Discussiones Mathematicae
General Algebra and Applications 45 (2025) 57–70
https://doi.org/10.7151/dmgaa.1471

STRUCTURES OF HALL SUBGROUPS OF FINITE METACYCLIC AND NILPOTENT GROUPS

Shrawani Mitkari[1,2] and Vilas Kharat

Department of Mathematics
S.P. Pune University, Pune 411007 India

e-mail: shrawaniin@unipune.ac.in
vilaskharat@unipune.ac.in

Abstract

In this paper, the structures of Hall subgroups of finite metacyclic and nilpotent groups are studied. It is proved that the collection of all Hall subgroups of a metacyclic group is a lattice and a group G is nilpotent if and only if its collection of Hall subgroups forms a distributive lattice. Also, lower semimodularity and complementation are studied in a collection of Hall subgroups of D_n for different values of n.

Keywords: group, Hall subgroup, lattice of subgroups, lower semimodular lattice, metacyclic group, nilpotent group.

2020 Mathematics Subject Classification: 20D25, 20D30, 20D40, 20E15, 20F22, 20K27, 06A06, 06A07, 06B20, 06B23.

1. Introduction and notation

Throughout this article, G denotes a finite group. It is known that the set of all subgroups of a given finite group G forms a lattice denoted by $L(G)$ with $H \wedge K = H \cap K$ and $H \vee K = \langle H, K\rangle$ for subgroups H, K of G. The interrelations between the theory of lattices and the theory of groups have been studied by many researchers, see Pálfy [10], Schmidt [12], Suzuki [14]. For the group theoretic concepts and notations, we refer to Birkhoff [1], Luthar and Passi [8], Schmidt [12].

[1]This work is supported by CSIR, under SRF (09/137(0621)/2019-EMR-I), Ministry of Sci. and Tech., Govt. of India.

[2]Corresponding author.

There are a few types of subgroups such as Hall subgroups whose collections may form lattices and these lattices can be used to study the properties of groups. Accordingly, a study for collection of Hall subgroups of metacyclic and nilpotent groups has been carried out.

The following notations are used throughout this article.

- $LH(G)$ – Collection of all Hall subgroups of G.
- $LN(G)$ – Collection of all normal subgroups of G, which is a sublattice of $L(G)$.
- $|G|$ – Order of G.
- $|L(G)|$ – Number of subgroups of G - Cardinality of $L(G)$.
- e – Neutral (Identity) element in G.
- $[m, r]$ – lcm of m and r.
- (m, r) – gcd of m and r.
- $\wedge_{LH}$ – g.l.b. in $LH(G)$.
- $\vee_{LH}$ – l.u.b. in $LH(G)$.
- $H \prec K$ – H is covered by K.
- D_n – Dihedral group of order $2n$: $\langle a,\ b \mid a^n = e = b^2,\ ba = a^{-1}b\rangle$.

The following definition of a Hall subgroup of a finite group is essentially due to Hall [6].

Definition 1.1 [6]**.** A *Hall subgroup* of a finite group is a subgroup whose order is coprime to its index.

Remark 1.2. Every Sylow p-subgroup of a finite group is a Hall subgroup.

The collection of Hall subgroups of a group is not necessarily a lattice, i.e., we have a group G in which $LH(G)$ does not form a lattice.

Consider $L(A_7)$ and its collection $LH(A_7)$ of all Hall subgroups of A_7. Note that, the subgroups $H = \langle(1\,2\,3)\ (2\,3\,4\,5\,6)\rangle$ and $K = \langle(1\,2\,3),\ (2\,3\,4\,5\,7)\rangle$ are isomorphic to A_6 and so Hall subgroups of A_7. Moreover, $H \wedge K = \langle(1\,2\,3),\ (2\,3\,4\,5)\rangle$ is isomorphic to A_5. Note that, $\left(|H \wedge K|, \frac{|G|}{|H \wedge K|}\right) = (120, 42) = 6$ and so $H \wedge K$ is not a Hall subgroup.

Also, the subgroups $T = \langle(2\ 3\ 4\ 5\ 6)\rangle$ and $S = \langle(2\ 4\ 3\ 5\ 6)\rangle$ are Sylow 5-subgroups of A_7. Note that, $T \vee S = H \wedge K$ which is not a Hall subgroup of A_7. Consequently, join of $T \vee_{LH} S$ as well as meet of $H \wedge_{LH} K$ does not exists and therefore $LH(A_7)$ is not a lattice.

Next consider, the lattice depicted in Fig 1.1 which is the Hasse diagram of $L(S_4)$. Note that, $LH(S_n) = L(S_n)$ for $n \leq 3$. The Hasse diagram of $LH(S_4)$

is depicted in Figure 1.2, and it is a lattice. Observe that for P_{28} and P_{27} in $LH(S_n)$, we have $P_{28} \wedge P_{27} = M_{18}$ in $L(S_4)$, but $M_{18} \notin LH(S_4)$ and as such, $LH(S_4)$ is not a sublattice of $L(S_4)$.

