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ON COAXIAL FILTERS OF ALMOST DISTRIBUTIVE LATTICES

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Abstract

In an Almost Distributive Lattice (ADL), coaxial filters and strongly coaxial filters are presented, and various characterization theorems of dually normal ADLs are given in terms of dual annihilators. Several characteristics of ADL coaxial filters are investigated. The concept of normal prime filters is presented, and its features are examined. For the class of all strongly coaxial filters to become a sublattice of the filter lattice, some equivalent conditions are derived.

Keywords: filter, dual annihilator, coaxial filter, strongly coaxial filter, dually normal ADL, normal prime filter.

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Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [9] and the concept of an ideal in an ADL was introduced analogous

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to that in a distributive lattice and it was observed that the set $PI(R)$ of all principal ideals of R forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. In [3], the authors thoroughly investigated certain significant properties of dual annihilators, dual annihilator filters and μ-filters of almost distributive lattices. In [8], the concepts of coaxial filters and strongly coaxial filters are introduced in a distributive lattice and studied its properties.

The notions of coaxial filters and strongly coaxial filters are introduced in this paper in terms of dual annihilators of ADLs, analogous to that in a distributive lattice. Dual annihilators and maximum ideals of ADLs are utilized to characterize dually normal ADLs once more. For each ADL filter to become a coaxial filter, a set of equivalent conditions is derived. The concept of normal prime filters is presented, and it can be seen that every normal prime filter is both a coaxial filter and a minimum prime filter. Some coaxial filter features are derived in terms of inverse homomorphic images and cartesian products. The concept of ADLs that are weakly dually normal is introduced. For every weakly dually normal ADL to become a dually normal ADL, some analogous requirements are derived. For each ADL filter to become a strongly coaxial filter, a set of equivalent conditions is derived. Finally, for the class of all strongly coaxial filters of an ADL to constitute a sublattice of the filter lattice, a set of analogous conditions is deduced.

1. Preliminaries

First, we recall certain definitions and properties of ADLs that are required in the paper. We begin with ADL definition as follows.

Definition [9]**.** An Almost Distributive Lattice with zero or simply ADL is an algebra $(R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$;

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;

(3) $(a \vee b) \wedge b = b$;

(4) $(a \vee b) \wedge a = a$;

(5) $a \vee (a \wedge b) = a$;

(6) $0 \wedge a = 0$;

(7) $a \vee 0 = a$, for all $a, b, c \in R$.

Example 1. Every non-empty set X can be regarded as an ADL as follows. Let

$a_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$a \vee b = \begin{cases} a & \text{if } a \neq a_0 \\ b & \text{if } a = a_0 \end{cases} \qquad a \wedge b = \begin{cases} b & \text{if } a \neq a_0 \\ a_0 & \text{if } a = a_0. \end{cases}$$

Then $(X, \vee, \wedge, a_0)$ is an ADL (where a_0 is the zero) and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $x, y \in R$, define $a \leq b$ if and only if $x = x \wedge y$ (or equivalently, $x \vee y = y$), then $\leq$ is a partial ordering on R.

It can be observed that an ADL[9] R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice. As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $x \in R$, $m \leq x \Rightarrow m = x$. The set of all maximal elements of an ADL R is denoted by $\mathcal{M}_{max.elt}$.

Theorem 2 [9]**.** *Let R be an ADL and $m \in R$. Then the following are equivalent:*

(1) *m is maximal with respect to $\leq$;*

(2) *$m \vee x = m$, for all $a \in R$;*

(3) *$m \wedge x = x$, for all $x \in R$;*

(4) *$x \vee m$ is maximal, for all $x \in R$.*

As in distributive lattices [1, 2], a non-empty subset U of an ADL R is called an ideal of R if $x \vee y \in U$ and $x \wedge a \in U$ for any $x, y \in U$ and $a \in R$. Also, a non-empty subset F of R is said to be a filter of R if $x \wedge y \in F$ and $a \vee x \in F$ for $x, y \in F$ and $a \in R$.

