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STUDY OF DIFFERENTIAL IDENTITIES IN 3-PRIME NEAR-RINGS

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Abstract

The main objective in the present paper is to describe the structure of a 3-prime near-ring $\mathcal{N}$ that satisfies certain algebraic identities involving g-derivation. In addition, and to show the necessity of the different hypotheses used in our results, we will present at the end of this work examples which illustrate that the restrictions imposed are not superfluous.

Keywords: 3-prime near-rings, g-derivation, multipliers, commutativity.

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1. Introduction

Throughout this paper, $\mathcal{N}$ will represent a left near-ring and $\mathcal{Z}(\mathcal{N})$ its multiplicative center. For $x, y \in \mathcal{N}$, the symbols $[x, y]$ and $x \circ y$ denote the commutator $xy - yx$ and the anti-commutator $xy + yx$, respectively. A near-ring $\mathcal{N}$ is 3-prime

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if $x\mathcal{N}y = \{0\}$, where $x, y \in \mathcal{N}$, implies $x = 0$ or $y = 0$. Also, $\mathcal{N}$ is 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$ implies $x = 0$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $g : \mathcal{N} \to \mathcal{N}$ is called a left multiplier, if $g(xy) = g(x)y$ for all $x, y \in \mathcal{N}$, likewise g is said to be a right multiplier if $g(xy) = xg(y)$ holds for all pairs $x, y \in \mathcal{N}$. Moreover, g is called a multiplier if g is both a left multiplier and a right multiplier. A g-derivation d_g on $\mathcal{N}$ is defined as an additive mapping on $\mathcal{N}$ verifying $d_g(xy) = d_g(x)g(y) + xd_g(y)$ for all $x, y \in \mathcal{N}$. Clearly, we can consider each derivation on $\mathcal{N}$ as a g-derivation associated with $g = id_{\mathcal{N}}$, but the converse is not true in general. Thereby, this work is essentially independent of all works involving derivations, which gives more advantage in the case where g is a multiplier of $\mathcal{N}$.

Differential identities and additive maps are fundamental in the study of prime rings and subsequently contribute to the understanding of their algebraic structure. In this context, Divinsky [10] proved that the simple Artinian ring is commutative if it has a non-trivial commuting automorphism. In 1957, Posner [11] proved that the existence of a nonzero centralizing derivation on a prime ring forces this ring to be commutative.

A few years later, several authors have subsequently refined and extended these results in various directions using suitably constrained additive mappings, as Jordan derivations, generalized derivations, semiderivations and (σ, τ)-derivations acting either on whole ring or on appropriate subsets of the ring (see [1, 4, 9] and [14] for reference where further references can be found). However, in the case of near-rings, this type of study was not known until 1987, when the researchers Bell and Mason published their article entitled. On derivations in near-rings (see [6]) in which they used the notion of derivation as defined for rings. Later, using some appropriate restrictions on 3-prime near-rings, interesting results between the commutativity of the near-ring $\mathcal{N}$ and certain special types of mappings on $\mathcal{N}$, were obtained by several authors (see for example, [5, 7, 8, 12] and [13]).

Our main in the present paper, is to continue this line of investigation by studying the commutativity criteria of 3-prime near-rings using the notion of g-derivations.

2. Main results

To prove our results, we present some lemmas including two important new ones. One of them studies the right multiplication of $d_g(x)g(y) + xd_g(y)$ by $g(z)$, where $x, y, z \in \mathcal{N}$. The other lemma treats the zero-symmetric property of $\mathcal{N}$.

Lemma 1 [2, Theorem 2.9]. *Let $\mathcal{N}$ be a 3-prime near-ring. If $[x,y] \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) [3, Lemma 1.2 (iii)] If $z \in \mathcal{Z}(\mathcal{N})$ and $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.

(ii) [6, Lemma 1.5] If $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.