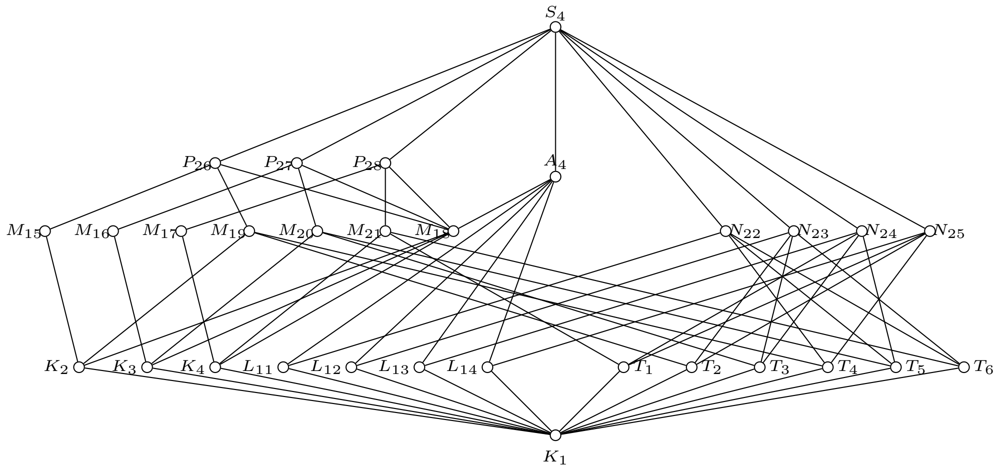

Figure 1.1. $L(S_4)$.

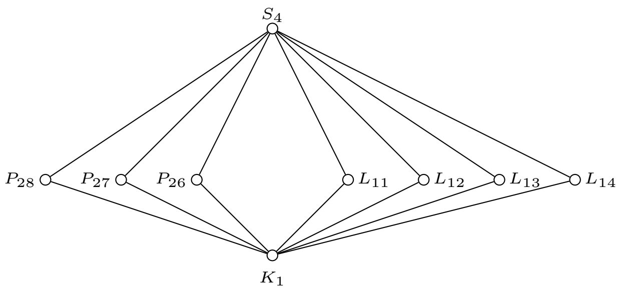

Figure 1.2. $LH(S_4)$.

So it is necessary to investigate the groups for which $LH(G)$ is a lattice and similarly, $LH(G)$ is a sublattice of $L(G)$. It is also worth studying some properties of $LH(G)$ in these situations.

Faigle, *et al.* (see [4, 11, 13]) studied strong lattices of finite length in which the join-irreducible elements play a key role.

For the following definition and other relevant definitions in lattice theory we refer to Birkhoff [1], Grätzer [5] and Stern [13].

Definition 1.3 [13]**.** An element j of a lattice L is called *join-irreducible* if, for all $x, y \in L$, $j = x \vee y$ implies $j = x$ or $j = y$.

For a lattice L of finite length $J(L)$ denotes the set of all non-zero join-irreducible elements.

We introduce the concept of join-irreducible subgroups as follows.

Definition 1.4. A subgroup of a group G is said to be *join-irreducible* if it is a join-irreducible element of $L(G)$.

We note that every cyclic subgroup of prime power order of a finite group is a join-irreducible subgroup.

From this fact and Lemma 2 of [15], the following Lemma follows.

Lemma 1.5. *A subgroup of a finite group is a join-irreducible subgroup if and only if it is a cyclic subgroup of prime power order.*

The following concept of a strong element was coined by Faigle [4]; see also [13].

Definition 1.6 [4]**.** Let L be a lattice of finite length. A join-irreducible element $j \neq 0$ is called a *strong element* if the following condition holds for all $x \in L$:
(St) $j \leq x \vee j^- \implies j \leq x$, where j^- denotes the uniquely determined lower cover of j.

A lattice is said to be *strong* if every join-irreducible element of it is strong.

Remark 1.7. The condition (St) in the definition of a strong element is equivalent to the following; see [13] for more details.

(St') For every $q < j \in J(L)$, $x \in L$, $j \leq x \vee q$ implies $j \leq x$.

The following characterization of strong lattices is due to Richter and Stern [11].

Theorem 1.8 [11]**.** *A lattice L of finite length is strong if and only if it does not contain a special pentagon sublattice with $j \in J(L)$.*

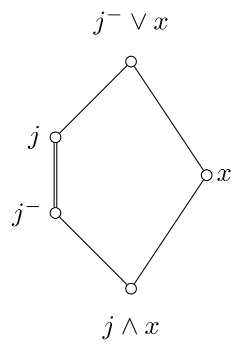

Figure 1.3. Special Pentagon.

Proof of the following Lemma follows from Theorem 1.8.

Lemma 1.9. *Let L be a finite lattice. If atoms are the only join-irreducible elements in L, then L is strong.*

Theorem 1.10. *Let G be a group, if $LH(G)$ is a lattice, then $LH(G)$ is strong.*

Proof. In view of the Lemma 1.9, it is sufficient to prove that only atoms are join-irreducible elements. Let $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$ and $J \in LH(G)$ a join-irreducible Hall subgroup. Then $|J| = p_t^{\alpha_t}$ for some prime $t \in \{1, 2, \ldots, m\}$ and $|J^-| = p_t^{\alpha_t - 1} \in L(G)$. Note that, $|J^-| = \{e\}$ in $LH(G)$. Consequently, if a subgroup J is join-irreducible in $LH(G)$ then it is an atom. ■

Note that, there exists a strong lattice which is not a Hall subgroup lattice of any finite group. Figure 1.4 depicts a strong lattice, which is not a $LH(G)$ for any finite group G.

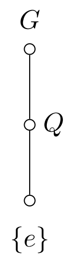

Figure 1.4. C_3

2. Hall subgroups in finite metacyclic groups

In this section, the collection of Hall subgroups of metacyclic group is investigated.

Following is the definition of a metacyclic group, see [2].

Definition 2.1 [2]**.** A finite group G is a *metacyclic* if it contains a cyclic normal subgroup N such that $\frac{G}{N}$ is also cyclic.

It is observed that a metacyclic group can be written $G = SN$ with $S \leq G$ and $N \trianglelefteq G$ such that both S and N are cyclic. Such a product is a metacyclic factorization of G.