The set $\mathfrak{I}(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $U, V \in U(R)$, $U \cap V$ is the infimum of U and V while the supremum is given by $U \vee V := \{x \vee y \mid x \in U,\ y \in V\}$. A proper ideal L of R is called a prime ideal if, for any $a, b \in R$, $a \wedge b \in L \Rightarrow a \in L$ or $b \in L$. A proper ideal (filter) L of R is called a prime ideal(filter) if, for any $a, b \in R$, $a \wedge b \in L (a \vee b \in L) \Rightarrow a \in L$ or $b \in L$. A proper ideal (filter) P of R is said to be maximal if it is not properly contained in any proper ideal (filter) of R. It can be observed that every maximal ideal (filter) of R is a prime ideal (filter). Every proper ideal (filter) of R is contained in a maximal ideal (filter). For any subset A of R the smallest ideal containing A is given by $(A] := \{(\bigvee_{i=1}^{n} e_i) \wedge a \mid e_i \in A, a \in R \text{ and } n \in \mathbb{N}\}$. If $A = \{e\}$, we write $(e]$ instead of $(A]$. Similarly, for any $A \subseteq R$, $[A) := \{a \vee (\bigwedge_{i=1}^{n} e_i) \mid e_i \in A, a \in R$ and $n \in \mathbb{N}\}$ is the smallest filter containing A. If $A = \{e\}$, we write $[e)$ instead of $[A)$. The set $\mathfrak{F}(R)$ of all filters of R forms a bounded distributive lattice, where $F \cap S$ is the infimum and $F \vee S = \{x \wedge y \mid x \in F,\ y \in S\}$ is the supremum in $\mathfrak{F}(R)$.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. Hence the set $PI(R)$ of all principal ideals of R is a sublattice of the distributive lattice $\mathfrak{I}(R)$ of ideals of R.

Theorem 3 [5]**.** *Let U be an ideal and F a filter of R such that $U \cap F = \emptyset$. Then there exists a prime ideal L such that $U \subseteq L$ and $L \cap F = \emptyset$.*

An ADL R is called a dually normal [6] if every prime ideal of R is contained in a unique maximal ideal of R. In that characterized topologically in terms of its maximal ideals and prime ideals. Some necessary and sufficient conditions for the space of maximal ideals to be dually normal are obtained.

Theorem 4 [7]**.** *A prime filter L of an ADL R with maximal elements is minimal if and only if to each $a \in L$ there exists $b \notin L$ such that $a \vee b$ is maximal element.*

For any subset G of an ADL R with maximal elements, the dual annihilator of G is define as the set $G^+ = \{a \in R \mid a \vee x \text{ is maximal, for all } x \in G\}$. For any subset G of R, G^+ is a filter of R with $G \cap G^+ \subseteq \mathcal{M}_{max.elt}$.

Lemma 5 [3]**.** *Let R be an ADL with maximal elements. For any subsets G and B of R, the following properties hold:*

(1) $G \subseteq B$ *implies* $B^+ \subseteq G^+$;

(2) $G \subseteq G^{++}$;

(3) $G^{+++} = G^+$;

(4) $G^+ = R$ *if and only if* $G \subseteq \mathcal{M}_{max.elt}$.

In case of filters, we have the following result.

Proposition 6 [3]**.** *Let R be an ADL R with maximal elements. For any filters F, S and T of R, the following properties hold:*

(1) $F^+ \cap F^{++} = \mathcal{M}_{max.elt}$;

(2) $F \cap S = \mathcal{M}_{max.elt}$ *implies* $F \subseteq S^+$;

(3) $(F \vee S)^+ = F^+ \cap S^+$;

(4) $(F \cap S)^{++} = F^{++} \cap S^{++}$.

It is clear that $([a))^+ = (a)^+$. Then clearly $(0)^+ = \mathcal{M}_{max.elt}$. The following corollary is a direct consequence of the above results.

Corollary 7 [3]**.** *Let R be an ADL with maximal elements. For any $x, y, z \in R$,*

(1) $x \leq y$ *implies* $(x)^+ \subseteq (y)^+$;

(2) $(x \wedge y)^+ = (x)^+ \cap (y)^+$;

(3) $(x \vee y)^{++} = (x)^{++} \cap (y)^{++}$;

(4) $(x)^{+} = R$ *if and only if* x *is maximal.*

A filter F of an ADL R with maximal elements is called a *dual annihilator filter* [3] if $F = F^{++}$. A filter F of an ADL R with maximal elements is called a μ-filter of R if $x \in F$ implies $(x)^{++} \subseteq F$ for all $x \in R$. Every dual annihilator filter of an ADL is a μ-filter.

2. Coaxial filters of ADLs

The notion of coaxial filters in ADLs is introduced in this section. The class of dually normal ADLs is defined by dual annihilators. For each ADL filter to become a coaxial filter, a set of analogous conditions is derived. Also, the notion of strongly coaxial filters in ADLs is introduced in this section. For the class of all strongly coaxial filters to become a sublattice to the filter lattice, a set of equivalent conditions is derived.

