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THE STRUCTURE OF LIE TRIPLE CENTRALIZERS ON PRIME RINGS AND APPLICATIONS

Sh. Goodarzi and A. Khotanloo[1]

Faculty of Mathematics Sciences and Statistics
Malayer University
P.O. Box 6517-95863, Malayer, Iran

e-mail: shima.goodarzi@stu.malayeru.ac.ir
a.khotanloo@malayeru.ac.ir

Abstract

Let $\mathcal{R}$ be an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P and ϕ be an additive map on $\mathcal{R}$ satisfying

$$\phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C],$$

for any $A,B,C \in \mathcal{R}$ whenever $AB=0$. In this paper, we study the structure of map ϕ and prove that ϕ on $\mathcal{R}$ is proper, i.e., has the form $\phi(A) = \lambda A + h(A)$, where $\lambda \in Z(\mathcal{R})$ and h is an additive map into its center vanishing at second commutators $[[A,B],C]$ with $AB=0$. Applying these results, we characterize generalized Lie triple derivations on $\mathcal{R}$. The obtained results can be used for some classical operator prime algebras such as standard operator algebras and factor von Neumann algebras, which generalize some known results.

Keywords: Lie triple centralizer, generalized Lie triple drivation, prime ring.

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1. Introduction

Assume $\mathcal{R}$ be an associative ring. Recall that an additive map $\delta : \mathcal{R} \to \mathcal{R}$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for all $a,b \in \mathcal{R}$. Suppose $[a,b] = ab - ba$ denote the Lie product and admit $a \circ b = ab + ba$ denote the Jordan product of elements $a,b \in \mathcal{R}$. An additive map δ on $\mathcal{R}$ to $\mathcal{R}$ is called a Lie derivation

[1]Corresponding author.

if it is a derivation for the Lie product, i.e., $\delta([a,b]) = [\delta(a),b] + [a,\delta(b)]$ for all $a,b \in \mathcal{R}$. Similarly, an additive map δ on $\mathcal{R}$ to itself is called a Jordan derivation if it satisfies $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$ for all $a,b \in \mathcal{R}$. An additive map Δ on $\mathcal{R}$ is said to be a generalized Lie derivation associated with the Lie derivation δ if

$$\Delta([a,b]) = [\Delta(a),b] + [a,\delta(b)], \qquad (a,b \in \mathcal{R}).$$

A Lie triple derivation is an additive map $\delta : \mathcal{R} \to \mathcal{R}$, which satisfies

$$\delta([[a,b],c]) = [[\delta(a),b],c] + [[a,\delta(b)],c] + [[a,b],\delta(c)], \qquad (a,b,c \in \mathcal{R}).$$

An additive map $\Delta : \mathcal{R} \to \mathcal{R}$ is said to be a generalized Lie triple derivation associated with the Lie triple derivation δ if

$$\Delta([[a,b],c]) = [[\Delta(a),b],c] + [[a,\Delta(b)],c] + [[a,b],\delta(c)], \qquad (a,b,c \in \mathcal{R}).$$

Every derivation is a Lie derivation and a Jordan derivation. Also, every Lie derivation is a generalized Lie derivation. Obviously, Lie derivations are Lie triple derivations. The known equation $[[a,b],c] = a \circ (b \circ c) - b \circ (a \circ c)$ for all $a,b,c \in \mathcal{R}$ it concludes that every Jordan derivation is also a Lie triple derivation. Lie triple derivations are generalized Lie triple derivations. However, the converse is not true in general. Therefore, the investigation of the structure of the generalized Lie triple derivations leads to the simultaneous characterization of both important classes of Jordan, Lie, and Lie triple derivations. These mappings are among the important cases in studying the structure of Lie algebras. Extensive studies have been performed to characterize these maps on different algebras, and here, for instance, we refer to [2, 5, 6, 30, 29] and the references therein.

An additive map $\phi : \mathcal{R} \to \mathcal{R}$ is said to be a Lie centralizer if

$$\phi([a,b]) = [\phi(a),b] = [a,\phi(b)], \qquad (a,b \in \mathcal{R}).$$

Also, An additive map ϕ on $\mathcal{R}$ into $\mathcal{R}$ is a Lie triple centralizer if

$$\phi([[a,b],c]) = [[\phi(a),b],c] = [[a,\phi(b)],c], \qquad (a,b \in \mathcal{R}).$$

Clearly, each Lie centralizer is a Lie triple centralizer, but the converse is not true in general. Therefore, the concept of Lie triple centralizer generalizes the concept of Lie centralizer. Additive map ϕ on $\mathcal{R}$ is called a Jordan centralizer if $\phi(a \circ b) = \phi(a) \circ b$ for all $a,b \in \mathcal{R}$ and every Jordan centralizer is also a Lie triple centralizer. By straightforward calculations, it can be checked that Δ is a generalized Lie (triple) derivation associated with the Lie derivation δ if and only if $\phi = \Delta - \delta$ is a Lie (triple) centralizer. Hence on a ring, if we determine the structure of the Lie (triple) centralizers and Lie (triple) derivations, then we can also characterize the structure of the generalized Lie (triple) derivations.

In the [19, 28], we see that the concept of Lie centralizer is a classical concept in other nonassociative algebras and the theory of Lie algebras. Determining the structure of Lie (triple) centralizers in the form of centralizers can be of great interest. In recent years, maps of Non-linear Lie centralizers on generalized matrix algebras to itself and Non-additive Lie centralizers on triangular rings, have been studied and investigated by many researchers, and the structure of these maps has been characterized into standard forms [12, 13, 16, 18, 22, 25].