Lemma 3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $-(x \circ y) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof. Obviously, if $\mathcal{N} = \{0\}$ then $\mathcal{N}$ is a commutative ring. So, in the following we treat the case when $\mathcal{N}$ is not zero. By hypotheses given, we have $-(x \circ y) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Taking $y = xy$ we get $x(-(x \circ y)) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$ which, because of Lemma 2(i), implies that

$$x \in \mathcal{Z}(\mathcal{N}) \text{ or } -(x \circ y) = 0 \text{ for all } x, y \in \mathcal{N}.$$

Suppose that there exists $x_0 \in \mathcal{N}$ such that $x_0 \notin \mathcal{Z}(\mathcal{N})$. From the previous relation, we conclude that $x_0 \circ y = 0$ for all $y \in \mathcal{N}$, that is $x_0 y = y(-x_0)$. Replacing y by yt, we get $x_0 yt = yt(-x_0) = y(-x_0)t$ for all $t, y \in \mathcal{N}$. It follows that $y[-x_0, t] = 0$ for all $t, y \in \mathcal{N}$. Substituting yz in place of y and using the fact that $\mathcal{N}$ is 3-prime, we obtain $y = 0$ or $-x_0 \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Since $\mathcal{N} \neq \{0\}$, we infer that $-x_0 \in \mathcal{Z}(\mathcal{N})$. On the other hand, we have $x_0 \circ (-x_0) = 0 = (-x_0)(x_0 + x_0)$. Left multiplying the second side by r, where $r \in \mathcal{N}$, we find that $(-x_0)r(x_0 + x_0) = 0$ which implies that $(-x_0)\mathcal{N}(x_0 + x_0) = \{0\}$ which, in view of the 2-torsion freeness and 3-primeness of $\mathcal{N}$, implies that $x_0 = 0$. But, the relation $0 \circ y = 0$ for all $y \in \mathcal{N}$ gives $0 \in \mathcal{Z}(\mathcal{N})$, a contradiction with our assumption that $x_0 \notin \mathcal{Z}(\mathcal{N})$. Consequently, $x \in \mathcal{Z}(\mathcal{N})$ for all $x \in \mathcal{N}$ and therefore $\mathcal{N}$ is a commutative ring by Lemma 2(ii). ∎

Lemma 4. *Let $\mathcal{N}$ be a near-ring admits a g-derivation d_g associated with a left multiplier g. Then*

$$(d_g(x)g(y) + xd_g(y))g(z) = d_g(x)g(y)z + xd_g(y)g(z) \text{ for all } x, y, z \in \mathcal{N}.$$

Proof. By the defining property of d_g, we have for all $x, y, z \in N$,

$$\begin{aligned} d_g((xy)z) &= d_g(xy)g(z) + xyd_g(z) \\ &= \big(d_g(x)g(y) + xd_g(y)\big)g(z) + xyd_g(z), \end{aligned} \tag{1}$$

and

$$\begin{aligned} d_g(x(yz)) &= d_g(x)g(yz) + xd_g(yz) \\ &= d_g(x)g(yz) + xd_g(y)g(z) + xyd_g(z). \end{aligned} \tag{2}$$

Comparing (1) and (2), we get the required result. ∎

Lemma 5. *A near-ring $\mathcal{N}$ admits a g-derivation d_g associated with a left multiplier g if and only if it is zero-symmetric.*

Proof. Suppose that $\mathcal{N}$ is a zero-symmetric near-ring. We can see that the identity map I_d on $\mathcal{N}$ is a 0-derivation on $\mathcal{N}$. Conversely, assume that $\mathcal{N}$ has a g-derivation d_g, we have for all $x, y \in \mathcal{N}$

$$\begin{aligned} d_g((x0)y) &= d_g(x0)g(y) + x0d_g(y) \\ &= 0g(y) + 0d_g(y). \end{aligned}$$

On the other side, we have

$$\begin{aligned} d_g(x(0y)) &= d_g(x)g(0)y + xd_g(0)g(y) + x0d_g(y) \\ &= 0y + 0g(y) + 0d_g(y). \end{aligned}$$

Now, comparing the two expressions of $d_g(x.0.y)$ and conclude. ∎

In this section, we give some new results and examples concerning the existence of g-derivations in near-rings which are not rings. We will also apply Lemma 5 several times without mentioning it. We begin by the following interesting result.