Definition. For any subset G of an ADL R, define

$$G^{\square} = \{a \in R \mid (x)^{+} \vee (a)^{+} = R \text{ for all } x \in G\}.$$

Clearly $\mathcal{M}^{\square}_{max.elt} = R$ and $R^{\square} = \mathcal{M}_{max.elt}$. For any $x \in R$, we denote $(\{x\})^{\square}$ by $(x)^{\square}$. Then it is obvious that $(0)^{\square} = \mathcal{M}_{max.elt}$ and $(m)^{\square} = R$, where $m \in \mathcal{M}_{max.elt}$. Clearly $G \cap G^{\square} \subseteq \mathcal{M}_{max.elt}$.

Proposition 8. *For any subset G of an ADL R with maximal element m, $G^{\square}$ is a filter of R.*

Proof. Clearly $m \in G^{\square}$. Let $a, b \in G^{\square}$. Then $(a)^{+} \vee (x)^{+} = R = (b)^{+} \vee (x)^{+}$, for all $x \in G$. Now $(a \wedge b)^{+} \vee (x)^{+} = \{(a)^{+} \cap (b)^{+}\} \vee (x)^{+} = \{(a)^{+} \vee (x)^{+}\} \cap \{(b)^{+} \vee (x)^{+}\} = R \cap R = R$. Hence $a \wedge b \in G^{\square}$. Let $a \in G^{\square}$. Then we get $(a)^{+} \vee (x)^{+} = R$, for all $x \in G$. Let b be any element of R. Since $a \leq a \vee b$, we get that $(a)^{+} \subseteq (b \vee a)^{+}$ and $R = (a)^{+} \vee (x)^{+} \subseteq (b \vee a)^{+} \vee (x)^{+}$. That implies $(b \vee a)^{+} \vee (x)^{+} = R$. Hence $b \vee a \in G^{\square}$. Therefore $G^{\square}$ is a filter of R. ■

Lemma 9. *For any two subsets G and B of an ADL R with maximal elements, the following properties hold:*

(1) $G \subseteq B$ *implies* $B^{\square} \subseteq G^{\square}$;

(2) $G \subseteq G^{\square\square}$;

(3) $G^{\square\square\square} = G^{\square}$;

(4) $G^{\square} = R$ *if and only if* $G \subseteq \mathcal{M}_{max.elt}$.

We get the following result easily when using the filters.

Proposition 10. *For any two filters F and S of an ADL R, $(F \vee S)^{\square} = F^{\square} \cap S^{\square}$.*

The following corollary is a direct consequence of the above results.

Corollary 11. *Let R be an ADL with maximal elements. For any $x, y \in R$, we have the the following:*

(1) *$x \leq y$ implies $(x)^{\square} \subseteq (y)^{\square}$;*

(2) *$(x \wedge y)^{\square} = (x)^{\square} \cap (y)^{\square}$;*

(3) *$(x)^{\square} = R$ if and only if x is maximal.*

For any filter F of an ADL R, it is easy to see that $F^{\square} \subseteq F^{+}$. However, a set of equivalent conditions is given for every filter to satisfy the reverse inclusion which is not true in general.

Example 12. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee$, $\wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee, \wedge)$ is an ADL. Consider a filter $F = \{1, 2, 6\}$. Clearly, $F^{+} = \{1, 2, 4, 5\}$ and $F^{\square} = \{1, 2, 4\}$. Hence $F^{+} \nsubseteq F^{\square}$.

The following result drives to another characterization of dually normal ADL.

Theorem 13. *Let R be an ADL with maximal elements. Then the following assertions are equivalent:*

(1) *R is a dually normal ADL;*

(2) *for any $x, y \in R$ with $x \vee y$ is a maximal element, $(x)^{+} \vee (y)^{+} = R$;*

(3) *for any filters F, S of R, $F \cap S = \mathcal{M}_{max.elt}$ if and only if $F \subseteq S^{\square}$;*

(4) *for any filter F of R, $F^{\square} = F^{+}$;*

(5) *for any $x \in R$, $(x)^{\square} = (x)^{+}$;*

(6) *for any two maximal ideals P and Q of R, there exist $x \notin P$ and $y \notin Q$ such that $x \wedge y = 0$.*

