If $u, v \in V(Q_k)$, then $d^+(e) = d^+(u) + d^+(v) - 2 = k - 1 + p + k - 1 + p - 2 = 2k + 2p - 4$. Futher, $d^-(e) = d^-(u) + d^-(v) = q + q = 2q$. Thus, $(d^+(e), d^-(e)) = (2k + 2p - 4, 2q)$.

Case 2. If $u \in V(V_p)$, $v \in V(Q_k)$, (or $u \in V(Q_k)$ and $v \in V(V_p)$) then $d^+(e) = d^+(u) + d^+(v) - 2 = k - 1 + p + k - 2 = 2k + p - 3$. Further, $d^-(e) = d^-(u) + d^-(v) = q + 0 = q$. Thus, $(d^+(e), d^-(e)) = (2k + p - 3, q)$.

Case 3. If $u \in V(V_q)$, $v \in V(Q_k)$ (or $u \in V(Q_k)$ and $v \in V(V_q)$) then $d^+(e) = d^-(u)+d^-(v)-2 = k+q-2 = k+q-2$. Further,$d^-(e) = d^+(u)+d^+(v) = 0 + k - 1 + p = k - 1 + p$. Thus, $(d^+(e), d^-(e)) = (k + q - 2, k + p - 1)$.

Therefore, the only values that the pair $(d^+(e), d^-(e))$ can take is $(2k + 2p - 4, 2q), (2k + p - 3, q)$ and $(k + q - 2, k + p - 1)$. Thus, the line-signed graph of $L(2,1)$-order sum signed graph is tri-regular. ■

The motivation for introducing the concept of edge code is taken from the color code concept from the color connections of a graph (see [5]). As seen in Proposition 4.2, the number of negative and positive edges differ for different sets of vertices. Thus, edge code helps one in determining whether or not the vertex belongs to the clique Q_k.

Definition 4.3. For a signed graph $S = (\Gamma, \sigma)$ with $V(S)$ being its vertex set and $E(S)$ being its edge set, we define an *edge code* of a vertex $v_i \in V(S)$ as a tuple $(e_i, e_{i+1}, \dots, e_{i+k}, e_{i+k+1}, e_{i+k+2}, \dots, e_r)$ such that $e_j \in E(S), j \in \{i, i+1, \dots, i+k\}$ is a positive edge incident to the vertex v_i and $e_s \in E(S), s \in \{i+k+1, i+k+2, \dots, r\}$ is a negative edge incident to the vertex v_i.

We know that in the line-signed graph $V(L(S)) = E(S)$, which implies that e_j, e_s that are edges in $\sigma(\Gamma_{os}(G))$ are the vertices in its line-signed graph $L(S)$.

Fixing our i to a fixed v_i, where $v_i \in V(Q_k) \subseteq V(\sigma(\Gamma_{os}(G)))$, we define a homomorphism from $L(2,1)$-order sum signed graph of a finite cyclic group with k generators and $2 \leq p \leq q \leq n-k$, to its line-signed graph as follows.

Theorem 4.4. The mapping $\psi : V(\sigma(\Gamma_{os}(G))) \longrightarrow V(L(S))$ defined by

$$\psi(v_i) = \begin{cases} e_j, & if \ \ v_i \in V(Q_k) \cup V(V_p) \\ e_s, & if \ \ v_i \in V(V_q) \end{cases}$$

is a homomorphism provided the vertices are mapped in a way such that both the adjacency and the sign of the edges are preserved.

Proof. As v_i is fixed, its edge code $(e_i, e_{i+1}, \dots, e_{i+k}, e_{i+k+1}, e_{i+k+2}, \dots, e_r)$ is a fixed tuple. Since all these edges are incident to the vertex v_i, they share a common vertex, which is v_i. From the construction of the line graph and by Definition 4.1, we can verify that the adjacency condition and the sign of the edges are preserved concerning the defined mapping ψ. Hence, ψ is a homomorphism. ■

Conclusion

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References

[1] B.D. Acharya, *Set-valuation of a signed digraph*, J. Combin. System Sci. **37** (2012).

[2] J. Amreen and S. Naduvath, *Order sum graph of a group*, Baghdad Sci. J. **20(1)** (2021) 0181–0188.
https://doi.org/10.21123/bsj.2022.6480

[3] A. Aniyan and S. Naduvath, *Induced signed graphs of some classes of graphs*, Proc. Jangjeon Math. Soc. **23** (2020) 283–291.
https://doi.org/10.17777/pjms2020.23.2.283

[4] D. Bankapur and S. Naduvath, *A study on* $L(2,1)$*-signed graphs*, Communicated.

[5] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133(1)** (2008) 85–98.
https://doi.org/ 10.21136/MB.2008.133947

[6] G. Chartrand and P. Zhang, Chromatic Graph Theory (Chapman and Hall/CRC, 2008).
https://doi.org/10.1201/9781584888017

[7] J.A. Gallian, Contemporary Abstract Algebra (Chapman and Hall/CRC, 2020).
https://doi.org/10.1201/9781003142331

[8] M.K. Gil, Contributions to Some Topics in Graph Theory and Its Applications, Ph.D. thesis (Indian Institute of Technology, Bombay, 1983).

[9] F. Harary, *On the notion of balance of a signed graph*, Michigan Math. J. **2(2)** (1953) 143–146.
https://doi.org/10.1307/mmj/1028989917

[10] R. Naserasr, S. Sen and E. Sopena, *The homomorphism order of signed graphs*, J. Comb. Math. Comb. Comput. (2020).

[11] R. Naserasr and E. Rollová and E. Sopena, *Homomorphisms of signed graphs*, J. Graph Theory **79(3)** (2015) 178–212.

[12] P. Ochem, A. Pinlou and S. Sen, *Homomorphisms of signed planar graphs* (2014). arXiv:1401.3308

[13] C. Wang, *The signed matchings in graphs*, Discuss. Math. Graph Theory **28** (2008) 477–486.

[14] D.B. West *et al.*, Introduction to Graph Theory (Prentice hall Upper Saddle River, 2001).

[15] T. Zaslavsky *et al.*, Graphs, Gain Graphs, and Geometry a.k.a. Signed Graphs and their Friends (Lecture Notes-Binghamton University, 2008).

[16] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. **4(1)** (1982) 47–74.
https://doi.org/10.1016/0166-218X(82)90033-6

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