Discussiones Mathematicae
General Algebra and Applications 45 (2025) 229–239
https://doi.org/10.7151/dmgaa.1476

ON BI-IDEALS OF ORDERED FULL TRANSFORMATION SEMIGROUPS

P.K. Minnumol[1] and P.G. Romeo

Department of Mathematics
Cochin University of Science and Technology (CUSAT)
Kochi, Kerala, India, 682022

e-mail: pkminnumol@gmail.com
romeo_parackal@yahoo.com

Abstract

In this paper we describe the Green's relations on the semigroup of bi-ideals of ordered full transformation semigroup $\mathcal{B}(\mathcal{T}_X)$ in terms of Green's relations of ordered full transformation semigroup $\mathcal{T}_X$.

Keywords: ordered semigroup, bi-ideals, natural partial order, full transformation semigroup, Green's relations.

2020 Mathematics Subject Classification: 20M17, 20M12, 06F05.

1. Introduction

The bi-ideals of semigroups were introduced by Good and Hughes in 1952. It is a special case of the (m,n)-ideal introduced by Lajos who gave a characterization of semigroup by the set of their bi-ideals (see cf. [3, 4, 5]). Further Kehayopulu studied the concept of bi-ideals in the case of ordered semigroups in [6], Mallick and Hansda [10] introduced a semigroup $\mathcal{B}(S)$ of all bi-ideals of an ordered semigroup S and they gave different charecterizations of $\mathcal{B}(S)$ for different regular subclasses of S. In this paper we describe the Green's relations on the semigroup of bi-ideals of an ordered full transformation semigroup.

[1]Corresponding author.

2. Preliminaries

First we recall several definitions and results which are needed in the sequel. An *ordered semigroup* $(S, \cdot, \leq)$ is a poset $(S, \leq)$ as well as a semigroup $(S, \cdot)$ such that for any $a, b, x \in S$, $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. For $A, B \subseteq S$, we have $AB = \{ab \,|\, a \in A \text{ and } b \in B\}$ and $(A] = \{x \in S \,|\, (\exists\, a \in A) \text{ and } x \leq a\}$ which is called *downward closure* of A.

Definition. Let S be an ordered semigroups. A nonempty subset A of S is called a *left (right) ideal* of S if $SA \subseteq A\,(AS \subseteq A)$ and $(A] = A$. A is called an *ideal* of S if it is both a left and right ideal of S.

Definition [8]**.** A nonempty subset A of an ordered semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$ and $(A] = A$. A bi-ideal A of S is called a *subidempotent bi-ideal* if $A^2 \subseteq A$.

Definition. An ordered semigroup S is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$ or equivalenty $a \in (aSa]$. An ordered semigroup S is called *intra-regular* if for every $a \in S$ there exists $x, y \in S$ such that $a \leq xa^2y$ or equivalenty $a \in (Sa^2S]$.

If S is a regular ordered semigroup then the bi-ideals and the subidempotent bi-ideals are the same.

Definition [10]**.** Let $(S, \cdot, \leq)$ be an ordered semigroup and $\mathcal{B}(S)$ be the collection all bi-ideals of S. Then $\mathcal{B}(S)$ together with the binary operation defined by

$$A * B = (AB]$$

is a semigroup and is called the semigroup of bi-deals of the ordered semigroup $(S, \cdot, \leq)$.

The semigroup of bi-ideals $\mathcal{B}(S)$ of an ordered semigroup S is significant in the study of structure of the semigroup as is evident from the following theorems.

Theorem 1 [10]**.** *Let S be an ordered semigroup. Then S is regular if and only if the semigroup $\mathcal{B}(S)$ of all bi-ideals is regular.*

Theorem 2 [10]**.** *An ordered semigroup S is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a band.*

The principal left ideal, right ideal, ideal and bi-ideal generated by $a \in S$ are denoted by $L(a), R(a), I(a), B(a)$ respectively and defined as follows

$$\begin{aligned} L(a) &= (a \cup Sa] \\ R(a) &= (a \cup aS] \\ I(a) &= (a \cup Sa \cup aS \cup SaS] \\ B(a) &= (a \cup aSa]. \end{aligned}$$

The *Green's relations* $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on an ordered semigroup S are given as follows (see cf. [7])

$$a\,\mathcal{L}\,b \text{ if and only if } L(a) = L(b)$$
$$a\,\mathcal{R}\,b \text{ if and only if } R(a) = R(b)$$
$$a\,\mathcal{J}\,b \text{ if and only if } I(a) = I(b)$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$
$$\mathcal{D} = \mathcal{L} \vee \mathcal{R}.$$

