Discussiones Mathematicae
General Algebra and Applications 45 (2025) 355–366
https://doi.org/10.7151/dmgaa.1478

REGULARITY AND GREEN'S RELATIONS ON GFIN(Γ) ⋊ G

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Abstract

Let X be a non-empty set, G a group with identity 1 and let $f : X \to G$ be a mapping. Denote the Cayley graph of the group G with respect to f by Γ. In this paper, we consider the set of all pairs (Γ', g) such that Γ' is a finite subgraph of Γ and $g \in V(\Gamma')$. This set is a semigroup under the semidirect product with respect to the natural action of G on the semilattice of subgraphs of Γ defined as follows: for every $g \in G$ and every subgraph $\Gamma', g\Gamma'$ is the subgraph of Γ such that

$$V(g\Gamma') = \{gh : h \in V(\Gamma')\} \text{ and } E(g\Gamma') = \{(gh, x) : (h, x) \in E(\Gamma')\}.$$

We denote this semigroup by $\mathrm{GFin}(\Gamma) \rtimes G$. Regularity and Green's relations for the semigroup $\mathrm{GFin}(\Gamma) \rtimes G$ are investigated. Moreover, we characterize the natural partial order on $\mathrm{GFin}(\Gamma) \rtimes G$.

Keywords: Cayley graph, semigroup, regularity, Green's relations, natural partial order.

2020 Mathematics Subject Classification: 18B40.

1. Introduction

Let S be a semigroup. Then S^1 is either the semigroup S if S is a monoid or the semigroup S with an identity adjoined if S has no identity. Green's relations on S are five equivalence relations defined as follows: for each $a, b \in S$,

$$a \mathcal{L} b \text{ if and only if } a = xb, b = ya \text{ for some } x, y \in S^1,$$

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Furthermore, we dually define the $\mathcal{R}$-relation as follows:

$$a\ \mathcal{R}\ b \text{ if and only if } a = bx, b = ay \text{ for some } x, y \in S^1,$$

Moreover, we define the $\mathcal{J}$-relation as follows:

$$a\ \mathcal{J}\ b \text{ if and only if } a = xby, b = uav \text{ for some } u, v, x, y \in S^1,$$

$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $D = \mathcal{L} \circ \mathcal{R}$.

Green's relations are important tools for understanding the behavior of divisibility in a semigroup. So, many researchers are interested in Green's relations on some spectial semigroups. See [5, 7, 10, 11, 12, 13] and [14].

An element a of a semigroup S is called regular if $a \in aSa$, that is, $a = axa$ for some $x \in S$. A semigroup S is called a regular semigroup if every element of S is regular. And for any semigroup S, we denote the set of all idempotents in S by $\mathcal{E}(S)$.

In 1980, Nambooripad [9] defined $\leq$ on regular semigroup S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in \mathcal{E}(S),$$

and he proved that $(S, \leq)$ is a partially ordered set.

Later, Mitsch [8] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

This order is called the natural partial order on S. It is a useful tool to visualize relationships between elements in a semigroup.

Let X be a non-empty set, let G be a group with identity 1 and let $f : X \to G$ be a function. By the Cayley graph Γ of G with respect to f, we means the directed graph whose vertex set $V(\Gamma)$ is G and whose edge set $E(\Gamma)$ is $G \times X$, where each $g \in G, x \in X$ denotes an edge with initial vertex g and terminal vertex $g(xf)$. In 1989, Magolis and Meakin [6] let G be X-generated as a group with respect to f, that is, each element of G is a product of elements of the forms xf and $(xf)^{-1}$ where $x \in X$. Let

$$M(X; f) = \left\{(\Gamma', g) : \Gamma' \text{ is a finite connected subgraph of } \Gamma \text{ with } 1, g \in V(\Gamma')\right\}.$$

This set is a semigroup under the semidirect product with the natural action G on the semilattice of all subgraphs of Γ with union operation, defined as follows: $g\emptyset = \emptyset$ where $\emptyset$ is an empty graph. For each non-empty subgraph Γ' of Γ and $g \in G$, let $g\Gamma'$ be the subgraph of Γ with

$$V(g\Gamma') = \{gh : h \in V(\Gamma')\} \text{ and } E(g\Gamma') = \{(gh, x) : (h, x) \in E(\Gamma')\}.$$

We call $M(X; f)$ the Margolis-Meakin expansion of G with respect to f. Green's relations and some characterizations on $M(X; f)$ were studied in [6].