Proof. (1) $\Rightarrow$ (2): Assume that R is a dually normal ADL. Then every prime ideal of R is contained in a unique maximal ideal of R. Let $x, y \in R$ with $x \vee y$ is maximal. Suppose $(x)^+ \vee (y)^+ \neq R$. Then there exists a prime ideal L such that $\{(x)^+ \vee (y)^+\} \cap L = \emptyset$. Then $L \vee (x]$ is an ideal of R such that $L \subseteq L \vee (x]$. Suppose $y \in L \vee (x]$. Then $y = f \vee x$ for some $f \in L$. Hence $f \vee x = (f \vee x) \vee x = y \vee x$. Since $x \vee y$ is maximal, we have that $t \vee a$ is maximal and which implies $f \in (x)^+ \subseteq (x)^+ \vee (y)^+$. Thus $f \in \{(x)^+ \vee (y)^+\} \cap L$, which is a contradiction. Therefore $y \notin L \vee (x]$, which means that $L \vee (x]$ is a proper ideal of R. Then there exists a maximal ideal P_1 such that $L \vee (x] \subseteq P_1$. Similarly, there exists a maximal ideal P_2 such that $L \vee (y] \subseteq P_2$. Since $x \vee y$ is maximal, we get $y \notin P_1$ and $x \notin P_2$. Therefore $P_1 \neq P_2$. Thus the prime ideal L is contained in two distinct maximal ideals, which is a contradiction to the hypothesis. Therefore $(x)^+ \vee (y)^+ = R$.

(2) $\Rightarrow$ (3): Assume condition (2). Let F and S be two filters of R. Suppose $F \cap S = \mathcal{M}_{max.elt}$. Let $a \in F$. For any $x \in S$, we get $a \vee x \in F \cap S = \mathcal{M}_{max.elt}$. Hence $a \vee x$ is maximal. By condition (2), we get $(a)^+ \vee (x)^+ = R$. Thus $a \in S^{\Box}$. Therefore $F \subseteq S^{\Box}$. Conversely, suppose that $F \subseteq S^{\Box}$. Let $a \in F \cap S$. Then $a \in F \subseteq S^{\Box}$. Hence $a \in S \cap S^{\Box} = \mathcal{M}_{max.elt}$, which means a is maximal. Therefore $F \cap S = \mathcal{M}_{max.elt}$.

(3) $\Rightarrow$ (4): Assume condition (3). Let F be a filter of R. Clearly $F^{\Box} \subseteq F^+$. Conversely, let $a \in F^+$. Hence, for any $x \in F$, we have

$$\begin{aligned} a \vee x \textit{ is maximal } &\Rightarrow [a) \cap [x) = \mathcal{M}_{max.elt} \\ &\Rightarrow [a) \subseteq (x)^{\Box} \qquad \text{by (3)} \\ &\Rightarrow [a) \subseteq (x)^{\Box} \text{ for all } x \in F \\ &\Rightarrow a \in F^{\Box} \end{aligned}$$

which gives that $F^+ \subseteq F^{\Box}$. Therefore $F^+ = F^{\Box}$.

(4) $\Rightarrow$ (5): It is obvious.

(5) $\Rightarrow$ (6): Assume condition (5). Let P and Q be two distinct maximal ideals of R. Choose $a \in P - Q$. Since $a \notin Q$, we get $Q \vee (a] = R$. Hence, $x \vee a$ is maximal, for some $x \in Q$. Since $x \vee a$ is maximal, by (5), we get $a \in (x)^+ = (x)^{\Box}$. Hence $(a)^+ \vee (x)^+ = R$. Then $0 \in (x)^+ \vee (a)^+$. Then there exist $e \in (x)^+$ and $f \in (a)^+$ such that $e \wedge f = 0$. Since $e \in (x)^+$ and $f \in (a)^+$, we get $e \vee x$ and $f \vee a$ are maximal elements. If $e \in Q$, then $e \vee x \in Q$, which is a contradiction. If $f \in P$, then $f \vee a \in P$, which is also a contradiction. Therefore there exist $f \notin P$ and $e \notin Q$ such that $e \wedge f = 0$.

(6) $\Rightarrow$ (1): Assume condition (6). Let L be a prime ideal of R. Let P_1 and P_2 be two maximal ideals of R such that $L \subseteq P_1$ and $L \subseteq P_2$. Suppose $P_1 \neq P_2$. By (6), there exist $a, b \in R$ such that $a \notin P_1$ and $b \notin P_2$ such that $a \wedge b = 0$. Since

$a \notin P_1$ and $b \notin P_2$, we get that $a \notin L$ and $b \notin L$. Therefore, we get $0 = a \wedge b \notin L$, which is a contradiction. Hence, L should be contained in a unique maximal ideal. Therefore R is a dually normal ADL. ∎

Definition. A filter F of an ADL R is called a *coaxial filter* if for all $a, b \in R, (a)^{\square} = (b)^{\square}$ and $a \in F$ imply that $b \in F$.

Clearly each $(a)^{\square}, a \in R$ is a coaxial filter of R. It is evident that any filter F of an ADL R is a coaxial filter if it satisfies $(a)^{\square\square} \subseteq F$ for all $a \in F$.