Theorem 6. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero g-derivation d_g associated with a multiplier g satisfying $d_g([[x,y],z]) = 0$ for all $x, y, z \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof. We divide the proof into two cases.

Case 1. If $g = 0$, by hypotheses given, we have $d_g([[x,y],z]) = 0$ for all $x, y, z \in \mathcal{N}$ which means that $d_g([x,y]z) = d_g(z[x,y])$ for all $x, y, z \in \mathcal{N}$. So, $[x,y]d_g(z) = zd_g([x,y])$ for all $x, y, z \in \mathcal{N}$. Replacing x by $[u,v]$ in the preceding relation, we obtain $[[u,v],y]d_g(z) = 0$ for all $u, v, y, z \in \mathcal{N}$; also putting $z = rt$ we find that $[[u,v],y]rd_g(t) = 0$ for all $u, v, y, r, t \in \mathcal{N}$ and hence $[[u,v],y]\mathcal{N}d_g(t) = \{0\}$ for all $u, v, y, t \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $d_g \neq 0$, we conclude that $[[u,v],y] = 0$ for all $u, v, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring by Lemma 1.

Case 2. Suppose that $g \neq 0$, we have

$$d_g([[x,y],z]) = 0 \text{ for all } x, y, z \in \mathcal{N} \tag{3}$$

that is,

$$d_g([x,y])g(z) + [x,y]d_g(z) = d_g(z)g([x,y]) + zd_g([x,y]) \text{ for all } x, y, z \in \mathcal{N}. \tag{4}$$

Replacing x by $[u,v]$ in (4) and using (3), we get

$$[[u,v],y]d_g(z) = d_g(z)g([[u,v],y]) \text{ for all } u, v, y, z \in \mathcal{N}. \tag{5}$$

Now, taking $z = x$ in (3), we infer that $d_g([x, y]x) = d_g(x[x, y])$ for all $x, y \in \mathcal{N}$ which can be written as

$$d_g([x, y])g(x) + [x, y]d_g(x) = d_g([x, xy]) \text{ for all } x, y \in \mathcal{N}. \tag{6}$$

Substituting $[u, v]$ for x in (6), where $u, v \in \mathcal{N}$, and invoking (3) we arrive at

$$[[u, v], y]d_g([u, v]) = 0 \text{ for all } u, v, y \in \mathcal{N}.$$

According to (5) and the last result, we conclude that

$$d_g([u, v])g([[u, v], y]) = 0 \text{ for all } u, v, y \in \mathcal{N}.$$

It follows that,

$$d_g([u, v])g([u, v])y = d_g([u, v])g(y)[u, v] \text{ for all } u, v, y \in \mathcal{N}. \tag{7}$$

Putting yt instead of y in (7), we get

$$d_g([u, v])g([u, v])yt = d_g([u, v])g(y)t[u, v] \text{ for all } u, v, t, y \in \mathcal{N}. \tag{8}$$

From (7) and (8), we can see that $d_g([u, v])g(y)[[u, v], t] = 0$ for all $u, v, t, y \in \mathcal{N}$. Substituting rys for y in latter expression, we obtain

$$d_g([u, v])rg(y)s[[u, v], t] = 0 \text{ for all } u, v, r, t, s, y \in \mathcal{N},$$

which reduces to

$$d_g([u, v])\mathcal{N}g(y)\mathcal{N}[[u, v], t] = \{0\} \text{ for all } u, v, t, y \in \mathcal{N}. \tag{9}$$