In a regular ordered semigroup S we have a particularly useful way of looking at the equivalences $\mathcal{L}$ and $\mathcal{R}$. If S is a regular ordered semigroup, for each $a \in S$ there exist $x \in S$ such that $a \leq axa$, hence $a \in (Sa]$ and $a \in (aS]$ and so the Green's equivalences for regular ordered semigroups are

$$a\,\mathcal{L}\,b \text{ if and only if } (Sa] = (Sb]$$
$$a\,\mathcal{R}\,b \text{ if and only if } (aS] = (bS].$$

Theorem 3. *Let a and b be elements of a regular ordered semigroup S. Then $a\,\mathcal{L}\,b$ if and only if there exists $x, y \in S$ such that $a \leq xb$ and $b \leq ya$. Also $a\,\mathcal{R}\,b$ if and only if there exists $u, v \in S$ such that $a \leq bu$ and $b \leq av$.*

Proof. Suppose that $a\,\mathcal{L}\,b$ then $(Sa] = (Sb]$. Clearly $a \in (Sb]$ and $b \in (Sa]$, which implies that $a \leq xb$ and $b \leq ya$ for some $x, y \in S$.

Conversely suppose that there exists $x, y \in S$ such that $a \leq xb$ and $b \leq ya$. Let $t \in (Sa]$, then $t \leq sa$ for some $s \in S$ and $t \leq sxb \Rightarrow t \in (Sb]$. Now let $t' \in (Sb]$, then $t' \leq s'b$ for some $s' \in S$ and $t' \leq s'ya \Rightarrow t' \in (Sa]$. Hence $(Sa] = (Sb]$. So $a\,\mathcal{L}\,b$. Similarly we can prove that $a\,\mathcal{R}\,b$ if and only if there exists $u, v \in S$ such that $a \leq bu$ and $b \leq av$. ■

3. Equivalence relation on semigroup of bi-ideals-$\mathcal{B}(S)$

Let $(S, \cdot, \leq)$ be an ordered semigroup and $\mathcal{B}(S)$ be the semigroup of bi-ideals of S. Next we proceed to define certain relations on the semigroup of bi-ideals of the ordered semigroup S by making use of the Green's relations on S which turns out to be equivalence relations on $\mathcal{B}(S)$ and we call them as Green's relations on $\mathcal{B}(S)$.

Theorem 4. *Let S be an ordered semigroup and $\mathcal{B}(S)$ be the semigroup of bi-ideals of S. For $A, B \in \mathcal{B}(S)$, define the relation $\mathcal{L}'$ on $\mathcal{B}(S)$ by $A\,\mathcal{L}'\,B$ if and only if for each $a \in A$ there exists some $b \in B$ such that $a\,\mathcal{L}_S b$ and vice-versa, where $\mathcal{L}_S$ is the Green's $\mathcal{L}$ relation on S. Then $\mathcal{L}'$ is an equivalence relation on $\mathcal{B}(S)$.*

Proof. Let $A\,\mathcal{L}'\,B$. Then for each $a \in A$ there exists $b \in B$ such that $a\,\mathcal{L}_S\,b$ and for each $b' \in B$ there exists $a' \in A$ such that $a'\,\mathcal{L}_S\,b'$. Consider $A, B, C \in$

$\mathcal{B}(S)$, clearly $A\,\mathcal{L}'\,A$ and when $A\,\mathcal{L}'\,B$ then $B\,\mathcal{L}'\,A$, i.e., $\mathcal{L}'$ is both reflexive and symmetric. Again for $A\,\mathcal{L}'\,B$ and $B\,\mathcal{L}'\,C$, for each $a \in A$ there is a $b \in B$ such that $a\,\mathcal{L}_S\,b$ and for each $b \in B$ there exists $c \in C$ such that $b\,\mathcal{L}_S\,c$. By the transitivity of $\mathcal{L}_S$ we have $a\,\mathcal{L}_S\,c$, i.e., for each $a \in A$ there exists $c \in C$ such that $a\,\mathcal{L}_S\,c$. Similarly it is seen that for each $c' \in C$ there exists $a' \in A$ such that $a'\,\mathcal{L}_S\,c'$. Hence $A\,\mathcal{L}'\,C$ and $\mathcal{L}'$ is an equivalence relation on $\mathcal{B}(S)$. ■

Throughout the paper, we denote $\mathcal{L}_{\mathcal{B}(S)}$, $\mathcal{R}_{\mathcal{B}(S)}$ as the Green's $\mathcal{L}$ relation and $\mathcal{R}$ relation on $\mathcal{B}(S)$ respectively. Next we have the following useful result.