Theorem 14. *The following assertions are equivalent in an ADL R:*

(1) *every filter is a coaxial filter;*

(2) *every principal filter is a coaxial filter;*

(3) *every prime filter is a coaxial filter;*

(4) *for $x, y \in R, (x)^{\square} = (y)^{\square}$ implies $[x) = [y)$.*

Proof. (1) ⇒ (2): It is clear.

(2) ⇒ (3): Assume that every principal filter is a coaxial filter. Let L be a prime filter of R. Suppose $(x)^{\square} = (y)^{\square}$ and $x \in L$. Then clearly $[x) \subseteq L$. Since $(x)^{\square} = (y)^{\square}$ and $[x)$ is a coaxial filter, we get that $y \in [x) \subseteq L$. Therefore L is a coaxial filter.

(3) ⇒ (4): Assume that every prime filter of R is a coaxial filter. Let $x, y \in R$ such that $(x)^{\square} = (y)^{\square}$. Suppose $[x) \neq [y)$. Without loss of generality assume that $[x) \nsubseteq [y)$. Consider $\Sigma = \{ F \in \mathfrak{F}(R) \mid x \vee y \in F \text{ and } x \notin F \}$. Clearly, Σ satisfies the hypothesis of the Zorn's Lemma and hence Σ has a maximal element, say L. We now prove that L is a prime filter in R. Let $a, b \in R$ be such that $a \notin L$ and $b \notin L$. Hence $L \subset L \vee [a)$ and $L \subset L \vee [b)$. Therefore by the maximality of L, $L \vee [a)$ and $L \vee [b)$ are not in Σ. Hence $x \in L \vee [a)$ and $x \in L \vee [b)$. Therefore, we have

$$\begin{aligned} x &\in \{ L \vee [a) \} \cap \{ L \vee [b) \} \\ &= L \vee \{ [a) \cap [b) \} \\ &= L \vee [a \vee b). \end{aligned}$$

If $a \vee b \in L$, then $x \in L \vee [a \vee b) = L$, which is a contradiction to that $x \notin L$. Thus we get $a \vee b \notin L$. Hence L is a prime filter. Therefore by hypothesis (3), we can get that L is a coaxial filter of R. Since $L \in \Sigma$, we get that $x \vee y \in L$ and $x \notin L$. Since L is prime, we get $y \in L$. Since $y \in L$ and L is coaxial, we get $x \in L$, which is a contradiction to $x \notin L$. Therefore $[x) = [y)$.

(4) ⇒ (1): Assume condition (4). Let F be a filter of R. Suppose $x, y \in R$ be such that $(x)^{\square} = (y)^{\square}$. Then by (4), we get that $[x) = [y)$. Suppose $x \in F$. Then we get $y \in [y) = [x) \subseteq F$. Therefore F is a coaxial filter of R. ∎

In the following, normal prime filters are introduced

Definition. A prime filter L of an ADL R is called a *normal prime filter* if to each $a \in L$, there exists $a' \notin L$ such that $(a)^{\square} \vee (a')^{\square} = R$.

Proposition 15. *Every normal prime filter is a minimal prime filter.*

Proof. Let L be a normal prime filter of an ADL R. Suppose $a \in L$. Since L is normal, there exists $a' \notin L$ such that $(a)^{\square} \vee (a')^{\square} = R$. Hence we get $R = (a)^{\square} \vee (a')^{\square} \subseteq (a \vee a')^{\square}$. Thus by Corollary 11(3), we get that $a \vee a'$ is maximal. Therefore L is a minimal prime filter of R. ■

In general, every minimal prime filter need not be a normal filter.

From the example-12, consider a prime filter $L = \{1, 2, 3, 6\}$. Clearly, we have that for any $a \in L$ there exists an element $a' \notin L$ such that $(a)^{\square} \vee (a')^{\square} = R$. Hence a prime filter L is not normal.

However, in the following, we establish a sufficient condition for every minimal prime filter to become a normal prime filter.

Proposition 16. *If R is a dually normal ADL, then every minimal prime filter of R is a normal prime filter.*

Proof. Assume that R is a dually normal ADL and L is a minimal prime filter of R. Let $a \in L$. Then there exists $a' \notin L$ such that $a \vee a'$ is maximal. Since R is a dually normal ADL, we get $(a)^{\square} \vee (a')^{\square} = (a)^{+} \vee (a')^{+} = R$. Therefore L is a normal prime filter in R. ■