In virtue of the 3-primeness of $\mathcal{N}$ and g is not zero, (9) shows that

$$d_g([u, v]) = 0 \text{ or } [u, v] \in \mathcal{Z}(\mathcal{N}) \text{ for all } u, v \in \mathcal{N}. \tag{10}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $[u_0, v_0] \in \mathcal{Z}(\mathcal{N})$. Taking $z = [u_0, v_0]t$ in (3), we get $d_g\Big([u_0, v_0]\big[[x, y], t\big]\Big) = 0$ for all $x, y, t \in \mathcal{N}$. By defining property of d_g and (3), the preceding equation gives $d_g\big([u_0, v_0]\big)g\big([[x, y], t]\big) = 0$ for all $x, y, t \in \mathcal{N}$, and hence

$$d_g([u_0, v_0])g([x, y])t = d_g([u_0, v_0])g(t)[x, y] \text{ for all } x, y, t \in \mathcal{N}.$$

Replacing t by tz, we infer that

$$d_g([u_0, v_0])g(t)[[x, y], z] = 0 \text{ for all } x, y, z, t \in \mathcal{N}. \tag{11}$$

Taking $t = rts$ in (11), we get $d_g([u_0, v_0])rg(t)s[[x, y], z] = 0$ for all $x, y, z, r, t, s \in \mathcal{N}$ which can be rewritten as

$$d_g([u_0, v_0])\mathcal{N}g(t)\mathcal{N}[[x, y], z] = \{0\} \text{ for all } x, y, z, t \in \mathcal{N}.$$

In the light of the 3-primeness of $\mathcal{N}$ and $g \neq 0$, we conclude that either

$$d_g([u_0, v_0]) = 0 \text{ or } [[x, y], z] = 0 \text{ for all } x, y, z \in \mathcal{N}.$$

So that, (10) yields

$$d_g([u, v]) = 0 \text{ or } [[x, y], z] = 0 \text{ for all } u, v, x, y, z \in \mathcal{N}. \tag{12}$$

(i) If $[[x, y], z] = 0$ for all $x, y, z \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring by Lemma 1.

(ii) Let

$$d_g([u, v]) = 0 \text{ for all } u, v \in \mathcal{N}. \tag{13}$$

Replacing v by uv in (13) and using it again, we get $d_g(u)g([u, v]) = 0$ for all $u, v \in \mathcal{N}$, that is $d_g(u)g(u)v = d_g(u)g(v)u$ for all $u, v \in \mathcal{N}$. Now, replacing v by vt in the last equation and applying it, we get $d_g(u)g(v)[u, t] = 0$ for all $u, v, t \in \mathcal{N}$. Putting svr instead of v, where $s, r \in \mathcal{N}$, we obtain $d_g(u)sg(v)r[u, t] = 0$ for all $u, s, r, v, t \in \mathcal{N}$ which, in view of the 3-primeness of $\mathcal{N}$ and $g \neq 0$, shows that

$$d_g(u) = 0 \text{ or } [u, t] = 0 \text{ for all } u, t \in \mathcal{N}. \tag{14}$$

Suppose there exists $u_0 \in \mathcal{N}$ such that $[u_0, t] = 0$ for all $t \in \mathcal{N}$, so that $u_0 \in \mathcal{Z}(\mathcal{N})$. In this case, replacing v by $u_0 v$ in (13), we get $0 = d_g(u_0[u, v]) = d_g(u_0)g([u, v])$ for all $u, v \in \mathcal{N}$. It follows that $d_g(u_0)g(u)v = d_g(u_0)g(v)u$ for all $u, v \in \mathcal{N}$; again taking $u = ut$ in the latter equation, we obtain $d_g(u_0)g(u)[t, v] = 0$ for all $u, v, t \in \mathcal{N}$. Now, replacing u by rus and using the 3-primeness of $\mathcal{N}$ together with $g \neq 0$, we obtain $d_g(u_0) = 0$ or $[t, v] = 0$ for all $t, v \in \mathcal{N}$ and therefore (14) shows that

$$d_g(u) = 0 \text{ or } [t, v] = 0 \text{ for all } u, v, t \in \mathcal{N}.$$

As $d_g \neq 0$, the preceding result forces $[t, v] = 0$ for all $t, v \in \mathcal{N}$, and hence $\mathcal{N}$ is a commutative ring by Lemma 2(ii). ■

The result of Theorem 6 does not remain valid if we replace the Lie product by the Jordan product. In fact, we obtain the following result.