Note that, Hall subgroups of a metacyclic group G are obtained with the help of its metacyclic factorization. and so we have the following result which is a Lemma 5.3 of [7].

Lemma 2.2 [7]**.** *Let G be a finite group with a metacyclic factorization $G = SN$, to each set π of primes, the subgroup $H = S_\pi N_\pi$ is the unique Hall π-subgroup of G such that $S_\pi = H \cap S$, $N_\pi = N \cap H$ and so $H = (H \cap S)(H \cap N)$.*

As observed, the collection of Hall subgroups of a finite group need not form a lattice in general but in case of metacyclic group it forms a lattice as the following result shows.

Theorem 2.3. *If G is a finite metacyclic group, then $LH(G)$ is a lattice. However, it is not necessarily a sublattice of $L(G)$.*

Proof. Let G be a finite metacyclic group, in order to show that $LH(G)$ is a lattice, we prove that given two Hall subgroups H and K of G, $H \wedge_{LH} K$ and $H \vee_{LH} K$ exist.

Case I. Let H and K be two distinct Hall π_1 and π_2-subgroups respectively corresponding to metacyclic factorization SN of G.

In view of Lemma 2.2, the subgroups $H = S_{\pi_1} N_{\pi_1}$ and $K = S_{\pi_2} N_{\pi_2}$ are the unique Hall π_1 and π_2-subgroups of G such that $S_{\pi_1} = H \cap S$, $N_{\pi_1} = H \cap N$, $S_{\pi_2} = K \cap S$, $N_{\pi_2} = K \cap N$. Therefore, $H = (H \cap S)(H \cap N)$ and $K = (K \cap S)(K \cap N)$. Now, for the set $\pi = \pi_1 \cap \pi_2$ of primes, there is the unique Hall π-subgroup say $T = S_\pi N_\pi = (T \cap S)(T \cap N)$. Note that, T is the unique largest Hall subgroup of G which is contained in both H and K. Consequently, $H \wedge_{LH} K = T$. Similarly, for the set $\pi' = \pi_1 \cup \pi_2$ of primes there is the unique Hall π'-subgroup say $R = S_{\pi'} N_{\pi'} = (R \cap S)(R \cap N)$. Note that, R is the unique smallest Hall subgroup of G which contains both H and K. Therefore, $H \vee_{LH} K = R$.

Case II. Let H and K be two distinct Hall π_1 and π_2-subgroups respectively corresponding to two different metacyclic factorizations SN and $S'N'$.

In view of the Lemma 2.2, $H = (H \cap S)(H \cap N) = S_{\pi_1} N_{\pi_1}$ and $K = (K \cap S')(K \cap N') = S'_{\pi_2} N'_{\pi_2}$. Furthermore, each one of H and K is an unique Hall π_1 and π_2-subgroups corresponding to two metacyclic factorizations SN and $S'N'$ respectively. Now, corresponding to each prime $p_i \in \pi_1$ there is the unique Sylow p_i-subgroup say P_i, corresponding to factorization SN of G and similarly, corresponding to each prime $p_j \in \pi_2$ there is the unique Sylow p_j-subgroup say Q_j, corresponding to factorization $S'N'$ of G.

Note that, the subgroup $H' = S_{\pi_1 \cap \pi_2} N_{\pi_1 \cap \pi_2}$ then H' is a subgroup of H. If H' is also a subgroup of K then H' is the largest Hall subgroup of G which is contained in both H and K. Consequently, $H \wedge_{LH} K = H'$. If H' is not a subgroup of K, then choose the set π of primes of $p_i \in \pi_1 \cap \pi_2$ such that each Sylow p_i-subgroup P_i of G contained in both H and K. Note that, if P is a Hall π-subgroup of H then $P \supseteq \vee P_i$. Since every non-trivial Hall subgroup is join of Sylow subgroups we have $P = \vee P_i$. And so, it is contained in both H and K.

As such P is the unique largest Hall subgroup of G corresponding to metacyclic factorization SN as well as $S'N'$ and so, $H \wedge_{LH} K = \vee_{p_i \in \pi} P_i$.

Similarly, choose the subgroup $H' = S_{\pi_1 \cup \pi_2} N_{\pi_1 \cup \pi_2}$ then H is the subgroup of H'. If K is also a subgroup of H' then H' is the smallest Hall subgroup of G which contains both H and K and therefore $H \vee_{LH} K = H'$. If K is not a subgroup of H', choose the least set π' of primes π' with $\pi_1 \cup \pi_2 \subseteq \pi'$ such that $H,\ K \subseteq \vee_{p_i \in \pi'} P_i$. Let R be a Hall subgroup of G such that $\vee_{p_i \in \pi'} P_i \subseteq R$ is the unique Hall π'-subgroup corresponding to metacyclic factorizations SN as well as $S'N'$. Note that, R is the least Hall subgroup which contains H and K and so, $H \vee_{LH} K = R$.

Hence $LH(G)$ is a lattice whenever G is metacyclic.

Consider a dihedral group D_n, which is metacyclic group. In [9] it is noted that $LH(D_n)$ is a lattice but not necessarily a sublattice of $L(D_n)$. ■

Remark 2.4. Note that, a metacyclic group G may not have a unique metacyclic factorization, e.g., D_n. However, if G has unique meatcyclic factorization then $LH(G)$ is a sublattice of $L(G)$, e.g. $\mathbb{Z}_{pq}$. Also, for every finite group G whose order is square-free, $LH(G)$ is a sublattice of $L(G)$.