Proposition 17. *Let L be a normal prime filter of an ADL R. Then for each $a \in L$, we have the following property:*

$$a \notin L \quad \textit{if and only if} \quad (a)^{\square} \subseteq L.$$

Proof. Let L be a normal prime filter of R and $a \in R$. Suppose $a \notin L$. Let $f \in (a)^{\square}$. Then $R = (f)^{+} \vee (a)^{+} \subseteq (f \vee a)^{+}$. Hence $f \vee a$ is maximal. Since L is prime and $a \notin L$, we must have $f \in L$. Therefore $(a)^{\square} \subseteq L$. Conversely, assume that $(a)^{\square} \subseteq L$. Suppose $a \in L$. Since L is normal prime, there exists $a' \notin L$ such that $(a)^{\square} \vee (a')^{\square} = R$. Hence $R = (a)^{\square} \vee (a')^{\square} \subseteq (a)^{+} \vee (a')^{+}$. Hence $a' \in (a)^{\square} \subseteq L$, which is a contradiction. Therefore $a \notin L$. ■

Theorem 18. *Every normal prime filter of an ADL is a coaxial filter.*

Proof. Let L be a normal prime filter of R. Suppose $a, b \in R$ such that $(a)^{\square} = (b)^{\square}$ and $a \in L$. Since L is normal, there exists $a' \notin L$ such that $(a)^{\square} \vee (a')^{\square} = R$. Hence $R = (a)^{\square} \vee (a')^{\square} = (b)^{\square} \vee (a')^{\square} \subseteq (b \vee a')^{\square}$. Hence by Corollary 11(3), we get $b \vee a'$ is maximal and $b \vee a' \in L$. Since L is prime and $a' \notin L$, it yields that $b \in L$. Therefore L is a coaxial filter. ■

We provide a necessary and sufficient condition for the inverse image of a coaxial filter to become a coaxial filter again in the following result.

Theorem 19. *Let f be a homomorphism of ADLs from R onto R'. Then the following conditions are equivalent:*

(1) *if S is a coaxial filter of R', then $f^{-1}(S)$ is a coaxial filter in R;*

(2) *for each $a \in R'$, $f^{-1}((a)^{\square})$ is a coaxial filter in R.*

Proof. (1) $\Rightarrow$ (2): Assume that $f^{-1}(S)$ is a coaxial filter in R for each coaxial filter S of R'. Since $(a)^{\square}$ is a coaxial filter in R' for each $a \in R'$, we get from (1) that $f^{-1}((a)^{\square})$ is a coaxial filter in R.

(2) $\Rightarrow$ (1): Assume that $f^{-1}((a)^{\square})$ is a coaxial filter in R for each $a \in R'$. Let S be a coaxial filter of R'. Then clearly $f^{-1}(S)$ is a filter in R. Let $a, b \in R$ be such that $(a)^{\square} = (b)^{\square}$ and $a \in f^{-1}(S)$. Then $f(a) \in S$. For any $x \in R'$, we get

$$\begin{aligned} x \in (f(a))^{\square} &\Leftrightarrow f(a) \in (x)^{\square} \\ &\Leftrightarrow a \in f^{-1}((x)^{\square}) \\ &\Leftrightarrow b \in f^{-1}((x)^{\square}) \qquad \text{since } f^{-1}((x)^{\square}) \text{ is coaxial in } R \\ &\Leftrightarrow f(b) \in (x)^{\square} \\ &\Leftrightarrow x \in (f(b))^{\square}. \end{aligned}$$

Hence $(f(a))^{\square} = (f(b))^{\square}$. Since $f(a) \in S$ and S is a coaxial filter, we get $f(b) \in S$. Hence $b \in f^{-1}(S)$. Therefore $f^{-1}(S)$ is a coaxial filter in R. ∎

The properties of direct products of ADL coaxial filters are discussed. First, we require the following lemma, whose proof is straightforward.

Lemma 20. *Let R_1 and R_2 be two ADLs. For any $(x, y), (z, d) \in R_1 \times R_2$, we have the following properties:*

(1) $(x, y)^+ = (x)^+ \times (y)^+$;

(2) $(x, y)^+ \vee (z, d)^+ = (x \vee z, y \vee d)^+$;

(3) $(x, y)^{\square} = (x)^{\square} \times (y)^{\square}$.