Theorem 5. *Let S be a regular ordered semigroup and $\mathcal{B}(S)$ the semigroup of bi-ideals of S. Then $\mathcal{L}_{\mathcal{B}(S)} \subseteq \mathcal{L}'$.*

Proof. For A, B be any two bi-ideals of S and let $(A, B) \in \mathcal{L}_{\mathcal{B}(S)}$. Then by the equivalent condition of the Green's $\mathcal{L}$- relation, there exists $X, Y \in \mathcal{B}(S)$ such that $A = X * B$ and $B = Y * A$. i.e $A = (XB]$ and $B = (YA]$.

If possible $(A, B) \notin \mathcal{L}'$. Then either there exists $a \in A$ such that $(a, b) \notin \mathcal{L}_S, \forall\, b \in B$, or there exists $b \in B$ such that $(a, b) \notin \mathcal{L}_S, \forall\, a \in A$. Suppose that there is an $a \in A$ such that $(a, b) \notin \mathcal{L}_S, \forall\, b \in B$, then for any $b \in B$, there doesnot exists x or $y \in S$ such that $a \leq xb$ and $b \leq ya$, which is a contradiction to the existence of either X or Y. Hence $(A, B) \in \mathcal{L}'$ and so $\mathcal{L}_{\mathcal{B}(S)} \subseteq \mathcal{L}'$. ■

However, the relation $\mathcal{R}'$ can be defined dually and can obtain the following Theorems.

Theorem 6. *Let S be a regular ordered semigroup and $\mathcal{B}(S)$ be the semigroup of bi-ideals of S. Suppose that $A, B \in \mathcal{B}(S)$. Define a relation $\mathcal{R}'$ on $\mathcal{B}(S)$ as $A\,\mathcal{R}'\,B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a\,\mathcal{R}_S\,b$ and vice-versa, where $\mathcal{R}_S$ is the Green's $\mathcal{R}$ relation on S. Then $\mathcal{R}'$ is an equivalence relation on $\mathcal{B}(S)$.*

Theorem 7. *Let S be a regular ordered semigroup and $\mathcal{B}(S)$ be the semigroup of bi-ideals of S. Then $\mathcal{R}_{\mathcal{B}(S)} \subseteq \mathcal{R}'$.*

Remark 8. The Theorem 5 holds even if we drop the ordered regularity. For, let S be an ordered semigroup and $\mathcal{B}(S)$ be the semigroup of bi-ideals of S. Suppose that $\mathcal{L}_{\mathcal{B}(S)}$ is the Green's $\mathcal{L}$ relation on $\mathcal{B}(S)$ and $\mathcal{L}'$ is the equivalence relation defined on $\mathcal{B}(S)$ as $A\,\mathcal{L}'\,B$ if and only if for each $a \in A$ there exists $b \in B$ such that $a\,\mathcal{L}_S\,b$ and vice-versa where $\mathcal{L}_S$ is the Green's $\mathcal{L}$ relation on S and $A, B \in \mathcal{B}(S)$. Then $\mathcal{L}_{\mathcal{B}(S)} \subseteq \mathcal{L}'$.

However the reverse inclusion $\mathcal{L}' \subseteq \mathcal{L}_{\mathcal{B}(S)}$ will not in general holds. For, consider the following example.

Example 9. The set $S = \{a, b, c, d, e\}$ together with multiplication table:

.	a	b	c	d	e
a	a	b	c	d	e
b	a	b	c	d	e
c	c	c	c	c	c
d	d	d	d	d	d
e	c	c	c	c	c

and patial order $\leq := \{(a,a),\ (b,b),\ (c,a),\ (c,b),\ (c,c),\ (c,d),\ (c,e),\ (d,d),\ (e,d),\ (e,b),(e,e)\}$ is an ordered semigroup which is not regular. The egg box diagram of S is as follows

a	b	$\mathcal{D}_1$

c	
d	$\mathcal{D}_2$.
e	

The bi-ideals of S are $B_1 = \{c\}$, $B_2 = \{c, e\}$, $B_3 = \{c, d, e\}$, $B_4 = \{a, c, d, e\}$, $B_5 = \{b, c, d, e\}$, $B_6 = S$. The Cayley table of the semigroup of bi-ideals $\mathcal{B}(S) = \{B_1, B_2, B_3, B_4, B_5, B_6\}$ is given below

$*$	B_1	B_2	B_3	B_4	B_5	B_6
B_1	B_1	B_1	B_1	B_1	B_1	B_1
B_2	B_1	B_1	B_1	B_1	B_1	B_1
B_3	B_3	B_3	B_3	B_3	B_3	B_3
B_4	B_3	B_3	B_3	B_4	B_5	B_6
B_5	B_3	B_3	B_3	B_4	B_5	B_6
B_6	B_3	B_3	B_3	B_4	B_5	B_6

The egg box diagram of $\mathcal{B}(S)$ is given as follows

B_2	$\mathcal{D}_1$

B_4	B_5	B_6	$\mathcal{D}_2$

B_1	$\mathcal{D}_3$
B_3	

Clearly $(B_1, B_2) \in \mathcal{L}'$ but $(B_1, B_2) \notin \mathcal{L}_{\mathcal{B}(S)}$. Hence $\mathcal{L}' \nsubseteq \mathcal{L}_{\mathcal{B}(S)}$.