Recently, [1] introduced a new semigroup defined as follows: let Γ be the Cayley graph of the group G with respect to $f : X \to G$. Let $\mathrm{Fin}(\Gamma)$ be the semigroup of all finite subgraphs of Γ without isolated vertices with $\emptyset$ adjoined under union operations. Then the cartesian product $\mathrm{Fin}(\Gamma) \times G$ is a semigroup under the semidirect product with respect to natural action of G on $\mathrm{Fin}(\Gamma)$, which assigns to each $\Gamma' \in \mathrm{Fin}(\Gamma)$, the graph $g\Gamma'$ with $V(g\Gamma') = \{gh : h \in V(\Gamma')\}$ and $E(g\Gamma') = \{(gh, x) : (h, x) \in E(\Gamma)\}$. We denote this semigroup by $\mathrm{Fin}(\Gamma) \rtimes G$. Clearly, Margolis-Meakin expansion of G with respect to f is a subsemigroup of $\mathrm{Fin}(\Gamma) \rtimes G$ for X-generated group G respect to f. The notion of expansion is central to semigroup theory. As Birget and Rhodes introduced in [2], it involves representing a semigroup as a homomorphic image of another semigroup, where the homomorphism preserves certain properties. Two prominent examples of expansions are the Birget-Rhodes prefix expansion (see [2] for more details) and the Margolis-Meakin expansion. In [1], they constructed a new expansion (viewed as a subsemigroup of $\mathrm{Fin}(\Gamma) \rtimes G$) which contains the Margolis-Meakin expansion. This allows the results in [6] to be recaptured, as shown in [1].

In our work, we define a new semigroup as follows: let Γ be the Cayley graph of the group G with respect to $f : X \to G$. Put

$$\mathrm{GFin}(\Gamma) \rtimes G = \left\{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } g \in V(\Gamma')\right\}.$$

Then $\mathrm{GFin}(\Gamma) \rtimes G$ is a semigroup under the semidirect product with respect to the same action of G on the semilattice of subgraphs of Γ.

It is clear that our semigroup is heavily inspired by [1] and has the old one as a subsemigroup. To facilitate the creation of new expansions in the future, in this paper, we begin by investigating the fundamental properties. Firstly, we consider regularity, which will indicate the complexity of the new semigroup. Subsequently, we investigate Green's relations within $\mathrm{GFin}(\Gamma) \rtimes G$ to categorize element groups associated with distinct ideal classes. Finally, we study the ordering of elements in our semigroup via the natural order.

2. PRELIMINARIES AND NOTATIONS

From now on, we let X be a non-empty set, G a group with identity 1 and Γ a Cayley graph of G with respect to mapping $f : X \to G$. For any graphs Γ_1 and Γ_2, we say that Γ_1 is a subgraph of Γ_2 and we write $\Gamma_1 \subseteq \Gamma_2$ if

$$V(\Gamma_1) \subseteq V(\Gamma_2) \text{ and } E(\Gamma_1) \subseteq E(\Gamma_2).$$

Let $\Gamma_1 \cup \Gamma_2$ be a graph with $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$.

Now, we will give an example and some characterizations of the semigroup $\mathrm{GFin}(\Gamma) \rtimes G$.

Example 1. Let $G = \{1, g, h, gh\}$ be the Klein four-group with identity 1 and $X = \{x, y\}$. Define $f : X \to G$ by $xf = g$ and $yf = h$. Then the following graph is the Cayley graph of G respect to f.

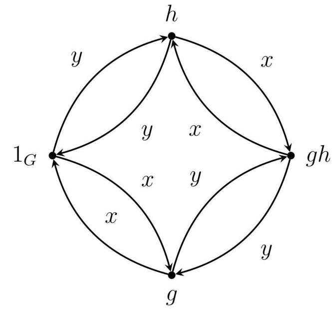

Figure 1. Cayley graph Γ.