Theorem 21. *Let $R = R_1 \times R_2$ be the product of ADLs $(R_1, \vee, \wedge, 0)$ and $(R_2, \vee, \wedge, 0)$. If F_1 and F_2 are coaxial filters of R_1 and R_2 respectively, then $F_1 \times F_2$ is a coaxial filter of $R_1 \times R_2$. Conversely, every coaxial filter of $R_1 \times R_2$ can be expressed as $F = F_1 \times F_2$ where F_1 and F_2 are coaxial filters of R_1 and R_2, respectively.*

Proof. Let F_1 and F_2 be the coaxial filters of R_1 and R_2 respectively. Then clearly $F_1 \times F_2$ is a filter of $R_1 \times R_2$. Let $x, z \in R_1$ and $y, d \in R_2$ be such that $(x,y)^{\square}=(z,d)^{\square}$ and $(x,y)\in F_1\times F_2$. Then $x\in F_1$ and $y\in F_2$. Since $(x,y)^{\square}=(z,d)^{\square}$, we get $(x)^{\square}\times(y)^{\square}=(z)^{\square}\times(d)^{\square}$ and hence $(x)^{\square}=(z)^{\square}$ and $(y)^{\square}=(d)^{\square}$. Since F_1 is a coaxial filter and $x\in F_1$, we get that $z\in F_1$. Similarly, we get $d\in F_2$. Hence $(z,d)\in F_1\times F_2$. Therefore $F_1\times F_2$ is a coaxial filter in $R_1\times R_2$.

Conversely, let F be a coaxial filter of $R_1\times R_2$. Suppose m_1 and m_2 are maximal elements of R_1 and R_2 respectively. Consider $F_1=\{x\in R_1\mid (x,m_2)\in F\}$ and $F_2=\{x\in R_2\mid (m_1,x)\in F\}$. Clearly, F_1 is a filter in R_1. Let $a,b\in R_1$ be such that $(a)^{\square}=(b)^{\square}$ and $a\in F_1$. Then $(a,m_2)\in F$. Since $(a)^{\square}=(b)^{\square}$, we get $(a,m_2)^{\square}=(a)^{\square}\times(m_2)^{\square}=(b)^{\square}\times(m_2)^{\square}=(b,m_2)^{\square}$. Since F is a coaxial filter in $R_1\times R_2$, we get $(b,m_2)\in F$. Hence $b\in F_1$. Therefore F_1 is a coaxial filter in R_1. Similarly, we can obtain that F_2 is a coaxial filter in R_2.

We now prove that $F=F_1\times F_2$. Clearly $F\subseteq F_1\times F_2$. Conversely, let $(x_1,x_2)\in F_1\times F_2$. Then $x_1\in F_1$ and $x_2\in F_2$. Hence $(x_1,m_2)\in F$ and $(m_1,x_2)\in F$. Hence $(x_1,0)=(m_1,0)\wedge(x_1,m_2)\in F$ and also $(0,x_2)=(0,m_2)\wedge(m_1,x_2)\in F$. Thus $(x_1,x_2)=(x_1,0)\vee(0,x_2)\in F$. Therefore $F_1\times F_2\subseteq F$. ■

We will now discuss the concept of weakly dually normal ADL.

Definition. An ADL R is called a *weakly dually normal* if it satisfies the property

$$(a)^{+}\vee(b)^{+}=(a)^{\square}\vee(b)^{\square},\ \text{for all } a,b\in R.$$

Every dually normal ADL is clearly a weakly dually normal ADL. In general, the reverse is not true. However, a set of equivalent conditions is derived for every weakly dually normal ADL to become a dually normal ADL in the following.

Theorem 22. *Let R be a weakly dually normal ADL. Then the following are equivalent:*

(1) *R is a dually normal ADL;*

(2) *for $a,b\in R,(a)^{\square}\vee(b)^{\square}=(a\vee b)^{\square}$;*

(3) *for $a,b\in R,\ a\vee b$ is maximal implies $(a)^{\square}\vee(b)^{\square}=R$.*

Proof. (1) ⇒ (2): Assume that R is a dually normal ADL. Let $a,b\in R$. Since R is dually normal, by Theorem 13, we get $(a)^{\square}\vee(b)^{\square}=(a)^{+}\vee(b)^{+}=(a\vee b)^{+}=(a\vee b)^{\square}$.

(2) ⇒ (3): It is clear.

(3) ⇒ (1): Assume that condition (3) is satisfied. Let $a,b\in R$ be such that $a\vee b$ is maximal. Since R is weakly dually normal, we get $R=(a)^{\square}\vee(b)^{\square}=(a)^{+}\vee(b)^{+}$. By Theorem 13, it yields that R is dually normal. ■

Corollary 23. *A weakly dually normal ADL in which every prime filter is normal is a dually normal ADL.*

Proof. Let R be a weakly dually normal ADL in which every prime filter is normal. Let $a, b \in R$ be such that $a \vee b$ is maximal. Suppose $(a)^{\square} \vee (b)^{\square} \neq R$. Then there exists a prime filter L such that $(a)^{\square} \vee (b)^{\square} \subseteq L$. Then $(a)^{\square} \subseteq L$ and $(b)^{\square} \subseteq L$. Since L is normal, by Proposition 17, we get $a \notin L$ and $b \notin L$. Hence $a \vee b$ is a maximal and $a \vee b \notin L$ which is a contradiction. Thus $(a)^{\square} \vee (b)^{\square} = R$. By the main theorem, R is a dually normal ADL. ■

The notion of strongly coaxial filters in ADLs is introduced in the following.