4. GREEN'S RELATIONS ON SEMIGROUP OF BI-IDEALS OF ORDERED FULL TRANSFORMATION SEMIGROUP

In the following we describe the semigroup of bi-ideals of ordered full transformation semigroup and define the Green's relations. For, consider full transformation semigroup $\mathcal{T}_X$ consisting of all maps from a finite set X into X with composition as binary operation and natural partial ordering given by

$$f \leq g \iff R(f) \subseteq R(g) \text{ and for some } \alpha \in \mathcal{T}_X,\ f = \alpha f = \alpha g,$$

where $R(f)$ and $R(g)$ are right ideal generated by $f,\ g \in \mathcal{T}_X$ respectively. Then $(\mathcal{T}_X, \leq)$ is a regular ordered semigroup (see cf. [2, 11]).

Lemma 10. *Let $\mathcal{T}_X$ be the full tranformation semigroup and $(\mathcal{T}_X, \leq)$ be an ordered full transformation semigroup on a set X. Suppose that $\mathcal{L}$ is the Green's $\mathcal{L}$ relation on semigroups. For $f, g \in \mathcal{T}_X$, then $f\,\mathcal{L}\,g$ in $(\mathcal{T}_X, \leq)$ if and only if $f\,\mathcal{L}\,g$ in $\mathcal{T}_X$.*

Proof. Suppose that $f\,\mathcal{L}\,g$ in $(\mathcal{T}_X, \leq)$. Then by Theorem 3, there exists $x, y \in \mathcal{T}_X$ such that $f \leq xg$ and $g \leq yf$. By definition of natural partial order, $f \leq xg$ implies there exists $\alpha \in \mathcal{T}_X$ such that $f = \alpha xg$. Then $Im\, f \subseteq Im\, g$. Similarly $g \leq yf$ implies $Im\, g \subseteq Im\, f$. Hence $f\,\mathcal{L}\,g$ in $(\mathcal{T}_X, \leq)$ implies $Im\, f = Im\, g$ and so $f\mathcal{L}\,g$ in $\mathcal{T}_X$.

Conversly suppose that $f\mathcal{L}\,g$ in $\mathcal{T}_X$, By equivalent condition of Green's $\mathcal{L}$ relation $\exists\ x, y \in \mathcal{T}_X$ such that $f = xg$ and $g = yf$. By the reflexivity of '$\leq$' we get $f \leq xg$ and $g \leq yf$. Hence $f\,\mathcal{L}\,g$ in $(\mathcal{T}_X, \leq)$. ■

Similary we obtain the following lemma.

Lemma 11. *Let $\mathcal{T}_X$ be the full tranformation semigroup on X and $(\mathcal{T}_X, \leq)$ be the ordered full transformation semigroup. Suppose that $\mathcal{R}$ is the Green's $\mathcal{R}$ relation on semigroups. For $f, g \in \mathcal{T}_X$,*

$$f\,\mathcal{R}\,g \in (\mathcal{T}_X, \leq) \iff f\,\mathcal{R}\,g \in \mathcal{T}_X.$$

Theorem 12 [10]**.** *Let S be a regular ordered semigroup and $\mathcal{R}(S)$, $\mathcal{L}(S)$, $\mathcal{B}(S)$ be the collection of right, left, and bi-ideals of S, respectively. Then $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are bands and $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

Theorem 13 [1]**.** *S is regular ordered semigroup if and only if for every right ideal R and left ideal L of $S, (RL] = R \cap L$.*

We have for any right ideal R and left ideal L of a regular ordered semigroup $(RL] = R \cap L$. Also $R * L = (RL]$ and any bi-ideal $B = R * L$. Subject to these obervations we have the following theorem.

Theorem 14. *Let S be a regular ordered semigroup and $\mathcal{R}(S)$, $\mathcal{L}(S)$, $\mathcal{B}(S)$ be the collection of right, left and bi-ideals of S respectively. Then any $B \in \mathcal{B}(S)$, $B = R \cap L$, where $R \in \mathcal{R}(S)$, $L \in \mathcal{L}(S)$.*

Corollary 15. *Let S be a regular ordered semigroup and $L(a), R(a), B(a)$ be the principal left, right and bi-ideal generated by $a \in S$, respectively. Then $B(a) = R(a) \cap L(a)$.*

Proof. Let $t \in B(a) \Rightarrow t \in (aSa] \Rightarrow t \leq axa$, for some $x \in S$. Then $t \leq a(xa)$ and $t \leq (ax)a \Rightarrow t \in R(a)$ and $t \in L(a) \Rightarrow t \in R(a) \cap L(a) \Rightarrow B(a) \subseteq R(a) \cap L(a)$.