In recent years, certain mappings that act as derivatives in local products have been investigated. One of the research paths in this field is the study of conditions in which the structure of derivatives on rings (algebras) can be determined by mappings that act on local products. Let $\mathcal{R}$ be a ring, in this case, an additive (a linear) map $\delta : \mathcal{R} \to \mathcal{R}$ is called derivable at a given point G in $\mathcal{R}$ if we have $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{R}$ with $ab = G$. These types of maps have been discussed by several researchers (see that [1, 3, 10, 11, 21, 32] and references therein). So far, few papers have worked on Lie triple derivations mappings that act on local products, and the authors have obtained results on operator algebras [23, 24]. An additive (a linear) map $\delta : \mathcal{R} \to \mathcal{R}$ is called Lie triple derivable at a given point $G \in \mathcal{R}$, if $\delta([[a, b], c]) = [[\delta(a), b], c]+[[a, \delta(b)], c]+ [[a, b], \delta(c)]$ for all $a, b, c \in \mathcal{R}$ with $ab = G$. In [30] the authors described the additive map $\delta : \mathcal{R} \to \mathcal{R}$, where $\mathcal{R}$ is a prime ring containing a non-trivial idempotent P satisfying

$$a, b \in \mathcal{R}, ab = 0 \Longrightarrow \delta([a, b]) = [\delta(a), b] + [a, \delta(b)],$$

Hereon, we say δ is a Lie derivation at zero products. Also, in order to characterize various mappings with these local features on different algebras, related works have been done in this field, we can see [20, 27, 30]. Recently authors have studied the characterization of Lie centralizers and generalized Lie derivations on non-unital triangular algebras through zero products [2]. Following their research, the authors working in this area have also obtained results, e.g. [8, 12, 15, 17, 26].

Now, considering the results obtained regarding derivations type maps in special products, it seems natural to address the problem of characterizing maps that are such as Lie triple centralizers or generalized Lie triple derivations at local acting. An additive (a linear) map $\phi : \mathcal{R} \to \mathcal{R}$ is called Lie n-centralizer at a given point $G \in \mathcal{R}$, if

$$\phi[[a, b], c] = [[\phi(a), b], c] = [[a, \phi(b)], c]$$

for all $a, b, c \in \mathcal{R}$ with $ab = G$. It is clear that each Lie triple centralizer satisfies Lie triple centralizer at zero product and the converse is, in general, not true (see Example 2.4 of [15]). Recently authors have studied the characterization of Lie centralizers and generalized Lie derivations on non-unital triangular algebras through zero products [2]. Following their research, the authors working in this area have also obtained results, e.g. [8, 15]. Also, the authors in [7, 9] characterize Lie triple mappings at zero product as well as at idempotent product on arbitrary von Neumann algebras. Suppose that exist $\lambda \in Z(\mathcal{R})$ and an additive map $h : \mathcal{R} \to Z(\mathcal{R})$ vanishing at every second commutator $[[A, B], C]$ when $AB = 0$ such that $\phi(A) = \lambda A + h(A)$ for any $A \in \mathcal{R}$. In this case, the additive mapping $\phi : \mathcal{R} \to \mathcal{R}$ defined by $\phi(A) = \lambda A + h(A)$ is a Lie triple centralizer, which is called the Lie triple centralizer with standard form (proper Lie triple centralizer). Note that, in general, every Lie triple centralizer is not necessarily a proper Lie triple centralizer (see Example 1.2 in [12]). In [12], Also Fadaee, Gharamani, and jing studied Lie triple centralizer $\phi : \mathcal{U} \to \mathcal{U}$ under some conditions on an unital generalized, and they showed that $\phi(A) = \lambda A + \psi(A)$, where ψ is a linear map from $\mathcal{U}$ into the center of $\mathcal{U}$ which annihilates all second commutators in commutators and λ is in the center of $\mathcal{U}$.

Now, with the idea from the studies mentioned above and as a continuation of the above works in this research, we determine the structure of additive maps on the unital prime rings that local act like Lie triple centralizers or generalized Lie triple derivations at zero products. Specifically, we consider the following conditions in additive maps ϕ and Δ on a unital prime ring $\mathcal{R}$

$$a, b, c \in \mathcal{R}, \quad ab = 0 \Longrightarrow \phi([[a, b], c] = [[\phi(a), b], c];$$

$$a, b, c \in \mathcal{R}, \quad ab = 0 \Longrightarrow \begin{cases} \Delta([[a, b], c]) = [[\Delta(a), b], c] + [[a, \Delta(b)], c] + [[a, b], \delta(c)] \\ \delta([[a, b], c]) = [[\delta(a), b], c] + [[a, \delta(b)], c] + [[a, b], \delta(c)]. \end{cases}$$

Firstly, in Section 2 we characterize the structure of the additive Lie triple centralizers at zero products (Theorem 2.1) and Lie triple centralizers (Theorem 2.2) on unital prime rings included a non-trivial idempotent and the above results are applied to some classical operator prime algebras such as standard operator algebras and factor von Neumann algebras (Corollaries 2.3–2.6). Also, in section 2 we characterize the structure of the additive Lie centralizers (Corollary 2.7) and Jordan centralizers (Corollary 2.8) on unital prime rings including a non-trivial idempotent and using these results we apply several classical examples of unital prime rings with nontrivial idempotents. In Section 3, we proved the main results. Finally, in Section 4 using the results above, we determine generalized Lie triple derivations at zero products and generalized Lie triple derivations on unital prime rings containing a non-trivial idempotent and also on factor von Neumann algebras and standard operator algebras (Theorem 4.2 and Corollaries 4.3–4.5).

Suppose that $\mathcal{R}$ is a prime ring, that is, for any $A, B \in \mathcal{R}$, quotation $A\mathcal{R}B = \{0\}$ implies $A = 0$ or $B = 0$. In this case, we denote the maximal right ring of quotients and the two-sided right ring of quotients of $\mathcal{R}$ by $\mathcal{Q}_{mr}(\mathcal{R})$ and $\mathcal{Q}_r(\mathcal{R})$, respectively. Note that $\mathcal{R} \subseteq \mathcal{Q}_r(\mathcal{R}) \subseteq \mathcal{Q}_{mr}(\mathcal{R})$. We say that he centre $\mathcal{C} = Z(\mathcal{Q}_r(\mathcal{R}))$ of $\mathcal{Q}_r(\mathcal{R})$ is the extended centroid of $\mathcal{R}$. We also know that the extended centroid of any prime ring is a field(To see more details, you can see [4]). On the other, we have $Z(\mathcal{R}) \subseteq \mathcal{C}$.