We let Γ_1 and Γ_2 be defined as follows:

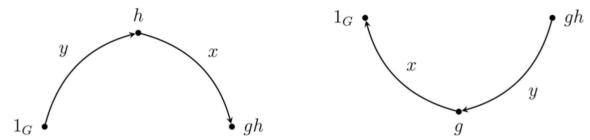

Figure 2. Γ_1 and Γ_2.

Then $(\Gamma_1, h), (\Gamma_2, gh) \in \mathrm{GFin}(\Gamma) \rtimes G$. We note that $(\Gamma_1, h)(\Gamma_2, gh) = (\Gamma_1 \cup h\Gamma_2, g)$ and $(\Gamma_2, gh)(\Gamma_1, h) = (\Gamma_2 \cup gh\Gamma_1, g)$. Since $h \in V(\Gamma_1 \cup h\Gamma_2) \setminus V(\Gamma_2 \cup gh\Gamma_1)$, we get that $\mathrm{GFin}(\Gamma) \rtimes G$ is not a commutative semigroup.

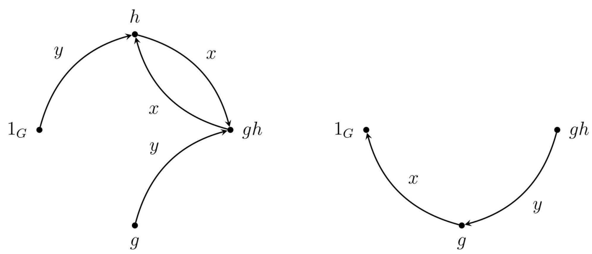

Figure 3. $\Gamma_1 \cup h\Gamma_2$ and $\Gamma_2 \cup gh\Gamma_1$.

Proposition 1. *Let* $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is an idempotent element if and only if* $g = 1$. *In this case, the set of all idempotent elements,* $\mathcal{E}(\mathrm{GFin}(\Gamma) \rtimes G)$ *is a commutative subsemigroup of* $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 2. *Let* G *be a finite group. Then* G *is isomorphic to a subgroup of* $\mathrm{GFin}(\Gamma) \rtimes G$.

Proof. Fix $x \in X$ and let Γ' be the graph with

$$V(\Gamma') = G \text{ and } E(\Gamma') = \{(g, x) : g \in G\}.$$

Define $\varphi : G \to \mathrm{GFin}(\Gamma) \rtimes G$ by

$$g\varphi = (\Gamma', g) \text{ for all } g \in G.$$

From the definition of Γ', we have $g\Gamma' = \Gamma'$ and $g \in V(\Gamma')$ for all $g \in G$. Hence we can verify that φ is an injective homomorphism. Therefore G is isomorphic to a subgroup of $\mathrm{GFin}(\Gamma) \rtimes G$. ■

We always use the following properties in this study. For finite subgraphs Γ', Γ'' of Γ and $g \in G$,

$$\text{if } g\Gamma' \subseteq \Gamma' \text{ then } g\Gamma' = \Gamma' \tag{1}$$

and

$$\text{if } \Gamma' \subseteq \Gamma'' \text{ then } g\Gamma' \subseteq g\Gamma''. \tag{2}$$

3. Regularity and Green's relations

First, we start regularity in $\mathrm{GFin}(\Gamma) \rtimes G$. Then, we characterize Green's relations on this semigroup.

Theorem 3. *Let* $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is regular if and only if* $1 \in V(\Gamma')$.

Proof. Assume that (Γ', g) is regular. Then there exists $(\Gamma'', h) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that

$$(\Gamma', g) = (\Gamma', g)(\Gamma'', h)(\Gamma', g) = (\Gamma' \cup g\Gamma'', gh)(\Gamma', g) = (\Gamma' \cup g\Gamma'' \cup gh\Gamma', ghg).$$

Thus $\Gamma' = \Gamma' \cup g\Gamma'' \cup gh\Gamma'$ and $g = ghg$. Since G is a group, we conclude $1 = gh$. Therefore $\Gamma' = \Gamma' \cup g\Gamma''$ and so $g\Gamma'' \subseteq \Gamma'$. This implies that $1 = gh \in V(g\Gamma'') \subseteq V(\Gamma')$.