Theorem 7. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. Then, there is no nonzero g-derivation d_g associated with a multiplier g satisfying $d_g((x \circ y) \circ z) = 0$ for all $x, y, z \in \mathcal{N}$.*

Proof. Assume that

$$d_g((x \circ y) \circ z) = 0 \text{ for all } x, y, z \in \mathcal{N}. \tag{15}$$

Which equivalent to

$$d_g((x \circ y)z) = -d_g(z(x \circ y)) \text{ for all } x, y, z \in \mathcal{N}.$$

Noting that in a left near-ring, we have $-(x + y) = -y - x$, hence the previous relation yields

$$d_g(x \circ y)g(z) + (x \circ y)d_g(z) = -zd_g(x \circ y) - d_g(z)g(x \circ y) \text{ for all } x, y, z \in \mathcal{N}. \tag{16}$$

Replacing x by $u \circ v$ in (16) and invoking (15), we obtain

$$((u \circ v) \circ y)d_g(z) = -d_g(z)g((u \circ v) \circ y) \text{ for all } u, v, y, z \in \mathcal{N}. \tag{17}$$

Case 1. If $g = 0$, (17) assures that $((u \circ v) \circ y)d_g(z) = 0$ for all $u, v, y, z \in \mathcal{N}$. Substituting rt for z in the last equation, we get $((u \circ v) \circ y)rd_g(t) = 0$ for all $u, v, y, r, t \in \mathcal{N}$ which can be written as $((u \circ v) \circ y)\mathcal{N}d_g(t) = \{0\}$ for all $u, v, y, t \in \mathcal{N}$. In view of $\mathcal{N}$ is 3-prime and $d_g \neq 0$, we infer that $(u \circ v) \circ y = 0$ for all $u, v, y \in \mathcal{N}$. Replacing y by yzt in the preceding relation and using it again, we get $(u \circ v) \circ yzt = yzt(-(u \circ v))$ for all $u, v, y, z, t \in \mathcal{N}$ which means that $yz(-(u \circ v))t = yzt(-(u \circ v))$ and then $yz[-(u \circ v), t] = 0$ for all $u, v, y, z, t \in \mathcal{N}$. So that, $y\mathcal{N}[-(u \circ v), t] = \{0\}$ for all $u, v, y, t \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$ and $\mathcal{N}$ is not zero, we obtain $-(u \circ v) \in \mathcal{Z}(\mathcal{N})$ for all $u, v \in \mathcal{N}$ and therefore, $\mathcal{N}$ is a commutative ring by Lemma 3.

Case 2. If $g \neq 0$. In this case, returning to (15) and replacing z by x, we get $d_g((x \circ y)x) = -d_g(x(x \circ y))$ for all $x, y \in \mathcal{N}$, which means that

$$d_g(x \circ y)g(x) + (x \circ y)d_g(x) = -d_g(x \circ xy) \text{ for all } x, y \in \mathcal{N}. \tag{18}$$

Replacing x by $u \circ v$ in (18) and using (15), we arrive at

$$((u \circ v) \circ y)d_g(u \circ v) = 0 \text{ for all } u, v, y \in \mathcal{N}. \tag{19}$$

According to (17), (19) assures that

$$d_g(u \circ v)g((u \circ v) \circ y) = 0 \text{ for all } u, v, y \in \mathcal{N}, \tag{20}$$

and hence

$$d_g(u \circ v)g(u \circ v)y = d_g(u \circ v)g(y)(-(u \circ v)) \text{ for all } u, v, y \in \mathcal{N}. \tag{21}$$