2. MAIN RESULTS AND COROLLARIES ON SOME CLASSICAL EXAMPLES OF PRIME RINGS

In this section, we present the main results of this paper. Throughout this section, it is assumed that $\mathcal{R}$ is an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P. In the following theorem, we give the structure of Lie triple centralizers on prime rings by acting on zero products.

Theorem 2.1. *Suppose $\mathcal{R}$ be an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P and let ϕ on $\mathcal{R}$ is an additive map. Then, the following statements are equivalent.*

(i) $A, B, C \in \mathcal{R}$, *with* $AB = 0 \Longrightarrow \phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C]$.

(ii) ϕ *on* $\mathcal{R}$ *is proper Lie triple centralizer (i.e., for* $A \in \mathcal{R}$, ϕ *has form* $\phi(A) = \lambda A + h(A)$, *where* λ *in center* $\mathcal{R}$ *and* $h : \mathcal{R} \to Z(\mathcal{R})$ *is an additive map vanishing at every second commutator* $[[A,B],C]$ *when* $AB = 0$.

According to Theorem 2.1, we characterize the structure of Lie triple centralizers on prime rings in the form of the following theorem.

Theorem 2.2. *Suppose $\mathcal{R}$ be an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P and let ϕ on $\mathcal{R}$ is an additive map. Then map ϕ is a Lie triple centralizer if and only if ϕ is a proper Lie triple centralizer.*

Now, we apply the 2.1 theorem to some classical examples of prime rings, such as the standard operator algebra and the von Neumann factor algebra, to determine the structure of Lie triple centralizer mappings, and we get some interesting results. For this, we will first have a review of these operator algebras.

Standard operator algebras

Suppose $\mathcal{X}$ be a Banach space over the real or complex field $\mathbb{F}$ with $\dim \mathcal{X} \geq 2$. In this case, we denote the algebra of all bounded operators and the ideal of all finite rank operators as $\mathcal{B}(\mathcal{X})$ and $\mathcal{F}(\mathcal{X})$, respectively. We remark that a standard operator algebra $\mathcal{A}$ is any subalgebra of $\mathcal{B}(\mathcal{X})$ which $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{A}$ and contain the identity operator I. It is clear $\mathcal{B}(\mathcal{X})$ is a unital standard operator algebra. We note that the extended centroid of the standard operator algebra $\mathcal{A}$ is equal to $Z(\mathcal{A}) = \mathbb{F}I$. Also, every standard operator algebra is a prime algebra and contains nontrivial idempotents.

Corollary 2.3. *Let $\mathcal{X}$ be a Banach space over the real or complex field $\mathbb{F}$ with dimension greater than* 2 *and $\mathcal{A}$ subalgebra of $\mathcal{B}(\mathcal{X})$ be a standard operator algebra. Suppose that ϕ on $\mathcal{A}$ is an additive map. Then, the following statements are equivalent.*

(i) $A, B, C \in \mathcal{A}$, *with* $AB = 0 \Longrightarrow \phi([[A, B], C]) = [[\phi(A), B], C] = [[A, \phi(B)], C]$.

(ii) *There exist $\lambda \in \mathbb{F}$ and a map h such that $\phi(A) = \lambda A + h(A)I$, where $h : \mathcal{A} \to \mathbb{F}I$ is an additive map vanishing on each second commutator $[[A, B], C]$ whenever $AB = 0$.*

Proof. The standard operator algebra $\mathcal{A}$ is an unital prime algebra that satisfies all the conditions of Theorem 2.1. ■

According to the explanations in this section and Corollary 2.3, we have the following result.

Corollary 2.4. *Let $\mathcal{X}$ be a Banach space over the real or complex field $\mathbb{F}$ with dimension greater than* 2 *and $\mathcal{A}$ subalgebra of $\mathcal{B}(\mathcal{X})$ be a standard operator algebra. Then an additive map ϕ on $\mathcal{A}$ is a Lie triple centralizer if and only if ϕ is a proper Lie triple centralizer.*

Factor von Neumann algebras

A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathbb{H}$ containing the identity I. A von Neumann algebra is a factor if its center is trivial. It is well known that every factor von Neumann algebras are unital prime algebras with nontrivial idempotents. It follows from these notes that each factor von Neumann algebra satisfies all conditions of Theorem 2.1.

Corollary 2.5. *Let $\mathcal{M}$ be a factor von Neumann algebra with* $\deg \mathcal{M} > 1$ *and let ϕ on $\mathcal{M}$ is an additive map. Then, the following statements are equivalent.*

(i) $A, B, C \in \mathcal{M}$, *with* $AB = 0 \Longrightarrow \phi([[A, B], C]) = [[\phi(A), B], C] = [[A, \phi(B)], C]$.

(ii) *There exist $\lambda \in \mathbb{C}$ and a map h such that $\phi(A) = \lambda A + h(A)I$, where $h : \mathcal{M} \to \mathbb{C}I$ is an additive map vanishing on each second commutator $[A, B]$ whenever $AB = 0$.*

According to the explanations in this section and Corollary 2.3, we have the following results.

Corollary 2.6. *Let $\mathcal{M}$ be a factor von Neumann algebra with deg $\mathcal{M} > 1$. Then an additive map $\phi : \mathcal{M} \to \mathcal{M}$ is a Lie triple centralizer if and only if ϕ is a proper Lie triple centralizer.*

Note that a Lie centralizer and Jordan centralizer must be a Lie triple centralizer. So the following corollary is immediate.