Now suppose that $B(a) = R \cap L$, for some $R \in \mathcal{R}(S)$, $L \in \mathcal{L}(S)$. Then $R(a) \subseteq R$ and $L(a) \subseteq L \Rightarrow R(a) \cap L(a) \subseteq R \cap L \Rightarrow R(a) \cap L(a) \subseteq B(a)$. Hence $B(a) = R(a) \cap L(a)$. ∎

In the light of Theorem 14 it can be seen that any bi-ideal B of $(\mathcal{T}_X, \leq)$ is of the form $B = R * L = (RL] = R \cap L$, where $R \in \mathcal{R}(\mathcal{T}_X)$, $L \in \mathcal{L}(\mathcal{T}_X)$.

The following examples describe the semigroup of bi-ideals of regular ordered semigroups $(\mathcal{T}_2, \leq))$ and $(\mathcal{T}_3, \leq))$ and it is shown that the relations $\mathcal{L}'$ and $\mathcal{R}'$ defined in Theorem 4 and Theorem 6 coincides with Green's relations in the semigroups of bi-ideals of $(\mathcal{T}_2, \leq)$ and $(\mathcal{T}_3, \leq)$.

Example 16. Bi-ideals of $(\mathcal{T}_2, \leq)$. Let $X = \{1, 2\}$, denote (i, j) for the mapping $1 \mapsto i, 2 \mapsto j$. Then $\mathcal{T}_2 = \{(1,1), (2,2), (1,2), (2,1)\}$ and it's Cayley table as well as natural partial order $\leq$ are given below

.	$(1,1)$	$(2,2)$	$(1,2)$	$(2,1)$
$(1,1)$	$(1,1)$	$(2,2)$	$(1,1)$	$(2,2)$
$(2,2)$	$(1,1)$	$(2,2)$	$(2,2)$	$(1,1)$
$(1,2)$	$(1,1)$	$(2,2)$	$(1,2)$	$(2,1)$
$(2,1)$	$(1,1)$	$(2,2)$	$(2,1)$	$(1,2)$

$\leq = \{(1,1) \leq (1,1), (2,2) \leq (2,2), (1,2) \leq (1,2), (2,1) \leq (2,1), (1,1) \leq (1,2), (1,1) \leq (2,1), (2,2) \leq (1,2), (2,2) \leq (2,1)\}$. Egg-box diagram of $(\mathcal{T}_2, \leq)$ is as follows

$(1,1)$ | $(2,2)$ — $\mathcal{D}_1$

$(1,2)$, $(2,1)$ — $\mathcal{D}_2$

Bi-ideals of $(\mathcal{T}_2, \leq)$ are $B_1 = \{(1,1)\}$, $B_2 = \{(2,2)\}$, $B_3 = \{(1,1), (2,2)\}$ and $B_4 = \{(1,1), (2,2), (1,2), (2,1)\}$. The Cayley table of $\mathcal{B}(\mathcal{T}_2) = \{B_1, B_2, B_3, B_4\}$, the semigroup of bi-ideals of $(\mathcal{T}_2, \leq)$ is given below

$*$	B_1	B_2	B_3	B_4
B_1	B_1	B_2	B_3	B_3
B_2	B_1	B_2	B_3	B_3
B_3	B_1	B_2	B_3	B_3
B_4	B_1	B_2	B_3	B_4

Egg-box diagram of $(\mathcal{B}(\mathcal{T}_2), *)$ is given as follows

B_1	B_2	B_3	$\mathcal{D}_1$
		B_4	$\mathcal{D}_2$.

The equivalence relations $\mathcal{L}'$ and $\mathcal{R}'$, defined in the Theorem 4 and Theorem 6, respectively are

$\mathcal{L}' = \{(B_1, B_1), (B_2, B_2), (B_3, B_3), (B_4, B_4)\}$

$\mathcal{R}' = \{(B_1, B_1), (B_2, B_2), (B_3, B_3), (B_4, B_4), (B_1, B_2), (B_1, B_3), (B_2, B_3), (B_2, B_1), (B_3, B_1), (B_3, B_2)\}$.