We note that,dihedral groups are metacyclic and so $LH(D_n)$ is a lattice. However, $LH(D_n)$ is a lattice is proved independently in [9] using the classification of the subgroups given in [3] as follows;

Theorem 2.5 [3]**.** *Every subgroup of D_n is cyclic or dihedral. A complete listing of the subgroups is as follows:*

(1) $\langle a^d \rangle$, *where $d|n$, with index $2d$,*

(2) $\langle a^k, a^i b \rangle$, *where $k|n$ and $0 \leq i \leq k-1$, with index k.*

Every subgroup of D_n occurs exactly once in this listing.

Remark 2.6. 1. A subgroup of D_n is said to be of *Type* (1) if it is cyclic subgroup as stated in (1) of Theorem 2.5.

2. A subgroup of D_n is said to be of *Type* (2) if it is dihedral subgroup as stated in (2) of Theorem 2.5.

A study of collection of Hall subgroups of D_n namely $LH(D_n)$ is carried out by Mitkari et. al. in [9], where the binary operations $\wedge_{LH}$ and $\vee_{LH}$ in $LH(D_n)$ are defined as per the classification of subgroups of D_n as follows.

Let $n = 2^{\alpha} \prod_{i=1}^{m} p_i^{\alpha_i}$.

1. If $T = \langle a^t \rangle$ for some $s,\ t \in \mathbb{N}$ and $S = \langle a^s \rangle$ are Hall subgroups of Type (1), then $T \vee_{LH} S = \langle a^g \rangle$ where $g = (s,t)$ and $T \wedge_{LH} S = \langle a^l \rangle$, where $l = [s,t]$.

2. If $T = \langle a^t \rangle$ is a Hall subgroup of Type (1) and $S = \langle a^s, a^i b \rangle$ is a Hall subgroups of Type (2) for some $s,\ t \in \mathbb{N}$, then $T \vee_{LH} S = \langle a^g, a^i b \rangle$ where $g = (s,t)$ and $T \wedge_{LH} S = \langle a^l \rangle$, where $l = [s,t]$.

3. If $T = \langle a^t, a^i b\rangle$ and $S = \langle a^s, a^j b\rangle$ are Hall subgroups of Type (2) for some $s,\ t \in \mathbb{N}$, then $T \vee_{LH} S = \langle a^g, a^i b\rangle$ where $g = \frac{g_1}{r}$ and $g_1 = (t, s, i-j),\ r = \left(\frac{2n}{g_1}, g_1\right)$ and

$$T \wedge_{LH} S = \begin{cases} \langle a^s\rangle, & \text{if } tx + sy = k - j \text{ has no integer solution} \\ \quad \text{where } s = \frac{2^{\alpha+1}n}{(|T|,|S|)} & \\ \langle a^d, a^{k-n_1x_0}b\rangle, & \text{if } tx + sy = k - j \text{ has an integer solution} \\ \quad \text{where } d = \frac{2n}{(|T|,|S|)} & \end{cases}$$

where (x_0, y_0) is an integer solution of an equation $tx + sy = k - j$.

Now, we establish some lattice theoretic property such as lower semimodularity, complementation, atomic covering condition and Mac-lanes exchange property in the subgroup lattice $LH(D_n)$.

Definition 2.7 [13]**.** A lattice L is said to be *lower semimodular*, for every $T,\ S \in L$, if $T \prec T \vee S$, then $T \wedge S \prec S$.

Theorem 2.8. *The lattice $LH(D_n)$ is lower semimodular.*

Proof. Let T and $S \in LH(D_n)$ be such that $T \prec T \vee S$.

Claim. $T \wedge S \prec S$.

Consider $n = 2^\alpha \prod_{i=1}^m p_i^{\alpha_i}$ where each p_i is an odd prime. Note that, if a Type (1) subgroup H of D_n generated by a^h is also a Hall subgroup, then it is necessary that $h = 2^\alpha \prod_{x\in M} p_x^{\alpha_x}$ for some subset $M \subseteq \{1, 2, \ldots, m\}$. Moreover, if a Type (2) subgroup H of D_n generated by $\{a^h, a^i b\}$ is also a Hall subgroup, then it is necessary that $h = \prod_{x\in N} p_x^{\alpha_x}$ for some subset $N \subseteq \{1, 2, \ldots, m\}$.

Case I. Let $T = \langle a^t\rangle$, where $t = 2^\alpha \prod_{x\in U\subseteq\{1,2,\ldots,m\}} p_x^{\alpha_x}$.

Subcase I(i). If $S = \langle a^s\rangle$ where $s = 2^\alpha \prod_{y\in V\subseteq\{1,2,\ldots,m\}} p_y^{\alpha_y}$ then $T \vee S = \langle a^g\rangle$ where $g = (s, t)$. In view of $T \prec T \vee S$, Note that,$\langle a^t\rangle \prec \langle a^g\rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{2^\alpha \prod_{x\in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$ and p_* is an odd prime dividing n with largest power α_*. We have $g|s$ (say $gk = s$ where $k \in \mathbb{Z}$) and $p_*^{\alpha_*} \nmid s$ since $T \not\subseteq S$.

Now $S \wedge T = \langle a^l\rangle$, where $l = [s, t] = [gk, gp_*^{\alpha_*}] = gkp_*^{\alpha_*} = sp_*^{\alpha_*}$ $(p_* \nmid s)$. Consequently, $T \wedge S = \langle a^{sp_*^{\alpha_*}}\rangle \prec \langle a^s\rangle$.