Corollary 2.7. *Suppose that $\phi : \mathcal{U} \to \mathcal{U}$ be an additive map. Let $\mathcal{U}$ be any of the following algebras.*

(a) *Unital prime ring with characteristic not 2 and containing a nontrivial idempotent P.*

(b) *Standard operator algebra on a complex Banach space X.*

(c) *Factor von Neumann algebra.*

Then an additive map ϕ on $\mathcal{U}$ into itself is a Lie centralizer if and only if ϕ is a proper Lie centralizer.

Corollary 2.8. *Suppose that $\phi : \mathcal{U} \to \mathcal{U}$ be additive map. Let $\mathcal{U}$ be any of the following algebras.*

(a) *Unital prime ring with characteristic not 2 and containing a nontrivial idempotent P.*

(b) *Standard operator algebra on a complex Banach space X.*

(c) *Factor von Neumann algebra.*

Then an additive map ϕ on $\mathcal{U}$ to $\mathcal{U}$ is a Jordan centralizer if and only if ϕ is a proper Jordan centralizer.

3. THE PROOF OF MAIN RESULTS

In this section, we will present the proof of the main result, Theorems 2.1 of this paper. First, we give the following lemma which is needed to prove the main result.

Lemma 3.1 [4, Theorem 1]**.** *Suppose that $\mathcal{R}$ be a prime ring, and let $AXB = BXA$ for any $A, B \in \mathcal{Q}_{mr}(\mathcal{R})$ and any $X \in \mathcal{R}$. Then A and B are $\mathcal{C}$-dependent.*

***Proof of Theorem* 2.1.** Let $P_1 = P$ be a nontrivial idempotent in $\mathcal{R}$, and $P_2 = I - P_1$. Set $\mathcal{R}_{ij} = P_i\mathcal{R}P_j$, $i, j = 1, 2$, then $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$.

The "if" part is obvious, we only check the "only if" part. We will organize the proof into a series of Claims.

Claim 1. $\phi(\mathcal{R}_{ij}) \subseteq \mathcal{R}_{ij}$, $1 \leq i \neq j \leq 2$.

For any $A_{12} \in \mathcal{R}_{12}$, since $P_2(A_{12}) = 0$, by the assumption we have

$$\begin{aligned}
\phi(A_{12}) &= \phi([[P_2, A_{12}], P_1]) \\
&= [[\phi(P_2), A_{12}], P_1] \\
&= [\phi(P_2)A_{12} - A_{12}\phi(P_2), P_1] \\
&= -A_{12}\phi(P_2)P_1 - P_1\phi(P_2)A_{12} + A_{12}\phi(P_2).
\end{aligned}$$

Multiplying above equation once from left and right to P_1, once from left and right to P_2, and once from left to P_2 and from right to P_2, we conclude that

$$P_1\phi(A_{12})P_1 = P_2\phi(A_{12})P_2 = P_2\phi(A_{12})P_1 = 0.$$

Now it is deduced from the previous equations $\phi(A_{12}) = P_1\phi(A_{12})P_2$. Consequently, $\phi(\mathcal{R}_{12}) \subseteq \mathcal{R}_{12}$.

For any $A_{21} \in \mathcal{R}_{21}$, since $P_1(A_{21}) = 0$, we have

$$\begin{aligned}
\phi(A_{12}) &= \phi([[P_1, A_{21}], P_2]) \\
&= [[\phi(P_1), A_{21}], P_2] \\
&= [\phi(P_1)A_{21} - A_{21}\phi(P_1), P_2] \\
&= -A_{21}\phi(P_1)P_2 - P_2\phi(P_1)A_{21} + A_{21}\phi(P_1).
\end{aligned}$$

Similar to the previous case can be seen $\phi(A_{21}) \in \mathcal{R}_{21}$.

Claim 2. *$\phi(\mathcal{R}_{ii}) \subseteq \mathcal{R}_{11} + \mathcal{R}_{22}$, for $i \in \{1, 2\}$.*

For any $A_{11} \in \mathcal{R}_{11}$ and $B_{22} \in \mathcal{R}_{22}$, since $A_{11}P_2 = P_1B_{22} = 0$, we have

$$0 = \phi([[A_{11}, P_2], P_1]) = [[\phi(A_{11}), P_2], P_1]$$

and

$$0 = \phi([[B_{22}, P_1], P_2]) = [[\phi(B_{22}), P_1], P_2]$$

which implies that

$$P_2\phi(A_{11})P_1 + P_1\phi(A_{11})P_2 = 0 \tag{1}$$

and

$$P_1\phi(B_{22})P_2 + P_2\phi(B_{22})P_1 = 0. \tag{2}$$

Multiplying (1) once from left to P_1 and once from left to P_2, we get $P_1\phi(A_{11})$ $P_2 = 0$ and $P_2\phi(A_{11})P_1 = 0$. Therefore,

$$\phi(A_{11}) = P_1\phi(A_{11})P_1 + P_2\phi(A_{11})P_2.$$

It is obtained by(2) and using similar methods above

$$\phi(B_{22}) = P_1\phi(B_{22})P_1 + P_2\phi(B_{22})P_2.$$

Claim 3. *For $i \in \{1, 2\}$, there exists a map $h_i : \mathcal{R}_{ii} \to Z(\mathcal{R})$ such that $P_j\phi(A_{ii})P_j = h_i(A_{ii})P_j$ $(1 \leq i \neq j \leq 2)$, holds for any $A_{ii} \in \mathcal{R}_{ii}$.*