Hence it is seen that the Green's relations $\mathcal{L}_{\mathcal{B}(\mathcal{T}_2)}$, $\mathcal{R}_{\mathcal{B}(\mathcal{T}_2)}$ on $\mathcal{B}(\mathcal{T}_2)$ coincides with $\mathcal{L}'$, $\mathcal{R}'$ and the Green's relations $\mathcal{L}_{\mathcal{B}(\mathcal{T}_2)}$, $\mathcal{R}_{\mathcal{B}(\mathcal{T}_2)}$ on $\mathcal{B}(\mathcal{T}_2)$ can be defined in terms of Green's relations $\mathcal{L}_{\mathcal{T}_2}$, $\mathcal{R}_{\mathcal{T}_2}$ on $(\mathcal{T}_2, \leq)$. From the Cayley table of $\mathcal{B}(\mathcal{T}_2)$ it is evident that every elements of $\mathcal{B}(\mathcal{T}_2)$ are idempotents. Hence $\mathcal{B}(\mathcal{T}_2)$ is a band. So by Theorem 2, $(\mathcal{T}_2, \leq)$ is both regular and intra- regular.

Example 17. Bi-ideals of $(\mathcal{T}_3, \leq)$. Let $X = \{1, 2, 3\}$. Similar to the case of $\mathcal{T}_2$, we shall denote (i, j, k) for the mapping $1 \mapsto i, 2 \mapsto j, 3 \mapsto k$ and order as the natural partial order. Then the egg box diagram of $\mathcal{T}_3$ is as below

(1,1,1)	(2,2,2)	(3,3,3)	$\mathcal{D}_1$				
			(1,2,2) (2,1,1)	(1,3,3) (3,1,1)	(2,3,3) (3,2,2)		
			(2,1,2) (1,2,1)	(3,1,3) (1,3,1)	(3,2,3) (2,3,2)	$\mathcal{D}_2$	
			(2,2,1) (1,1,2)	(3,3,1) (1,1,3)	(3,3,2) (2,2,3)		
						(1,2,3)(2,3,1) (3,1,2)(1,3,2) (3,2,1)(2,1,3)	$\mathcal{D}_3$.

The set of all right ideals of $\mathcal{T}_3$, $\mathcal{R}(\mathcal{T}_3) = \{\ \mathcal{D}_1, \mathcal{T}_3,$ union of $\mathcal{D}_1$ and rows of $\mathcal{D}_2\}$. The principal left ideals are $L_1 = \{(1,1,1)\}$, $L_2 = \{(2,2,2)\}$, $L_3 = \{(3,3,3)\}$, $L_4 = \{(1,1,1),(2,2,2)\} \cup \{$first column of $\mathcal{D}_2\ \}$, $L_5 = \{(1,1,1),(3,3,3)\} \cup \{$second column of $\mathcal{D}_2\ \}$, $L_6 = \{(2,2,2),(3,3,3)\} \cup \{$third column of $\mathcal{D}_2\ \}$ and $\mathcal{T}_3$. So the set of all left ideals of $\mathcal{T}_3$, $\mathcal{L}(\mathcal{T}_3) = \{\mathcal{T}_3, L_i,$ all possible unions of $L_i's\}$. Hence the bi-ideals of $\mathcal{T}_3$ are $\mathcal{B}(\mathcal{T}_3) = \{\ B = R \cap L\ /\ R \in \mathcal{R}(\mathcal{T}_3),\ L \in \mathcal{L}(\mathcal{T}_3)\}$.

Consider the equivalence relation $\mathcal{L}'$ on $\mathcal{B}(\mathcal{T}_3)$, i.e., $B_i\, \mathcal{L}'\, B_j$ if and only if for each $b_i \in B_i$ there exists $b_j \in B_j$ such that $b_i\, \mathcal{L}_{\mathcal{T}_3}\, b_j$ and for each $b'_j \in B_j$ there exists $b'_i \in B_i$ such that $b'_i\, \mathcal{L}_{\mathcal{T}_3}\, b'_j$. We can show that $\mathcal{L}' \subseteq \mathcal{L}_{\mathcal{B}(\mathcal{T}_3)}$. For let