Subcase II(ii). Let $S = \langle a^{s'}, a^i b\rangle$ for some subset $M \subseteq \{1, 2, \ldots, m\}$ where $s' = \prod_{y\in W\subseteq\{1,2,\ldots,m\}} p_y^{\alpha_y}$ such that $T \prec T \vee S$. Note that, $T \vee S = \langle a^g, a^i b\rangle$ where $g = (s', t)$. Since $T \prec T \vee S$ we have $\langle a^t\rangle \prec \langle a^g, a^i b\rangle$ if and only if $g = \frac{t}{2^\alpha} = \prod_{x\in U} p_x^{\alpha_x}$. As $g|s'$ ((say $gk = s'$ where $k \in \mathbb{Z}$), i.e., $\prod_{x\in U} p_x^{\alpha_x} | \prod_{y\in W} p_y^{\alpha_y}$ and so $\prod_{x\in U} p_x^{\alpha_x} \prod_{q\in X\subseteq W} p_q^{\alpha_q} = \prod_{y\in W} p_y^{\alpha_y}$. Now consider $T \wedge S = \langle a^l\rangle$ where

$l = [s', t] = [gk, 2^\alpha g] = 2^\alpha gk = 2^\alpha s'$ $(2 \nmid s')$. Consequently, $T \wedge S = \langle a^{2^\alpha s'} \rangle \prec \langle a^{s'}, a^i b \rangle = S$, as $\frac{|S|}{|S \wedge T|} = 2^{\alpha+1}$.

Case II. Let $T = \langle a^t, a^i b \rangle$ where $t = \prod_{x \in U} p_x^{\alpha_x}$.

Subcase II(i). Let $S = \langle a^s \rangle$ where $s = 2^\alpha \prod_{y \in V} p_y^{\alpha_y}$ such that $T \prec T \vee S$. We have $T \vee S = \langle a^g, a^i b \rangle$ where $g = (s, t)$. Since $T \prec T \vee S$, we have $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Note that, $g|s$ ((say $gk = s$ where $k \in \mathbb{Z}$) and $T \not\subset S$ which implies $p_*^{\alpha_*} \nmid s$.

Now consider $S \wedge T = \langle a^l \rangle$ where $l = [s, t] = [gq, gp_*^{\alpha_*}] = gqp_*^{\alpha_*} = sp_*^{\alpha_*}$ $(p_*^{\alpha_*} \nmid s)$. Consequently, $T \wedge S = \langle a^{sp_*^{\alpha_*}} \rangle \prec \langle a^s \rangle = S$.

Subcase II(ii). Let S be a dihedral subgroup with $|S| = |T|$ and $T \prec T \vee S$. Then $S = \langle a^t, a^j b \rangle$. Note that,$S \vee T = \langle a^g, a^i b \rangle = \langle a^g, a^j b \rangle$. Since $T \prec T \vee S$, we have $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Note that, $i,\ j \leq t$ and so $i - j \leq t$. Consider the equation $tx_1 + tx_2 = i - j$ for $x_1,\ x_2 \in \mathbb{Z}$ and this equation does not have a solution as $i - j \leq t,\ t \nmid i - j$. Therefore, $T \wedge S$ is a cyclic subgroup, suppose that $T \wedge S = \langle a^l \rangle$ where $l = \frac{2^{\alpha+1} n}{(|T|, |S|)} = \frac{2^{\alpha+1} n}{(\frac{2n}{t}, \frac{2n}{t})} = t 2^\alpha$. Therefore, $S \wedge T = \langle a^{t 2^\alpha} \rangle$. Note that, $\frac{|S|}{|S \wedge T|} = 2^{\alpha+1}$ and hence $T \wedge S \prec S$ for such choice of S and T.

Now suppose S be a dihedral subgroup such that $|T| \neq |S|$ and $T \prec T \vee S$, say $S = \langle a^{s'}, a^j b \rangle$ where $s' = \prod_{y \in V} p_y^{\alpha_y}$ for some $y \in V \subseteq \{1, 2, \ldots, m\}$. Note that, $S \vee T = \langle a^g, a^i b \rangle$ where $g = \frac{g_1}{r}$ and $g_1 = (t, s, i - j), r = \left(\frac{2n}{g_1}, g_1\right)$. Since $T \prec T \vee S$ we have $\langle a^t, a^i b \rangle \prec \langle a^g, a^i b \rangle$ if and only if $g = \frac{t}{p_*^{\alpha_*}} = \frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}$. Now as $g|s'$ and $g|i - j$ there exists $\alpha, \beta \in \mathbb{Z}$ we have $\alpha g = i - j$ and $\beta g = s'$. Consider the equation $tx_1 + sx_2 = i - j$, i.e., $g(p_*^{\alpha_*})x_1 + g(\beta)x_2 = g\alpha$, i.e., $(p_*^{\alpha_*})x_1 + (\beta)x_2 = \alpha$.

We have two cases: $p_*^{\alpha_*} \nmid \beta$ and $p_*^{\alpha_*}|\beta$ and we contend that in each case $T \wedge S \prec S$.

Suppose that, $p_*^{\alpha_*} \nmid \beta$, then $(p_*^{\alpha_*}, \beta) = 1$. Therefore, the equation $(p_*^{\alpha_*})x_1 + \beta x_2 = \alpha$ will always have a solution. In this case $T \wedge S = \langle a^d, a^z b \rangle$, where $d = \frac{2n}{\left(\frac{2n}{\prod_{x \in U} p_x^{\alpha_x}}, \frac{2n . p_*^{\alpha_*}}{\prod_{x \in U} p_x^{\alpha_x} \beta}\right)} = \beta \prod_{x \in U} p_x^{\alpha_x}$. Note that, $\frac{|S|}{|S \wedge T|} = p_*^{\alpha_*}$. Consequently, $T \wedge S \prec S$.