For any $A_{11} \in \mathcal{R}_{11}$, $B_{22} \in \mathcal{R}_{22}$, and $C_{ij} \in \mathcal{R}_{ij}$ $(1 \leq i \neq j \leq 2)$, since $A_{11}B_{22} = B_{22}A_{11} = 0$, we see

$$0 = \phi([[A_{11}, B_{22}], P_1]) = [[\phi(A_{11}), B_{22}], C_{21}]$$

and

$$0 = \phi([[B_{22}, A_{11}], P_2]) = [[\phi(B_{22}), A_{11}], C_{12}].$$

Considering the above equations, and using Claim 2, we arrive at

$$(P_2\phi(A_{11})P_2B_{22} - B_{22}P_2\phi(A_{11})P_2)C_{21} = 0$$

and

$$(P_1\phi(B_{22})P_1A_{11} - A_{11}P_1\phi(B_{22})P_1)C_{12} = 0.$$

Since R is prime, we conclude that $P_2\phi(A_{11})P_2 \in Z(\mathcal{R}_{22})$ and $P_1\phi(B_{22})P_1 \in Z(\mathcal{R}_{11})$. Thus $P_2\phi(A_{11})P_2AP_2 = P_2AP_2\phi(A_{11})P_2$ for any $A \in \mathcal{R}$ and $P_1\phi(B_{22})$ $P_1BP_1 = P_1BP_1\phi(B_{22})P_1$ for any $B \in \mathcal{R}$. Therefore Lemma 3.1, there exists unique elements $\lambda_1, \lambda_2 \in \mathcal{C}$, such that $P_2\phi(A_{11})P_2 = \lambda_1 P_2$ and $P_1\phi(B_{22})P_1 = \lambda_2 P_1$. Moreover, since $\mathcal{C}$ is feild, it is clear that $\lambda_1, \lambda_2 \in Z(\mathcal{R})$. We now define the maps $h_1 : \mathcal{R}_{11} \to Z(\mathcal{R})$ by $h_1(A_{11}) = \lambda_1$ and $h_2 : \mathcal{R}_{22} \to Z(\mathcal{R})$ by $h_2(B_{22}) = \lambda_2$. Given the uniqueness of λ_1 and λ_2, we know that the maps h_1 and h_2 are well-defined and additive. Also

$$P_2\phi(A_{11})P_2 = h_1(A_{11})P_2, \quad \text{and} \quad P_1\phi(B_{22})P_1 = h_2(B_{22})P_1.$$

Now, for any $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{R}$, we define linear maps $h : \mathcal{R} \to Z(\mathcal{R})$ and $\psi : \mathcal{R} \to \mathcal{R}$ by

$$h(A) = h_1(A_{11}) + h_2(A_{22}), \quad \text{and} \quad \psi(A) = \phi(A) - h(A).$$

By Claims 1–3, it is clear that $\psi(\mathcal{R}_{ij}) \subseteq \mathcal{R}_{ij}$, $\psi(\mathcal{R}_{ii}) \subseteq \mathcal{R}_{ii}$ and $\psi(\mathcal{R}_{ij}) = \phi(\mathcal{R}_{ij})$, $1 \leq i \neq j \leq 2$.

Claim 4. *ψ is an additive centralizer.*

We divide the proof into the following four Steps.

Step 1. $\psi(A_{ii}B_{ij}) = \psi(A_{ii})B_{ij} = A_{ii}\psi(B_{ij})$ for all $A_{ii} \in \mathcal{R}_{ii}$ and $B_{ij} \in \mathcal{R}_{ij}$, $1 \leq i \neq j \leq 2$.

In fact, for any $A_{ii} \in \mathcal{R}_{ii}$ and $B_{ij} \in \mathcal{R}_{ij}$, since $B_{ij}A_{ii} = 0$, we have

$$\begin{aligned}
\psi(A_{ii}B_{ij}) &= \phi(A_{ii}B_{ij}) \\
&= \phi([[B_{ij}, A_{ii}], P_i]) \\
&= [[\phi(B_{12}), A_{11}], P_i] \\
&= A_{ii}\phi(B_{ij}) \\
&= A_{ii}\psi(B_{ij})
\end{aligned}$$

and

$$\begin{aligned}\psi(A_{ii}B_{ij}) &= \phi(A_{ii}B_{ij})\\ &= \phi([[B_{ij}, A_{ii}], P_i])\\ &= [[B_{ij}, \phi(A_{ii})], P_i]\\ &= \phi(A_{ii})B_{ij}\\ &= \psi(A_{ii})B_{ij}.\end{aligned}$$

Hence, we obtain

$$\psi(A_{ii}B_{ij}) = A_{ii}\psi(B_{ij}) = \psi(A_{ii})B_{ij}. \tag{3}$$

Step 2. $\psi(A_{ij}B_{jj}) = \psi(A_{ij})B_{jj} = A_{ij}\psi(B_{jj})$ for all $A_{ij} \in \mathcal{R}_{ij}$ and $B_{jj} \in \mathcal{R}_{jj}$, $1 \leq i \neq j \leq 2$.

For any$A_{ij} \in \mathcal{R}_{ij}$ and $B_{jj} \in \mathcal{R}_{jj}$, since $B_{jj}A_{ij} = 0$, and with the similar argument Step 1, one can easily check that Step 2 is hold.

Step 3. $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii} = A_{ii}\psi(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathcal{R}_{ii}$, $i = 1, 2$.

For any $A_{ii}, B_{ii} \in \mathcal{R}_{ii}$ and any $S_{ij} \in \mathcal{R}_{ij}$, by Step 1, we have

$$\psi(A_{ii}B_{ii}S_{ij}) = \psi(A_{ii}B_{ii})S_{ij},$$

on other hands

$$\psi(A_{ii}B_{ii}S_{ij}) = A_{ii}\psi(B_{ii}S_{ij}) = A_{ii}\psi(B_{ii})S_{ij}.$$

It can be seen from the combination of the above two equations that $\psi(A_{ii}B_{ii})S_{ij} = A_{ii}\psi(B_{ii})S_{ij}$ holds for all $S_{ij} \in \mathcal{R}_{ij}$. It follows that $\psi(A_{ii}B_{ii}) = A_{ii}\psi(B_{ii})$ since $\mathcal{R}$ is prime. Also for any $A_{ii}, B_{ii} \in \mathcal{R}_{ii}$ and any $S_{ji} \in \mathcal{R}_{ji}$, by Step 2, we get

$$\psi(S_{ji}A_{ii}B_{ii}) = S_{ji}\psi(A_{ii}B_{ii}),$$

on other hands

$$\psi(S_{ji}A_{ii}B_{ii}) = \psi(S_{ji}A_{ii})B_{ii} = S_{ji}\psi(A_{ii})B_{ii},$$

Comparing the above two equations and since $\mathcal{R}$ is prime, we see that $\psi(A_{ii}B_{ii}) = \psi(A_{ii})B_{ii}$.