$(B_i, B_j) \in \mathcal{L}'$. Since each bi-ideal of $\mathcal{T}_3$ is the disjoint union of $\mathcal{H}$-classes, choose b_{i1} from one of those $\mathcal{H}$-class of B_i. So $b_{i1} \in B_i$ and $B_i \, \mathcal{L}' \, B_j \Rightarrow \exists b_{j1} \in B_j$ such that $b_{i1} \, \mathcal{L}_{\mathcal{T}_3} \, b_{j1}$. By Theorem 3 $\exists \, x, y \in \mathcal{T}_3$ such that $b_{i1} \leq x b_{j1}$ and $b_{j1} \leq y b_{i1}$. In particular we can find x_1 from $\mathcal{D}$-class of b_{i1} such that $b_{i1} \leq x_1 b_{j1}$. similarly choose b_{i2} from another $\mathcal{H}$-class of B_i and find x_2 from $\mathcal{D}$-class of b_{i2} such that $b_{i2} \leq x_2 b_{j2}$ and so on. Then we can easily verify that the union of those $\mathcal{H}$-class of x_{ik} is a bi-ideal of $\mathcal{T}_3$, i.e., $X = \cup \, \mathcal{H}(x_{ik}) \in \mathcal{B}(\mathcal{T}_3)$. Also $B_i = X * B_j$. Similarly we get $B_j = Y * B_i$. Hence $(B_i, B_j) \in \mathcal{L}_{\mathcal{B}(\mathcal{T}_3)}$. So $\mathcal{L}' \subseteq \mathcal{L}_{\mathcal{B}(\mathcal{T}_3)}$. We already proved in the Theorem 5 that $\mathcal{L}_{\mathcal{B}(\mathcal{T}_3)} \subseteq \mathcal{L}'$. Hence $\mathcal{L}_{\mathcal{B}(\mathcal{T}_3)} = \mathcal{L}'$, i.e., the Green's relation $\mathcal{L}_{\mathcal{B}(\mathcal{T}_3)}$ on $\mathcal{B}(\mathcal{T}_3)$ is coincides with $\mathcal{L}'$. Similarly $\mathcal{R}_{\mathcal{B}(\mathcal{T}_3)}$ is same as $\mathcal{R}'$. Hence the Green's relations $\mathcal{L}_{\mathcal{B}(\mathcal{T}_3)}$, $\mathcal{R}_{\mathcal{B}(\mathcal{T}_3)}$ on $\mathcal{B}(\mathcal{T}_3)$ can be defined in terms of Green's relations $\mathcal{L}_{\mathcal{T}_3}$, $\mathcal{R}_{\mathcal{T}_3}$ on $\mathcal{T}_3$.

We can extend this result to semigroup of bi-ideals of ordered full transformation semigroup $(\mathcal{T}_X, \leq)$ with $|X| = n$. The Green's relations on $\mathcal{B}(\mathcal{T}_X)$ is defined as follows.

Definition. Given $B_1, B_2 \in \mathcal{B}(\mathcal{T}_X)$, $B_1 \, \mathcal{L}_{\mathcal{B}(\mathcal{T}_X)} \, B_2$ if and only if for each $b_1 \in B_1$ there exists $b_2 \in B_2$ such that $b_1 \, \mathcal{L}_{\mathcal{T}_X} \, b_2$ and for each $b_1' \in B_1$ there exists $b_2' \in B_2$ such that $b_1' \, \mathcal{L}_{\mathcal{T}_X} \, b_2'$.

Definition. Given $B_1, B_2 \in \mathcal{B}(\mathcal{T}_X)$, $B_1 \, \mathcal{R}_{\mathcal{B}(\mathcal{T}_X)} \, B_2$ if and only if for each $b_1 \in B_1$ there exists $b_2 \in B_2$ such that $b_1 \, \mathcal{R}_{\mathcal{T}_X} \, b_2$ and for each $b_1' \in B_1$ there exists $b_2' \in B_2$ such that $b_1' \, \mathcal{R}_{\mathcal{T}_X} \, b_2'$.

Theorem 18. *Let $f, g \in (\mathcal{T}_X, \leq)$ and $B(f), B(g)$ be the principal bi-ideal generated by f, g respectively. Then $f \, \mathcal{L}_{\mathcal{T}_X} \, g$ if and only if $B(f) \, \mathcal{L}_{\mathcal{B}(\mathcal{T}_X)} \, B(g)$.*

Proof. Suppose that $f \, \mathcal{L}_{\mathcal{T}_X} \, g$. Then $Im\, f = Im\, g$. Let $t \in B(f)$. Then $t \leq fhf$ for some $h \in (\mathcal{T}_X, \leq)$. By the definition of natural partial order $\exists \alpha \in (\mathcal{T}_X, \leq)$ such that $t = \alpha fhf \Rightarrow Im\, t \subseteq Im\, f \Rightarrow Im\, t \subseteq Im\, g$. We have to find $t' \in B(g)$ such that $Im\, t = Im\, t'$. Let $Im\, g = \{x_1, x_2, \ldots, x_l\} \subseteq X$ and $Im\, t = \{x_{i1}, x_{i2}, \ldots, x_{ik}\} \subseteq Im\, g$. Define a map $y : X \to X$ as follows. Each $x_{ij} \mapsto y_{ij}$, where y_{ij} is an element of set of pre images of $x_{ij}, j = 1, 2, \ldots, k$ under the map g and $X - \{x_{i1}, x_{i2}, \ldots, x_{ik}\} \mapsto y_{i1}$. Now define $t' = gyg$. Then $t' \in B(g)$ and $Im\, t = Im\, t' \Rightarrow t \mathcal{L}_{\mathcal{T}_X} t'$. Similarly we can prove that for each $q \in B(g)$, there exists $q' \in B(f)$ such $q \, \mathcal{L}_{\mathcal{T}_X} \, q'$. So by above definition $B(f) \, \mathcal{L}_{\mathcal{B}(\mathcal{T}_X)} \, B(g)$.