Conversely, assume that $1 \in V(\Gamma')$. Note that $g^{-1} = g^{-1}1 \in V(g^{-1}\Gamma')$. Thus $(g^{-1}\Gamma', g^{-1}) \in \mathrm{GFin}(\Gamma) \rtimes G$. We see that

$$(\Gamma', g)(g^{-1}\Gamma', g^{-1})(\Gamma', g) = (\Gamma' \cup gg^{-1}\Gamma', gg^{-1})(\Gamma', g) = (\Gamma', 1)(\Gamma', g) = (\Gamma', g).$$

Hence, we conclude that (Γ', g) is regular. ■

The above theorem verifies that for a non-trivial group G, $\mathrm{GFin}(\Gamma) \rtimes G$ is not a regular semigroup. Next, we find its maximal regular subsemigroup.

Theorem 4. *Let $T = \{(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G : 1 \in V(\Gamma')\}$. Then T is the maximum regular subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$. Moreover, T is an inverse semigroup.*

Proof. It follows from Theorem 3 that T is the set of all regular elements in $\mathrm{GFin}(\Gamma) \rtimes G$. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in T$. Then

$$(\Gamma_1, g_1)(\Gamma_2, g_2) = (\Gamma_1 \cup g_1\Gamma_2, g_1g_2).$$

Since $g_1g_2 \in V(g_1\Gamma_2)$, we get that T is closed. Therefore, T is the maximum regular subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$. By Proposition 1, T is an inverse semigroup. ■

It is well-known that the $\mathcal{H}$-class containing an idempotent element e in a semigroup S forms a subgroup of S. This subgroup is a subset of S that is closed under the same multiplication, and the element e becomes the identity of the group. We can apply this principle to construct subgroups of $\mathrm{GFin}(\Gamma) \rtimes G$. Note that while $\mathrm{GFin}(\Gamma) \rtimes G$ lacks an identity element, it contains multiple idempotent elements. Thus, we can construct several subgroups within $\mathrm{GFin}(\Gamma) \rtimes G$. Now, we start by examining the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 5. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1) \mathrel{\mathcal{L}} (\Gamma_2, g_2)$ if and only if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or ($g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2, g_1g_2^{-1} \in V(\Gamma_1)$ and $g_2g_1^{-1} \in V(\Gamma_2)$).*

Proof. Assume that $(\Gamma_1, g_1) \mathrel{\mathcal{L}} (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. There exist $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_4, g_4)(\Gamma_1, g_1)$. This implies that $\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2$ and $g_1 = g_3g_2$. Hence $g_1g_2^{-1} = g_3 \in V(\Gamma_3) \subseteq V(\Gamma_1)$ and $g_1g_2^{-1}\Gamma_2 = g_3\Gamma_2 \subseteq \Gamma_1$ which means $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$. Similarly, we obtain that $g_2g_1^{-1} \in V(\Gamma_2)$ and $g_1^{-1}\Gamma_1 \subseteq g_2^{-1}\Gamma_2$, which means that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1) \mathrel{\mathcal{L}} (\Gamma_2, g_2)$. Assume that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, $g_1g_2^{-1} \in V(\Gamma_1)$ and $g_2g_1^{-1} \in V(\Gamma_2)$. It is clear that $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in \mathrm{GFin}(\Gamma) \rtimes G$. By assumption, we have

$$(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) = (\Gamma_1 \cup g_1g_2^{-1}\Gamma_2, g_1g_2^{-1}g_2) = (\Gamma_1 \cup g_1g_1^{-1}\Gamma_1, g_1) = (\Gamma_1, g_1)$$

and

$$(\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) = (\Gamma_2 \cup g_2g_1^{-1}\Gamma_1, g_2g_1^{-1}g_1) = (\Gamma_2 \cup g_2g_2^{-1}\Gamma_2, g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\ \mathcal{L}\ (\Gamma_2, g_2)$. ■