Step 4. $\psi(A_{ij}B_{ji}) = \psi(A_{ij})B_{ji} = A_{ij}\psi(B_{ji})$ for all $A_{ij} \in \mathcal{R}_{ij}$ and $B_{ji} \in \mathcal{R}_{ji}$, $1 \leq i \neq j \leq 2$.

Let $A_{ij} \in \mathcal{R}_{ij}$ and $B_{ji} \in \mathcal{R}_{ji}$, $1 \leq i \neq j \leq 2$. It follows from Steps 1, 2 and, 3 that

$$\psi(A_{ij}B_{ji}) = \psi(P_iA_{ij}B_{ji}) = \psi(P_i)A_{ij}B_{ji} = \psi(A_{ij})B_{ji},$$

and

$$\psi(A_{ij}B_{ji}) = \psi(A_{ij}B_{ji}P_i) = A_{ij}B_{ji}\psi(P_i) = A_{ij}\psi(B_{ji}).$$

In Steps 1–4, it is easy to check that ψ is an additive centralizer. In other words, the claim of Claim 4 is obtained.

Claim 5. *$h([[A, B], C]) = 0$ for all $A, B, C \in \mathcal{R}$ with $AB = 0$.*

In fact, for any $A, B, C \in \mathcal{R}$ with $AB = 0$, we have

$$\begin{aligned} h([[A, B], C]) &= \phi([[A, B], C]) - \psi([[A, B], C]) \\ &= [[\phi(A), B], C] - \psi([[A, B], C]) \\ &= [[\psi(A) + h(A), B], C] - \psi([[A, B], C]) \\ &= [[\psi(A), B], C] - \psi([[A, B], C]) \\ &= 0. \end{aligned}$$

Claim 6. *The theorem holds.*

Indeed, By Claims 1–6, $\phi(A) = \psi(A) + h(A)$ for any $A \in \mathcal{R}$. Since ψ is a centralizer on $\mathcal{R}$, for all $A \in \mathcal{R}$ we have

$$\psi(A) = \psi(AI) = A\psi(I), \quad \psi(A) = \psi(IA) = \psi(I)A.$$

Hence, $\psi(I) \in Z(\mathcal{R})$. Set $\lambda = \psi(I)$. So λ in center $\mathcal{R}$ and $\psi(A) = \lambda A$ for any $A \in \mathcal{R}$. Therefore,we show that $\phi(A) = \lambda A + h(A)$ for any $A \in \mathcal{R}$, where $\lambda \in Z(\mathcal{R})$ and h vanishes at second commutators $[[A, B], C]$ for all $A, B, C \in \mathcal{R}$ with $AB = 0$. Here the proof of one side of the theorem is complete.

The converse proof is trivial.

4. AN APPLICATIONS: CHARACTERIZATION OF GENERALIZED LIE DERIVATIONS ON PRIM RINGS

In this section, as an application of the 2.1 theorem, we determine the Lie triple derivations on prim rings by acting on zero products. To present the main result of this section, we need the following theorem, which was proved in [31].

To the main result of this section, we need the following theorem, which is proved in [31].

Theorem 4.1. *Let $\mathcal{R}$ be an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P and $P\mathcal{R}P$, $(1 - P)\mathcal{R}(1 - P)$ are noncommutative. Suppose δ on $\mathcal{R}$ be a map, then δ is a Lie triple derivation if only if there exists an additive derivation $d : \mathcal{R} \to \mathcal{R}$ and a map $h : \mathcal{R} \to Z(\mathcal{R})$ satisfying $h([[A, B], C]) = 0$ for all $A, B, C \in \mathcal{R}$ such that $\delta(A) = d(A) + h(A)$ for all $A \in \mathcal{R}$.*

The following theorem, which is a result of Theorem 2.2 and Theorem 4.1, actually generalizes Theorem 4.1.

Theorem 4.2. *Let $\mathcal{R}$ be an unital prime ring with characteristic not 2 and containing a nontrivial idempotent P and $P\mathcal{R}P$, $(1-P)\mathcal{R}(1-P)$ are noncommutative. Then the following statements are equivalent.*

(i) *$\Delta : \mathcal{R} \to \mathcal{R}$ be a generalized Lie triple derivation associated with the Lie triple derivation $\delta : \mathcal{R} \to \mathcal{R}$.*

(ii) *There exist derivation $d : \mathcal{R} \to \mathcal{R}$, additive maps $h, h_1 : \mathcal{R} \to Z(\mathcal{R})$ and an element λ in center $\mathcal{R}$ such that*

$$\Delta(A) = d(A) + h(A) + \lambda A, \quad \delta(A) = d(A) + h_1(A), \qquad (A \in \mathcal{R})$$

where $h([[A,B],C]) = h_1([[A,B],C]) = 0$ for all $A, B, C \in \mathcal{R}$.

Proof. Since (ii)⇒(i) is clear, it suffices to prove (i)⇒(ii). Therefore, by Theorem 4.1, there exist derivation $d : \mathcal{R} \to \mathcal{R}$, additive maps $h_1 : \mathcal{R} \to Z(\mathcal{R})$ such that $\delta = d + h_1$ and $h_1([[A,B],C]) = 0$ for all $A, B, C \in \mathcal{R}$. By assumption, for the additive map $\phi = \Delta - \delta$ on $\mathcal{R}$, we have

$$\phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C], \qquad (A, B, C \in \mathcal{R}).$$

Thus, by Theorem 2.2, there exist λ in center $\mathcal{R}$ and addtive map h_2 on $\mathcal{R}$ such that $\phi = \lambda I + h_2$ where $h_2(A) \in Z(\mathcal{R})$ for all $A \in \mathcal{R}$ and $h_2([[A,B],C]) = 0$ for all $A, B, C \in \mathcal{R}$. Suppose that $h = h_1 + h_2$. Thus, $h : \mathcal{R} \to Z(\mathcal{R})$ is a addtive map that $h([[A,B],C]) = 0$ for all $A, B, C \in \mathcal{R}$. Thus, we have

$$\Delta(A) = \delta(A) + \phi(A) = d(A) + h_1(A) + \lambda A + h_2(A) = d(A) + h(A) + \lambda A$$

for all $A \in \mathcal{R}$. This completes the proof. ■

According to the explanations of the previous section and the above Theorem, we have the following results.