Now, replacing y by yt in (21) and using it, we find that

$$d_g(u \circ v)g(y)[-(u \circ v), t] = 0 \text{ for all } u, v, t, y \in \mathcal{N}. \tag{22}$$

Putting rys instead of y in (22), we get

$$d_g(u \circ v)rg(y)s[-(u \circ v), t] = 0 \text{ for all } u, v, r, t, s, y \in \mathcal{N},$$

thereby obtaining

$$d_g(u \circ v)\mathcal{N}g(y)\mathcal{N}[-(u \circ v), t] = \{0\} \text{ for all } u, v, t, y \in \mathcal{N}. \tag{23}$$

Since $\mathcal{N}$ is 3-prime and $g \neq 0$, (23) gives

$$d_g(u \circ v) = 0 \text{ or } -(u \circ v) \in \mathcal{Z}(\mathcal{N}) \text{ for all } u, v \in \mathcal{N}. \tag{24}$$

Suppose there exist two elements $u_0, v_0 \in \mathcal{N}$ such that $-(u_0 \circ v_0) \in \mathcal{Z}(\mathcal{N})$. Replacing z by $(-(u_0 \circ v_0))z$ in (15), we get

$$d_g\Big((-(u_0 \circ v_0))\big((x \circ y) \circ z\big)\Big) = 0 \text{ for all } x, y, z \in \mathcal{N}.$$

Invoking (15), the latter result shows that

$$d_g(-(u_0 \circ v_0))g\big((x \circ y) \circ z\big) = 0 \text{ for all } x, y, z \in \mathcal{N}.$$

By additivity of g, it follows that

$$d_g(-(u_0 \circ v_0))g(x \circ y)z = -d_g(-(u_0 \circ v_0))g(z)(x \circ y) \text{ for all } x, y, z \in \mathcal{N}. \tag{25}$$

Replacing z by tz in (25) and using it again, we arrive at $d_g(-(u_0 \circ v_0))g(t)[-(x \circ y), z] = 0$ for all $x, y, z, t \in \mathcal{N}$. Taking $t = rts$ in the last equation, we get $d_g(-(u_0 \circ v_0))rg(t)s[-(x \circ y), z] = 0$ for all $x, y, z, r, t, s \in \mathcal{N}$ which means that $d_g(-(u_0 \circ v_0))\mathcal{N}g(t)\mathcal{N}[-(x \circ y), z] = \{0\}$ for all $x, y, z, t \in \mathcal{N}$. In view of the 3-primeness of $\mathcal{N}$, we conclude that

$$d_g(-(u_0 \circ v_0)) = 0 \text{ or } -(x \circ y) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}.$$

So that, (24) yields

$$-(x \circ y) \in \mathcal{Z}(\mathcal{N}) \text{ or } d_g(u \circ v) = 0 \text{ for all } u, v, x, y \in \mathcal{N}. \tag{26}$$

(i) If the first condition of (26) holds for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring by Lemma 3.

(ii) Suppose that the second part of (26) is verified, i.e

$$d_g(x \circ y) = 0 \text{ for all } x, y \in \mathcal{N}. \tag{27}$$

Substituting xy for y in (27) and using it again, we get $d_g(x)g(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, that is $d_g(x)g(x)y = -d_g(x)g(y)x$ for all $x, y \in \mathcal{N}$. Now, putting yt instead of y in the latter equation, we get $d_g(x)g(y)[-x, t] = 0$ for all $x, y, t \in \mathcal{N}$, again let $y = rys$ where $r, s \in \mathcal{N}$, we obtain $d_g(x)rg(y)s[-x, t] = 0$ for all $x, y, r, t, s \in \mathcal{N}$. In the light of the 3-primeness of $\mathcal{N}$, we find that for each $x \in \mathcal{N}$, we have either