Now suppose that $p_*^{\alpha_*}|\beta$. If the equation $(p_*^{\alpha_*})x_1 + \beta x_2 = \alpha$ for $x_1, x_2 \in \mathbb{Z}$ has a solution, then $p_*^{\alpha_*}|\alpha$. Now as $\left(\frac{\prod_{x \in U} p_x^{\alpha_x}}{p_*^{\alpha_*}}, p_*^{\alpha_*}\right) = 1$ implies $\prod_{x \in U} p_x^{\alpha_x} | i - j$ and also $\prod_{x \in U} p_x^{\alpha_x} | s'$. Consequently, $T \vee S = \langle a^g, a^i b \rangle = \langle a^t, a^i b \rangle = T$ (as $g_1 = (t, s', i - j) = t$ and $r = \left(\frac{2n}{g_1}, g_1\right) = 1$ then $g = \frac{g_1}{r} = g_1 = t$) which is not true since $T \prec T \vee S$. Therefore $p_*^{\alpha_*} \nmid \alpha$ and so the equation does not have a solution. As such $S \wedge T$ is not a Type (2) subgroup of D_n and we must have $S \wedge T = \langle a^l \rangle$, for $l = \frac{2^{\alpha+1}n}{\left(\frac{2n}{\prod_{x\in U} p_x^{\alpha_x}}, \frac{2n.p_*^{\alpha_*}}{\prod_{x\in U} p_x^{\alpha_x} \prod p_q^{\alpha_q}}\right)} = \frac{2^{\alpha}.\prod_{x\in U} p_x^{\alpha_x} \prod p_q^{\alpha_q}}{p_*^{\alpha_*}} = 2^{\alpha} s'$. Therefore, $\langle a^l \rangle = \langle a^{2^{\alpha} s'} \rangle \prec \langle a^{s'}, a^j b \rangle = S$. Note that, $\frac{|S|}{|T\wedge S|} = 2^{\alpha+1}$ and hence $T \wedge S \prec S$ for such choice of S and T. ■

A lattice is said to be complemented if every element has a complement. In what follows, we have a Theorem about $LH(D_n)$.

Theorem 2.9. *Let D_n be the dihedral group with $2n$ elements where $n = 2^{\alpha} \prod_{i=1}^{m} p_i^{\alpha_i}$. Then, the lattice $LH(D_n)$ is complemented.*

Proof. In order to show that $LH(D_n)$ is complemented, it is sufficient to show that every cyclic Hall subgroup has a complement in $LH(D_n)$.

Note that, if a cyclic subgroup $\langle a^h \rangle$ is also a Hall subgroup, then it is necessary that $h = 2^{\alpha} \prod_M p_x^{\alpha_x}$ such that $x \in M \subseteq \{1, 2, \ldots, m\}$. Moreover, if a dihedral subgroup $\langle a^h, a^i b \rangle$ is also a Hall subgroup, then it is necessary that $h = \prod_N p_x^{\alpha_x}$ such that $x \in N \subseteq \{1, 2, \ldots, m\}$.

Let $A = \langle a^k \rangle$ be a cyclic Hall subgroup, then $k = 2^{\alpha} \prod_U p_x^{\alpha_x}$ such that $x \in U \subseteq \{1, 2, \ldots, m\}$. Choose the subgroup $B = \langle a^t, a^i b \rangle$ where $t = \frac{n}{k}$. But then $g = (k, t) = 1$ and so $A \vee B = \langle a^g, a^i b \rangle = D_n$. Moreover $l = [k, t] = n$ this implies $A \wedge B = \langle a^l \rangle = \langle a^n \rangle = I$. Therefore, every cyclic Hall subgroup has complement and so every dihedral Hall subgroup has a complement. ■

It is known that the number of subgroups of D_n for $n \geq 3$ is $|L(D_n)| =$ Number of divisors of n + Sum of divisors of n. Along the same line, we have the following formula for the number of Hall subgroups of D_n, i.e., $|LH(D_n)|$.

Theorem 2.10. *For any $n \geq 3$, $|LH(D_n)| = 2^z + \prod_{m=1}^{z} (1 + p_m^{\alpha_m})$ where $n = 2^{\alpha} \prod_{m=1}^{z} p_m^{\alpha_m}$, where p is prime and z is the number of odd primes dividing n.*

Proof. Let $n = 2^{\alpha} \prod_{m=1}^{z} p_m^{\alpha_m}$, p being prime. If H is a cyclic Hall subgroup of D_n, then $|H| = \prod_{x\in S\subseteq\{1,2,\ldots,z\}} p_x^{\alpha_x}$ and $|H|$ is not a multiple of 2. Note that, number of subgroups whose order is divisible by single odd prime is given by $\binom{z}{1}$. Similarly, number of subgroups whose order contains exactly two odd prime factors is given by $\binom{z}{2}$. Consequently, number of cyclic Hall subgroups= $\binom{z}{0} + \binom{z}{1} + \binom{z}{2} + \binom{z}{3} + \cdots + \binom{z}{z} = 2^z$.