Example 2. Let $G = \{1, g, h, gh\}$ be the Klein four-group with identity 1 and $X = \{x, y\}$. Define $f : X \to G$ by $xf = g$ and $yf = h$. We let Γ_1 and Γ_2 be subgraph of Γ with $V(\Gamma_1) = \{1, g, h\}, V(\Gamma_2) = \{g, h, gh\}, E(\Gamma_1) = \{(1, x)\}$ and $E(\Gamma_2) = \{(gh, x)\}$. Then $(\Gamma_1, g), (\Gamma_2, h) \in \mathrm{GFin}(\Gamma) \rtimes G$. It is easy to see that $g^{-1}\Gamma_1 = h^{-1}\Gamma_2$ and $gh \notin V(\Gamma_1)$. Hence $((\Gamma_1, g), (\Gamma_2, h)) \notin \mathcal{L}$ via Theorem 5.

Theorem 6. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\ \mathcal{R}\ (\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Proof. Assume that $(\Gamma_1, g_1)\ \mathcal{R}\ (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. Then there exist $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_3, g_3)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_4, g_4)$. Thus $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_3$ and $\Gamma_2 = \Gamma_1 \cup g_1\Gamma_4$. It follows that $\Gamma_2 \subseteq \Gamma_1$ and $\Gamma_1 \subseteq \Gamma_2$. Hence $\Gamma_2 = \Gamma_1$.

Suppose that $\Gamma_1 = \Gamma_2$. Note that $g_2^{-1}g_1 \in V(g_2^{-1}\Gamma_1)$ and $g_1^{-1}g_2 \in V(g_1^{-1}\Gamma_2)$. This means that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. We see that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_2 \cup \Gamma_1, g_1) = (\Gamma_1, g_1)$$

and

$$(\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_1 \cup \Gamma_2, g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\ \mathcal{R}\ (\Gamma_2, g_2)$. ■

As an immediate consequence of the previous theorems, we get the following result.

Theorem 7. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\ \mathcal{H}\ (\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2, g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1g_2^{-1}, g_2g_1^{-1} \in V(\Gamma_1)$).

Theorem 8. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *or (there exists* $g \in V(\Gamma_2)$ *such that* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2, g_1g^{-1} \in V(\Gamma_1)$ *and* $gg_1^{-1} \in V(\Gamma_2)$*).*

Proof. Assume that $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$. Then there exists $(\Gamma_3, g_3) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1)\ \mathcal{L}\ (\Gamma_3, g_3)$ and $(\Gamma_3, g_3)\ \mathcal{R}\ (\Gamma_2, g_2)$. From Theorem 5 and 6, we get that $[(\Gamma_1, g_1) = (\Gamma_3, g_3)$ or $(g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_3,\ g_1g_3^{-1} \in V(\Gamma_1),\ g_3g_1^{-1} \in V(\Gamma_3))]$ and $\Gamma_3 = \Gamma_2$. Therefore $g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_2$ where $g_3 \in V(\Gamma_2)$.

Conversely, if $\Gamma_1 = \Gamma_2$, then $(\Gamma_1, g_1)\ \mathcal{R}\ (\Gamma_2, g_2)$. Since $\mathcal{R} \subseteq \mathcal{D}$, we have $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$. Suppose that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2, g_1g^{-1} \in V(\Gamma_1)$ and $gg_1^{-1} \in V(\Gamma_2)$ for some $g \in V(\Gamma_2)$. Then $(\Gamma_1, g_1)\ \mathcal{L}\ (\Gamma_2, g)$ and we note that $(\Gamma_2, g)\ \mathcal{R}\ (\Gamma_2, g_2)$. Hence $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$. ■

Theorem 9. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then*

$$(\Gamma_2, g_2) \in (\mathrm{GFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)(\mathrm{GFin}(\Gamma) \rtimes G)$$

if and only if there exists $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in (\mathrm{GFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)(\mathrm{GFin}(\Gamma) \rtimes G)$. Then $(\Gamma_2, g_2) = (\Gamma_3, g_3)(\Gamma_1, g_1)(\Gamma_4, g_4)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$. This implies that $\Gamma_2 = \Gamma_3 \cup g_3\Gamma_1 \cup g_3g_1\Gamma_4$ and $g_2 = g_3g_1g_4$. Therefore $g_3\Gamma_1 \subseteq \Gamma_2$ and $g_3 \in V(\Gamma_3) \subseteq V(\Gamma_2)$.