Corollary 4.3. *Suppose that $\Delta : \mathcal{U} \to \mathcal{U}$ and $\delta : \mathcal{U} \to \mathcal{U}$ be additive maps. Let $\mathcal{U}$ be any of the following algebras.*

(a) *Standard operator algebra on a complex Banach space X.*

(b) *Factor von Neumann algebra.*

Δ is a generalized Lie triple derivation associated with the Lie triple derivation δ if and only if there exist the additive maps $d : \mathcal{U} \to \mathcal{U}$, $h, h_1 : \mathcal{U} \to Z(\mathcal{U})$ and an element $\lambda \in Z(\mathcal{U})$ such that

$$\Delta(A) = d(A) + h(A) + \lambda A, \quad \delta(A) = d(A) + h_1(A), \qquad (A \in \mathcal{U})$$

where d is a derivation and $h([[A,B],C]) = h_1([[A,B],C]) = 0$ for all $A, B, C \in \mathcal{U}$.

To the next corollary, we need the following theorem, which is proved in [23].

Theorem 4.4. *Let $\mathcal{M}$ be a factor von Neumann algebra with dimension greater than 1 acting on a Hilbert space and a linear map $\delta : \mathcal{M} \to \mathcal{M}$ satisfying*

$$\delta([[A,B,C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] = [[A,B],\Delta(C)],$$

for all $A,B,C \in \mathcal{M}$ with $AB = 0$. Then there exist an operator $M \in \mathcal{M}$ and a linear map $h : \mathcal{M} \to \mathbb{C}I$ vanishing at every second commutator $[[A,B],C]$ when $AB = 0$ such that

$$\delta(A) = AM - MA + h(A),$$

for any $A \in \mathcal{M}$.

The following results are a generalization of Theorem 4.4.

Corollary 4.5. *Let $\mathcal{M}$ be a factor von Neumann algebra with dimension greater than 1 acting on a Hilbert space. Suppose that $\Delta : \mathcal{M} \to \mathcal{M}$ and $\delta : \mathcal{M} \to \mathcal{M}$ be additive maps. Then the following statements are equivalent.*

(i) *Δ and δ satisfy the following conditions.*

$$\Delta([[A,B],C]) = [[\Delta(A),B],C] + [[A,\delta(B)],C] = [[A,B],\Delta(C)];$$

$$\delta([[A,B],C]) = [[\delta(A),B],C] + [[A,\delta(B)],C] = [[A,B],\Delta(C)];$$

for all $A,B,C \in \mathcal{M}$ with $AB = 0$.

(ii) *There exist additive maps $d : \mathcal{M} \to \mathcal{M}$, $h, h_1 : \mathcal{M} \to \mathbb{C}I$ and are elements $M, T \in \mathbb{C}I$ such that*

$$\Delta(A) = AM - TA + h(A), \quad \delta(A) = d(A) + h_1(A), \quad (A \in \mathcal{M})$$

where d is a derivation and $h([[A,B],C]) = h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{M}$ with $AB = 0$.

Proof. Since (ii)⇒(i) is clear, it suffices to prove (i)⇒(ii). Therefore, by Theorem 4.4, there exist an operator $M \in \mathcal{M}$ and a linear map $h_1 : \mathcal{M} \to \mathbb{C}I$ such that $\delta(A) = AM - MA + h_1(A)$, and $h_1(A) \in \mathbb{C}I$ for all $A \in \mathcal{R}$ and $h_1([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{R}$ with $AB = 0$. By assumption, for the additive map $\phi = \Delta - \delta$ on $\mathcal{R}$, we have

$$\phi([[A,B],C]) = [[\phi(A),B],C] = [[A,\phi(B)],C], \qquad (A,B,C \in \mathcal{R}).$$

By Theorem 2.1, there exist $R \in \mathbb{C}I$ and addtive map h_2 on $\mathcal{R}$ such that $\phi = \lambda I + h_2$ where $h_2(A) \in \mathcal{C}I$ for all $A \in \mathcal{R}$ and $h_2([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{R}$ with $AB = 0$. Suppose that $h = h_1 + h_2$. Thus, $h : \mathcal{R} \to \mathcal{C}I$ is a addtive map that $h([[A,B],C]) = 0$ for all $A,B,C \in \mathcal{R}$. Set $T = M + R$. Thus, we have

$$\Delta(A) = \delta(A) + \phi(A) = AM - MA + h_1(A) + RA + h_2(A) = AM - TA + h(A)$$

for all $A \in \mathcal{R}$. This completes the proof. ∎

REFERENCES

[1] J. Alaminos, J. Extremera, A.R. Villena and M. Brešar, *Characterizing homomorphisms and derivations on C^*-algebras*, in: Proceedings of Royal Society Edinburgh Section A: Mathematics **137(1)** (2007) 1–7.
https://doi.org/10.1017/S0308210505000090

[2] R. Behfar and H. Ghahramani, *Lie maps on triangular algebras without assuming unity*, Mediterranean J. Math. **18(5)** (2021) 1–28.
https://doi.org/10.1007/s00009-021-01836-z

[3] M. Brešar, *Characterizing homomorphisms, derivations and multipliers in rings with idempotents*, in: Proceedings of the Royal Society of Edinburgh Section A: Mathematics **137(1)** (2007) 9–21.
https://doi.org/10.1017/S0308210504001088