Now consider a dihedral Hall subgroup H then $|H| = 2^{\alpha+1} \prod_{x\in S\subseteq\{1,2,\ldots,z\}} p_x^{\alpha_x}$. If H_1 be a dihedral Hall subgroup whose order is divisible by single odd prime say p_1, then $H_1 = \left\langle a^{\prod_{m=2}^{z} p_m^{\alpha_m}}, a^i b \right\rangle$ and number of subgroups whose order is equal to order of H_1 is $\prod_{m=2}^{z} p_m^{\alpha_m}$. Consequently, the number of all such subgroups whose order is divisible by exactly single odd prime is equal to $\sum_{x\in S\subset\{1,2,\ldots,z\}} \prod p_x^{\alpha_x}$ such that $|S| = z-1$. Similarly, if H_2 is a dihedral Hall subgroup whose order is divisible by exactly two odd prime factors, say p_1 and p_2, then $H_2 = \left\langle a^{\prod_{m=3}^{z} p_m^{\alpha_m}}, a^i b \right\rangle$ and the number of subgroups whose order is equal to order of H_2 is $\prod_{m=3}^{z} p_m^{\alpha_m}$. Consequently, number of all such subgroups whose order contains exactly two odd primes is equal to $\sum_{x\in S\subset\{1,2,\dots,z\}} \prod p_x^{\alpha_x}$ such that $|S| = z-2$. As such, number of all such dihedral Hall subgroups considering the number of prime divisors involved is given by $\sum_{m=1}^{z} p_m^{\alpha_m} + \sum_{x\in S_1\subset\{1,2,\dots,z\}} \prod p_x^{\alpha_x} + \sum_{x\in S_2\subset\{1,2,\dots,z\}} \prod p_x^{\alpha_x} + \sum_{x\in S\subset\{1,2,\dots,z\}} \prod p_x^{\alpha_x} + \cdots + \sum_{x\in S_{z-1}\subset\{1,2,\dots,z\}} \prod p_x^{\alpha_x} + 1 = \prod_{m=1}^{z} \left(1+p_m^{\alpha_m}\right)$, where $|S_i| = z-i$ for $i = 1, 2, \dots, z-1$.

Therefore, number of Hall subgroups of $D_n = |LH(D_n)| = 2^z + \prod_{m=1}^{z} (1+p_m^{\alpha_m})$, whenever $n = 2^\alpha \prod_{m=1}^{z} p_m^{\alpha_m}$. ■

3. HALL SUBGROUPS OF FINITE NILPOTENT GROUPS

In this section, properties of collection of Hall subgroups of finite nilpotent groups are investigated.

We recall the following characterization, see Grätzer [5].

Theorem 3.1. *A modular lattice is distributive if and only if it does not a sublattice isomorphic to diamond ($\mathcal{M}_3$).*

Remark. For every Hall subgroup K of G, $LH(K)$ is a sublattice of $LH(G)$ whenever $LH(G)$ is a lattice.

Theorem 3.2. *Let G be a finite group. Then $LH(G)$ is a distributive lattice if and only if G is a nilpotent group.*

Proof. Let G be a finite nilpotent group, we first show that $LH(G)$ is a sublattice of $L(G)$. Let $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$ and the subgroups H, K are Hall subgroups of G. Note that, G is nilpotent if and only if it is direct product of its Sylow p-subgroups, i.e., $G = G_1 \times G_2 \times \cdots \times G_m = \prod_{i=1}^{m} G_i$, where each G_i is the Sylow p_i-subgroup of G. Also, Note that,each G_i is unique being part of direct product and so normal in G.

Claim I. $H \wedge K$ is a Hall subgroup.

Let $H = \prod_{i\in S_1} G_i$ and $K = \prod_{i\in S_2} G_i$ such that $S_1,\ S_2 \subseteq \{1, 2, \dots, m\}$ are unique of its order being normal in G. But then the subgroup $H \cap K = T = \prod_{i\in S_1\cap S_2} G_i$ is the Hall subgroup of G and so $H \cap K$ is a Hall subgroup.

Claim II. $H \vee K$ is a Hall subgroup.

Let $H = \prod_{i\in S_1} G_i$ and $K = \prod_{i\in S_2} G_i$ such that $S_1, S_2 \subseteq \{1, 2, \dots, m\}$ are unique of its order being normal in G. But then the subgroup $\langle H, K\rangle = T =$

$\prod_{i \in S_1 \cup S_2} G_i$ is the Hall subgroup of G and so $\langle H, K \rangle$ is a Hall subgroup. This proves that $LH(G)$ is a sublattice of $L(G)$.

Note that, each Hall subgroup is normal as it is join of Sylow p-subgroups and every Sylow p-subgroup is unique as G is direct product of its Sylow p-subgroups being nilpotent. Consequently, $LH(G)$ is a sublattice of $LN(G)$ which implies that $LH(G)$ is modular since $LN(G)$ is a modular lattice and sublattice of modular lattice is modular. We show that $LH(G)$ does not contain diamond ($\mathcal{M}_3$) as its sublattice.

Suppose $LH(G)$ contains a diamond as its sublattice. Note that,the five subgroups $H_i,\ i \in \{1, 2, \ldots, 5\}$ in M_3 as depicted in Figure 3.1. The each one of the five subgroups are of different orders these are of different orders.

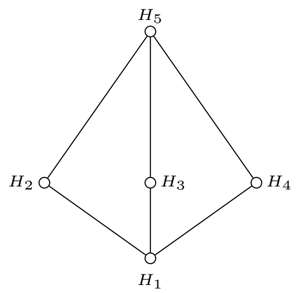

Figure 3.1. Figure $\mathcal{M}_3$.

Now $H_2 \vee H_3 = H_2H_3 = H_4 \vee H_3 = H_4H_3 = H_2 \vee H_4 = H_4H_2$. Consequently, $|H_4H_3| = |H_4H_2| = |H_2H_3| = |H_5|$, but then $|H_4H_3| = \frac{|H_4||H_3|}{|H_4 \cap H_3|} = |H_4H_2| = \frac{|H_4||H_2|}{|H_4 \cap H_2|}$ which implies $|H_2| = |H_3|$, a contradiction.