Conversely, suppose that $B(f) \, \mathcal{L}_{\mathcal{B}(\mathcal{T}_X)} \, B(g)$. By equivalent condition of Green's relation in semigroup, $\exists \, X, Y \in \mathcal{B}(\mathcal{T}_X)$ such that $B(f) = X * B(g)$ and $B(g) = Y * B(f) \Rightarrow B(f) = (XB(g)]$ and $B(g) = (YB(f)]$. Then $f \leq xt$ and $g \leq yt'$, where $x \in X, y \in Y, t \in B(g)$ and $t' \in B(f)$. Also $t \in B(g) \Rightarrow t \leq gqg$ and $t' \in B(f) \Rightarrow t' \leq fq'f$, $(q, q' \in (\mathcal{T}_X, \leq))$. So $f \leq xgqg$ and $g \leq yfq'f \Rightarrow f \leq (xgq)g$ and $g \leq (yfq')f$. Hence by Theorem 3, $f \, \mathcal{L}_{\mathcal{T}_X} \, g$. ■

Consider the semigroup $(\mathcal{B}(\mathcal{T}_X),*)$. Let $B \in \mathcal{B}(\mathcal{T}_X)$. Since every right and left ideals are idempotents in $\mathcal{B}(\mathcal{T}_X)$ (Theorem 12), if B is left or right ideal then $\mathcal{L}_{\mathcal{B}(\mathcal{T}_X)}$-class and $\mathcal{R}_{\mathcal{B}(\mathcal{T}_X)}$-class of B contains the idempotent B. If B is neither a left ideal nor a right ideal then B is of the form $B = R \cap L$, where $R \in \mathcal{R}(\mathcal{T}_X)$, $L \in \mathcal{L}(\mathcal{T}_X)$. It is evident that $B\,\mathcal{L}_{\mathcal{B}(\mathcal{T}_X)}\,L$ and $B\,\mathcal{R}_{\mathcal{B}(\mathcal{T}_X)}\,R$. Hence $\mathcal{L}_{\mathcal{B}(\mathcal{T}_X)}$-class of B contains the idempotent L and $\mathcal{R}_{\mathcal{B}(\mathcal{T}_X)}$-class of B contains the idempotent R. So $(\mathcal{B}(\mathcal{T}_X),*)$ is a regular semigroup.

Acknowledgement

First author wishes to thank Cochin University of Science and Technology (CUSAT) for providing financial support. Second author wishes to thank Council of Scientific and Industrial Research (CSIR) INDIA for providing financial support.

References

[1] Y. Cao, *Characterization of regular ordered semigroups by quasi-ideals*, Vietnam J. Math. **30(3)** (2002) 239–250.

[2] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vol. I, Mathematical surveys, No. 7 (American Mathematical Society, Providence, R.I., 1961).

[3] R.A. Good and D.R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc. **58** (1952) 624–625.
https://doi.org/10.1090/S0002-9904-1952-09650-6

[4] S. Lajos, *On (m; n)-ideals of semigroups*, Abstract of Second Hunger. Math. Congr. I, **1960** (1960) 42–44.

[5] S. Lajos, *On the bi-ideals in semigroups*, Proc. Japan Acad. **45(8)** (1969) 710–712.
https://doi.org/10.3792/pja/1195520625

[6] N. Kehayopulu, *Note on bi-ideals in ordered semigroups*, Pure Math. Appl. **6(4)** (1995) 333–344.

[7] N. Kehayopulu, *Ideals and Green's relations in ordered semigroups*, Internat. J. Math. and Math. Sci. (2006) 1–8.
https://doi.org/10.1155/IJMMS/2006/61286

[8] N. Kehayopulu, J.S. Ponizovskii and M. Tsingelis, *Bi-ideals in ordered semigroups and ordered groups*, J. Math. Sci. **112(4)** (2002).
https://doi.org/10.1023/A:1020347003781

[9] N. Kehayopulu and M. Tsingelis, *On left regular ordered semigroups*, South. Asian Bull. Math. **25** (2002) 609615. https://doi.org/10.1007/s100120200005

[10] S. Mallick and K. Hansda, *On the semigroup of bi-ideals of an ordered semigroup*, Kragujevac J. Math. **47(3)** (2023) 339–345.
https://doi.org/10.46793/KgJMat2303.339M

[11] K.S.S. Nambooripad, Theory of Regular Semigroups (Sayahna Foundation, Trivandrum, 2018).

Received 11 December 2023
Revised 15 July 2024
Accepted 15 July 2024