Conversely, assume that there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. Then $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. From assumption, we then have

$$(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) = (\Gamma_2 \cup g\Gamma_1 \cup \Gamma_2, g_2) = (\Gamma_2, g_2).$$

Thus $(\Gamma_2, g_2) \in (\mathrm{GFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)(\mathrm{GFin}(\Gamma) \rtimes G)$. ■

It is well-known that for a finite semigroup, we have $\mathcal{D} = \mathcal{J}$ and in general we only have $\mathcal{D} \subseteq \mathcal{J}$. The following theorem verifies that $\mathcal{D}$ and $\mathcal{J}$ are identical on $\mathrm{GFin}(\Gamma) \rtimes G$ although the semigroup is infinite.

Theorem 10. *On* $\mathrm{GFin}(\Gamma) \rtimes G, \mathcal{D} = \mathcal{J}$.

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1)\ \mathcal{J}\ (\Gamma_2, g_2)$. There exist $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5), (\Gamma_6, g_6) \in (\mathrm{GFin}(\Gamma) \rtimes G)^1$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$. Otherwise, there are 6 cases to consider.

Case 1. $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5)$ and (Γ_6, g_6) are not the identity. By Theorem 9, there exists $h_1 \in V(\Gamma_1)$ such that $h_1\Gamma_2 \subseteq \Gamma_1$ and there exists $h_2 \in V(\Gamma_2)$ such that $h_2\Gamma_1 \subseteq \Gamma_2$. Then $h_2h_1\Gamma_2 \subseteq h_2\Gamma_1 \subseteq \Gamma_2$. From (1), we obtain that

$$h_2h_1\Gamma_2 = h_2\Gamma_1 = \Gamma_2.$$

Similarly, we get $h_1h_2\Gamma_1 = h_1\Gamma_2 = \Gamma_1$. Since $g_1 \in V(\Gamma_1) = V(h_1\Gamma_2)$, we have $g_1 = h_1k$ for some $k \in V(\Gamma_2)$. This implies that

$$g_1^{-1}\Gamma_1 = (h_1k)^{-1}\Gamma_1 = k^{-1}h_1^{-1}\Gamma_1 = k^{-1}h_1^{-1}h_1\Gamma_2 = k^{-1}\Gamma_2.$$

Note that $g_1k^{-1} = h_1kk^{-1} = h_1 \in V(\Gamma_1)$ and $kg_1^{-1} = k(h_1k)^{-1} = kk^{-1}h_1^{-1} = h_1^{-1}$. Since $\Gamma_2 = h_2\Gamma_1$, we have $h_2^{-1}\Gamma_2 = h_2^{-1}h_2\Gamma_1 = \Gamma_1$. Thus $1 = h_2^{-1}h_2 \in V(h_2^{-1}\Gamma_2) = V(\Gamma_1)$. From $\Gamma_1 = h_1\Gamma_2$, we obtain that $h_1^{-1}\Gamma_1 = \Gamma_2$. This means that $h_1^{-1} \in V(h_1^{-1}\Gamma_1) = V(\Gamma_2)$. By Theorem 8, we then have $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Case 2. $(\Gamma_4, g_4) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. We get that $(\Gamma_1, g_1)\ \mathcal{L}\ (\Gamma_2, g_2)$. Since $\mathcal{L} \subseteq \mathcal{D}$, we obtain $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Case 3. $(\Gamma_3, g_3) = (\Gamma_5, g_5)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_6, g_6)$ these mean $(\Gamma_1, g_1)\ \mathcal{R}\ (\Gamma_2, g_2)$. From $\mathcal{R} \subseteq \mathcal{D}$, we obtain $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Case 4. $(\Gamma_3, g_3) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. Thus $g_5\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_1$. From (1), we have $\Gamma_1 = \Gamma_2$. It follows form Theorem 8 that $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Similarly, if $(\Gamma_4, g_4) = (\Gamma_5, g_5)$ is the identity, we conclude that $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Case 5. (Γ_4, g_4) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. Thus $g_3 = g_1 g_2^{-1}$ and $g_1 g_2^{-1}\Gamma_2 \subseteq \Gamma_1$. This means that $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$ by (2). Since $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$ and Theorem 9, there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. This implies that $g_2^{-1}g\Gamma_1 \subseteq g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$. From (1), we have $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. Note that $g_1 g_2^{-1} = g_3 \in V(\Gamma_1)$. Since $g_2^{-1}g\Gamma_1 = g_2^{-1}\Gamma_2$, we have $g\Gamma_1 = \Gamma_2$. Then $g \in V(\Gamma_2) = V(g\Gamma_1)$ which implies $1 \in V(\Gamma_1)$. From $g_2 g_1^{-1}\Gamma_1 = \Gamma_2$, we get that $g_2 g_1^{-1} \in V(\Gamma_2)$. Hence $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$ by Theorem 8.