$$d_g(x) = 0 \text{ or } -x \in \mathcal{Z}(\mathcal{N}). \tag{28}$$

Suppose there exists $x_0 \in \mathcal{N}$ such that $-x_0 \in \mathcal{Z}(\mathcal{N})$. Replacing y by $(-x_0)y$ in (27), we get $d_g((-x_0)(x \circ y)) = 0$ for all $x, y \in \mathcal{N}$ and hence $d_g(-x_0)g(x)y = -d_g(-x_0)g(y)x$ for all $x, y \in \mathcal{N}$. Taking $y = yt$ in the last equation, we obtain $d_g(-x_0)g(y)[-x, t] = 0$ for all $x, y, t \in \mathcal{N}$; a second time, replacing y and x by rys and $-x$, respectively, and using the 3-primeness of $\mathcal{N}$, we obtain $d_g(-x_0) = 0$ or $[x, t] = 0$ for all $x, t \in \mathcal{N}$. Consequently, (28) shows that

$$d_g(y) = 0 \text{ or } [x, t] = 0 \text{ for all } x, y, t \in \mathcal{N}.$$

As $d_g \neq 0$, then from the previous result, we can see that $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$ and hence $\mathcal{N}$ is a commutative ring by Lemma 2(ii).

Now, returning to our hypotheses and using the fact that $\mathcal{N}$ is a commutative ring, we find that $d_g(4(xyz)) = 0$ for all $x, y, z \in \mathcal{N}$. By the 2-torsion freeness of $\mathcal{N}$, we get $d_g(xyz) = 0$ for all $x, y, z \in \mathcal{N}$ which implies that $d_g(xyzt) = 0$ for all $x, y, z, t \in \mathcal{N}$. So, $d_g(xyz)g(t) + xyzd_g(t) = 0$ and hence, $xyzd_g(t) = 0$ for all $x, y, z, t \in \mathcal{N}$ which, in view of the 3-primeness of $\mathcal{N}$, contradicts our original hypotheses. ■

The following example shows that the 3-primeness of $\mathcal{N}$ in the Theorem 6 and Theorem 7 cannot be omitted.

Example 8. Let $(\mathcal{M}, +)$ be an any group and let us define the multiplicative law on $\mathcal{M}$, noted ., as follows: $x.y = y$ for all $x \in \mathcal{M} \setminus \{0\}, y \in \mathcal{M}$ and $0.y = y.0 = 0$ for all $y \in \mathcal{M}$. Define $\mathcal{N}$ and the maps $g, d_g : \mathcal{N} \to \mathcal{N}$ by: $\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in \mathcal{M} \right\}$, $g \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d_g \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x.y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We can see that $\mathcal{N}$ and $\mathcal{M}$ are left near-rings, which $\mathcal{N}$ is not 3-prime, g is a multiplier, and d_g is a nonzero g-derivation that satisfies $d_g([[A, B], C]) = 0$ and $d_g((A \circ B) \circ C) = 0$ for all $A, B, C \in \mathcal{N}$. However, $\mathcal{N}$ is a noncommutative left near-ring.

In the next example, we prove that the condition: $d_g([[x,y],z]) = \bar{0}$ for all $x,y,z \in \mathcal{N}$ in Theorem 6 is necessary.

Example 9. Let $(\mathbb{Z}/2\mathbb{Z},+)$ be the usual group. Let us define $*$ in $\mathbb{Z}/2\mathbb{Z}$ as follows: $x*y = y$ for all $x,y \in \mathbb{Z}/2\mathbb{Z}$. Then, $(\mathbb{Z}/2\mathbb{Z},+,*)$ is a noncommutative 3-prime left near-ring. Let us define a map g and the g-derivation $d_g : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ by: $g(x) = \bar{0}$ and $d_g(x) = x$ for all $x \in \mathbb{Z}/2\mathbb{Z}$. The condition $d_g([[x,y],z]) = \bar{0}$ for all $x,y,z \in \mathbb{Z}/2\mathbb{Z}$ is not verified because $d_g([[\bar{1},\bar{0}],\bar{0}]) = \bar{1} \neq \bar{0}$.

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