[4] M. Brešar, M.A. Chebotar and W.S. Martindale III, Functional Identities (Basel, Birkhauser, 2006).
https://doi.org/10.1007/978-3-7643-7796-0

[5] D. Benkovič, *Lie triple derivations of unital algebras with idempotents*, Linear and Multilinear Algebra **63(1)** (2015) 141–165.
https://doi.org/10.1080/03081087.2013.851200

[6] D. Benkovič, *Generalized Lie derivations of unital algebras with idempotents*, Operators and Matrices, **12(2)** (2018) 357–367.
https://doi.org/10.7153/oam-2018-12-23

[7] B. Fadaee and H. Ghahramani, *Additive mappings on von Neumann algebras acting as Lie triple centralizer via local actions and related mappings*, Acta Sci. Math. (Szeged) (2024).
https://doi.org/10.1007/s44146-024-00123-z

[8] B. Fadaee and H. Ghahramani, *Lie centralizers at the zero products on generalized matrix algebras*, J. Algebra and its Appl. (2021).
https://doi.org/10.1142/S0219498822501651

[9] B. Fadaee and H. Ghahramani, *On Lie triple centralizers of von Neumann algebras*, 25 July 2023, Preprint (Version 1) available at Research Square.
https://doi.org/10.21203/rs.3.rs-3189873/v1

[10] B. Fadaee and H. Ghahramani, *Jordan left derivations at the idempotent elements on reflexive algebras*, **92(3–4)** (2018) 261–275.
https://doi.org/10.5486/PMD.2018.7785

[11] B. Fadaee and H. Ghahramani, *Linear maps on C^*-algebras behaving like (anti-) derivations at orthogonal elements*, Bull. Malaysian Math. Sci. Soc. **43(3)** (2020) 2851–2859.
https://doi.org/10.1007/s40840-019-00841-6

[12] B. Fadaee, H. Ghahramani and W. Jing, *Lie triple centralizers on generalized matrix algebras*, Quaestiones Math. (2022).
https://doi.org/https://doi.org/10.21203/rs.3.rs-3189873/v1

[13] A. Fošner and W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory, Advances in Operator Theory **4(2)** (2019) 342–350. https://doi.org/10.15352/aot.1804-1341

[14] A. Fošner, F. Wei and Z.-K. Xiao, *Nonlinear Lie-type derivations of von Neumann algebras and related topics*, Colloquium Math. **1(132)** (2013) 53–71. https://doi.org/10.4064/cm132-1-5

[15] H. Ghahramani and W. Jing, *Lie centralizers at zero products on a class of operator algebras*, Ann. Funct. Anal. **12(2)** (2021) 1–12. https://doi.org/10.1007/s43034-021-00123-y

[16] P. Ghimire, *Linear Lie centralizers of the algebra of dominant block upper triangular matrices*, Linear and Multilinear Algebra **70** (2022) 5040–5051. https://doi.org/10.1080/03081087.2021.1902465

[17] Sh. Goodarzi and A. Khotanloo, *Lie Centralizers and generalized Lie derivations on prime rings by local actions*, Commun. Algebra (2023). https://doi.org/10.1080/00927872.2023.2228429

[18] A. Jabeen, *Lie(Jordan) centralizers on generalized matrix algebras*, Commun. Algebra **49(1)** (2021) 278–291. https://doi.org/10.1080/00927872.2020.1797759

[19] N. Jacobson, Lie Algebras (Interscience Publishers, 1962).

[20] P. Ji, W. Qi and X. Sun, *Characterizations of Lie derivations of factor von Neumann algebras*, Linear and Multilinear Algebra, **61(3)** (2013) 417–428. https://doi.org/10.1080/03081087.2012.689982

[21] W. Jing, S.J. Lu and P.T. Li, *Characterisations of derivations on some operator algebras*, Bull. Aust. Math. Soc. **66(2)** (2002) 227–232. https://doi.org/https://doi.org/10.1017/S0004972700040077

[22] L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear and Multilinear Algebra, **70(14)** (2022) 2693–705. https://doi.org/10.1080/03081087.2020.1810605

[23] L. Liu, *Lie Triple Derivations on Factor Von Nemman algebras*, Bull. Korean Math. Soc. **52(2)** (2015) 581–591. https://doi.org/10.4134/BKMS.2015.52.2.581

[24] L. Liu, *Lie Triple Derivations on von Neumann Algebras*, Chinese Ann. Math. Ser. B **39(5)** (2018) 817–828. https://doi.org/10.1007/s11401-018-0098-0

[25] L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear and Multilinear Algebra, (2020). https://doi.org/10.1080/03081087.2020.1810605

[26] L. Liu and K. Gao, *Characterizations of Lie centralizers of triangular algebras*, Linear and Multilinear Algebra, (2022). https://doi.org/10.1080/03081087.2022.2104788

[27] F. Lu and W. Jing, *Characterizations of Lie derivations of B(X)*, Linear Algebra and Its Appl. **432(1)** (2010) 89–99.
https://doi.org/10.1016/j.laa.2009.07.026

[28] K. McCrimmon, A Taste of Jordan Algebras (Springer-Verlag, 2004).

[29] C.R. Miers, *Lie triple derivations of von Neumann algebras*, Proc. Amer. Math. Soc. **71(1)** (1978) 57–61.
https://doi.org/10.1090/S0002-9939-1978-0487480-9

[30] X. Qi and J. Hou, *Characterization of Lie derivations on prime rings*, Commun. Algebra **39(10)** (2011) 3824–3835.
https://doi.org/10.1080/00927872.2010.512588

[31] X. Qi, *Characterization of multiplicative Lie triple derivations on rings*, Abstract and Appl. Anal. **2014** (2014).
https://doi.org/10.1155/2014/739730

[32] J. Zhu and S. Zhao, *Characterizations of all-derivable points in nest algebras*, Proc. Amer. Math. Soc. **141(7)** (2013) 2343–2350.
https://doi.org/10.1090/S0002-9939-2013-11511-X

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