Similarly, if (Γ_6, g_6) is the identity, we conclude that $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$.

Case 6. (Γ_5, g_5) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_6, g_6)$. Thus $\Gamma_2 = \Gamma_1 \cup g_1\Gamma_6$ and $g_2 = g_1 g_6$. From Theorem 9, there exits $g \in V(\Gamma_1)$ such that $g\Gamma_2 \subseteq \Gamma_1$. We obtain that $g\Gamma_2 \subseteq \Gamma_1 \subseteq \Gamma_2$. From (1), we have $\Gamma_1 = \Gamma_2$ which means $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$ via Theorem 8.

Similarly, if (Γ_3, g_3) is the identity, we conclude that $(\Gamma_1, g_1)\ \mathcal{D}\ (\Gamma_2, g_2)$. ■

4. Natural partial order

From Theorem 3, $\mathrm{GFin}(\Gamma) \rtimes G$ is not, in general, a regular semigroup. Then the natural order $\leq$ on $\mathrm{GFin}(\Gamma) \rtimes G$ is defined as follows: for $(\gamma_1, g_1), (\gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$,

$$(\Gamma_1, g_1) \leq (\Gamma_2, g_2) \text{ if } (\Gamma_1, g_1) = (\Gamma_2, g_2) \text{ or}$$

$$(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2) = (\Gamma_2, g_2)(\Gamma_4, g_4) \text{ and } (\Gamma_1, g_1) = (\Gamma_1, g_1)(\Gamma_4, g_4)$$

for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$. Next, we characterize a natural partial order on $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 11. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ if and only if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or $(\Gamma_2 \subseteq \Gamma_1, g_1 = g_2$ and $1 \in V(\Gamma_1))$.*

Proof. Assume that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. Thus there exists $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that

$$(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2) = (\Gamma_2, g_2)(\Gamma_4, g_4) \text{ and } (\Gamma_1, g_1) = (\Gamma_1, g_1)(\Gamma_4, g_4).$$

Then $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2) = (\Gamma_2 \cup g_2\Gamma_4, g_2g_4)$ and $(\Gamma_1, g_1) = (\Gamma_1 \cup g_1\Gamma_4, g_1g_4)$. This means that $1 = g_4$ and $g_1 = g_2$. Then $g_3 = 1$ and we get that $1 \in V(\Gamma_3) \subseteq V(\Gamma_1)$. Clearly, $\Gamma_2 \subseteq \Gamma_1$.

Conversely, if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. We suppose that $\Gamma_2 \subseteq \Gamma_1, g_1 = g_2$ and $1 \in V(\Gamma_1)$. Then $1 = g_2^{-1}g_2 \in V(g_2^{-1}\Gamma_1)$. Thus $(g_2^{-1}\Gamma_1, 1) \in \mathrm{GFin}(\Gamma) \rtimes G$. Clearly, $(\Gamma_1, 1)(\Gamma_2, g_2) = (\Gamma_1 \cup \Gamma_2, g_2) = (\Gamma_1, g_1), (\Gamma_2, g_2)(g_2^{-1}\Gamma_1, 1) = (\Gamma_2 \cup \Gamma_1, g_1) = (\Gamma_1, g_1)$ and $(\Gamma_1, g_1)(g_2^{-1}\Gamma_1, 1) = (\Gamma_1 \cup g_2g_2^{-1}\Gamma_1, g_1) = (\Gamma_1, g_1)$. Hence $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. ∎