Conversely, suppose that $LH(G)$ is a distributive lattice. We contend that, G is direct product of its Sylow p-subgroups. If not, then there exists a prime p such that $p||G|$ and a Sylow p-subgroup of G is not normal. Let P_1 and P_2 be two Sylow p-subgroups of G, then these are also Hall subgroups.

Note that, $|G|$ is divisible by at least two primes since every finite group with prime power order is nilpotent.

Case I. Let $|G| = p^\alpha q^\beta$ where p, q are distinct primes. Choose a subgroup Q of G such that Q is a Sylow q-subgroup, which is also a Hall subgroup. Note that,$P_1 \wedge_{LH} Q = P_2 \wedge_{LH} Q = P_1 \wedge_{LH} P_2 = \{e\}$ and $P_1 \vee_{LH} Q = P_2 \vee_{LH} Q = P_1 \vee_{LH} P_2 = G$. Moreover P_1, P_2, Q Hall subgroup. Consequently, $LH(G)$ contains sublattice $S = \{\{e\}, P_1, P_2, Q, G\}$ isomorphic to M_3, a contradiction to the fact that $LH(G)$ is distributive.

Case II. Let $|G| = p^\alpha q_1^{\beta_1} \cdots q_m^{\beta_m}$ where p, q_i's are distinct primes. Since $LH(G)$ is a lattice, $P_1 \vee_{LH} P_2 = T$ is a Hall subgroup of G, let $|T| = p^\alpha \prod_{i \in X} q_i^{\beta_i}$

for a subset $X \subseteq \{1, 2, \ldots, m\}$. Note that, if there exists a Hall subgroup Q of order $\prod_{i\in X} q_i^{\beta_i}$ then this subgroup is such that $p \nmid |Q|$ is a co-atom in $LH(T)$. If not, then consider a subgroup Q which is Hall subgroup with order $\prod_{i\in Y\subset X} q_i^{\beta_i}$. Such Q exists, since at least we have a Sylow q_i-subgroup which is a Hall subgroup. Also, such Q is co-atom in $LH(T)$ and $p \nmid |Q|$.

Now, consider the subset $\{\{e\}, P_1, P_2, Q, T\}$ with $P_1 \wedge_{LH} Q = P_2 \wedge_{LH} Q = P_1 \wedge_{LH} P_2 = \{e\}$ and $P_1 \vee_{LH} Q = P_2 \vee_{LH} Q = P_1 \vee_{LH} P_2 = T$, which forms a sublattice isomorphic to M_3 of $LH(T)$ and so, $LH(T)$ is not distributive. Consequently, $LH(G)$ is not distributive, a contradiction.

Therefore, G is direct product of its Sylow p-subgroups and so nilpotent. ∎

In the next Lemma the number of Hall subgroups of finite nilpotent groups is obtained.

Lemma 3.3. *Let G be a finite nilpotent group and $|G| = \prod_{i=1}^{m} p_i^{\alpha_i}$, then $|LH(G)| = 2^m$.*

Proof. Note that, if G is a finite nilpotent group and π is any set of primes, then G has a Hall π-subgroup. Moreover, by Theorem 3.2, we have the unique Hall π-subgroup for each set π of primes. Consequently, the number of distinct Hall subgroups of G is $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = 2^m$. ∎

REFERENCES

[1] G. Birkhoff, Lattice Theory, Amer. Math. Soc. **25** (Providence, R.I., 1967).

[2] H. Chen, Y. Xiong and Z. Zhu, *Automorphisms of metacycylic groups*, Czechoslov. Math. J. **68(3)** (2018) 803–815.
https://doi.org/10.21136/CMJ.2017.0656-16

[3] K. Conrad, Dihedral Groups II.
https://kconrad.math.uconn.edu/blurbs/grouptheory/dihedral2.pdf

[4] U. Faigle, *Geometries on partially ordered sets*, J. Combin. Theory, Ser. B **28** (1980) 26–51.
https://doi.org/10.1016/0095-8956(80)90054-4

[5] G. Grätzer, General Lattice Theory, Academic press (New York, 1978).

[6] P. Hall, *Theorems like Sylow's*, Proc. London Math. Soc. **22(6)** (1956) 286–304.
https://doi.org/10.1112/plms/s3-6.2.286

[7] Hyo-Seob Sim, *Metacyclic groups of odd order*, Proc. London Math. Soc. s3–69 (1994) 47–71.

[8] I.S. Luthar and I.B.S. Passi, Algebra Volume 1: Groups, Narosa Publishing House Pvt. Ltd. (New Delhi, 1999).

[9] S. Mitkari, V. Kharat and S. Ballal, *On some subgroup lattices of dihedral, alternating and symmetric groups*, Discuss. Math. Gen. Algebra Appl. **43(2)** (2023) 309–326.
https://doi.org/10.7151/dmgaa.1425

[10] P.P. Pálfy, *Groups and lattices*, London. Math. Soc. Lecture Note Ser. **305** (2003) 428–454.

[11] G. Richter and M. Stern, *Strongness in* (*semimodular*) *lattices of finite length*, Wiss. Z. Univ. Halle. **33** (1984) 73–77.

[12] R. Schmidt, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics 14 de Gruyter (Berlin, 1994).

[13] M. Stern, Semimodular Lattices (Cambridge University Press, 1999).

[14] M. Suzuki, Structure of a Group and Structure of its Lattice of Subgroups (Springer Verlag, Berlin, 1956).

[15] P.M. Whitman, *Groups with a cyclic group as a lattice homomorph* Ann. Math. **49(2)** (1948) 347–351.
https://doi.org/10.2307/1969283

Received 3 August 2023
Revised 23 February 2024
Accepted 23 February 2024