Theorem 12. *The natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$ *is right compatible.*

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. By Theorem 11, we have $\Gamma_2 \subseteq \Gamma_1, g_1 = g_2$ and $1 \in V(\Gamma_1)$. Let $(\Gamma_3, g_3) \in \mathrm{GFin}(\Gamma) \rtimes G$. We will show that $(\Gamma_1, g_1)(\Gamma_3, g_3) \leq (\Gamma_2, g_2)(\Gamma_3, g_3)$. Since $\Gamma_2 \subseteq \Gamma_1, g_1 = g_2$ and $1 \in V(\Gamma_1)$, we have $\Gamma_2 \cup g_2\Gamma_3 \subseteq \Gamma_1 \cup g_1\Gamma_3, g_1g_3 = g_2g_3$ and $1 \in V(\Gamma_1 \cup g_1\Gamma_3)$. Thus $(\Gamma_1, g_1)(\Gamma_3, g_3) \leq (\Gamma_2, g_2)(\Gamma_3, g_3)$. We conclude that $\leq$ is right compatible. ∎

Example 3. Let $G = \{1, g, h, gh\}$ be the Klein four-group with identity 1 and $X = \{x, y\}$. Define $f : X \to G$ by $xf = g$ and $yf = h$. Consider the subgraphs Γ_1, Γ_2 and Γ_3 of Γ defined as follow: $V(\Gamma_1) = \{1, g, gh\}, E(\Gamma_1) = \{(1, x), (g, y)\}, V(\Gamma_2) = \{g, gh\}, E(\Gamma_2) = \{(g, y)\}, V(\Gamma_3) = \{h, gh\}$ and $E(\Gamma_3) = \{(gh, x)\}$. Clearly, $(\Gamma_1, g) \leq (\Gamma_2, g)$ (by Theorem 11). Since $E(\Gamma_3 \cup h\Gamma_1) \neq E(\Gamma_3 \cup h\Gamma_2)$ and $1 \notin V(\Gamma_3 \cup h\Gamma_1)$, we obtain that $(\Gamma_3, h)(\Gamma_1, g) \not\leq (\Gamma_3, h)(\Gamma_2, g)$. Hence the natural partial order on $\mathrm{GFin}(\Gamma) \rtimes G$ is not left compatible.

Theorem 13. *Let* $g \in G$ *and let* $\emptyset_g$ *be the graph with*

$$V(\emptyset_g) = \{g\} \text{ and } E(\emptyset_g) = \emptyset.$$

Then the following statements hold:

(1) $(\emptyset_g, g)$ *is the maximal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(3) *If* Γ *is infinite, then* $\mathrm{GFin}(\Gamma) \rtimes G$ *has no minimal element under the natural partial order.*

Proof. (1) and (2) are obvious.

(3) Assume that Γ is infinite and let $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. Since $\Gamma' \neq \Gamma$, we have $V(\Gamma') \neq V(\Gamma)$ or $E(\Gamma') \neq E(\Gamma)$.

Case 1. $V(\Gamma') \neq V(\Gamma)$. Choose $h \in V(\Gamma) \setminus V(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{1, h\}$ and $E(\Gamma'') = E(\Gamma')$. Therefore $(\Gamma'', g) \leq (\Gamma', g)$ and $(\Gamma'', g) \neq (\Gamma', g)$.

Case 2. $E(\Gamma') \neq E(\Gamma)$. Choose $(h, x) \in E(\Gamma) \setminus E(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{1, h, hxf\}$ and $E(\Gamma'') = E(\Gamma') \cup \{(h, x)\}$. So $(\Gamma'', g) \leq (\Gamma', g)$ and $(\Gamma'', g) \neq (\Gamma', g)$.

Hence $\mathrm{GFin}(\Gamma) \rtimes G$ has no minimal element under the natural partial order. ■

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Received 31 May 2024
Revised 12 November 2024
Accepted 13 November 2024