Definition. For any filter F of an ADL R, define $\xi(F)$ as

$$\xi(F) = \{a \in R \mid (a)^{\square} \vee F = R\}.$$

The following lemma is an immediate consequence from the above definition.

Lemma 24. *For any two filters F, S of an ADL R, we have*

(1) $\xi(F) \subseteq F$;

(2) $F \subseteq S$ *implies* $\xi(F) \subseteq \xi(S)$;

(3) $\xi(F \cap S) = \xi(F) \cap \xi(S)$.

Proof. (1) Let $a \in \xi(F)$. Then $(a)^{\square} \vee F = R$. Hence $a = x \wedge y$ for some $x \in (a)^{\square} \subseteq (a)^{+}$ and $y \in F$. Then $a \vee x$ is maximal and $a \vee y \in F$. Thus $a = a \vee a = a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) = a \vee y \in F$. Therefore $\xi(F) \subseteq F$.

(2) and (3) can be easily verified. ■

In general, $\xi(\xi(F))$ and F need not be the same for any filter F of an ADL. It can be seen in the following example:

From the example-12, consider a filter $F = \{1, 2, 6\}$. We have that $\xi(F) = \{1, 2, 7\}$ and hence $\xi(\xi(F)) = \{1, 2, 7\}$. Therefore $\xi(\xi(F)) \neq F$.

Proposition 25. *For any filter F of an ADL R with maximal elements, $\xi(F)$ is a filter of R.*

Proof. Clearly $m \in \xi(F)$, for any maximal element m of R. Let $a, b \in \xi(F)$. Then $(a)^{\square} \vee F = R$ and $(b)^{\square} \vee F = R$. Hence $(a \wedge b)^{\square} \vee F = \{(a)^{\square} \cap (b)^{\square}\} \vee F = \{(a)^{\square} \vee F\} \cap \{(b)^{\square} \vee F\} = R$. Hence $a \wedge b \in \xi(F)$. Let $a \in \xi(F)$. Then $(a)^{\square} \vee F = R$. Let b be any element of R. Since $a \leq a \vee b$, we get $(a)^{\square} \subseteq (b \vee a)^{\square}$. Then $R = (a)^{\square} \vee F \subseteq (b \vee a)^{\square} \vee F$. Thus $b \vee a \in \xi(F)$. Therefore $\xi(F)$ is a filter of R. ■

Definition. A filter F of an ADL R is called *strongly coaxial* if $F = \xi(F)$.

Proposition 26. *Every strongly coaxial filter is a coaxial filter.*

Proof. Let F be a strongly coaxial filter of an ADL R. Then $F = \xi(F)$. Let $a, b \in R$ be such that $(a)^\square = (b)^\square$ and $a \in F = \xi(F)$. Then clearly $(a)^\square \vee F = R$. Hence $(b)^\square \vee F = R$ and so $b \in \xi(F) = F$. Thus F is a coaxial filter of R. ∎

In general, the converse of the above proposition is not true. In the following theorem, however, we establish a set of equivalent conditions for every ADL filter to become strongly coaxial.

Theorem 27. *Consider the following assertions in an ADL R:*

(1) *every prime filter is normal;*

(2) *every filter is strongly coaxial;*

(3) *every prime filter is strongly coaxial.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If R is a weakly dually normal ADL, then all the above conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): Assume that every prime filter is normal. Let F be a filter of R. Clearly $\xi(F) \subseteq F$. Let $a \in F$. Suppose $(a)^\square \vee F \neq R$. Then there exists a prime filter L of R such that $(a)^\square \vee F \subseteq L$. Hence $(a)^\square \subseteq L$ and $a \in F \subseteq L$. Since L is normal and $(a)^\square \subseteq L$, by Proposition 17, we get that $a \notin L$, which is a contradiction to that $a \in L$. Hence $(a)^\square \vee F = R$. Thus $a \in \xi(F)$. Therefore F is strongly coaxial.

(2) $\Rightarrow$ (3): It is obvious.

Suppose that R is a weakly dually normal ADL.

(3) $\Rightarrow$ (1): Assume that every prime filter is strongly coaxial. Let L be a prime filter of R. Then by our assumption, $\xi(L) = L$. Let $a \in L$. Then $(a)^\square \vee L = R$. Hence $x \wedge y = 0$ for some $x \in (a)^\square$ and $y \in L$. Since $x \in (a)^\square$ and R is a weakly dually normal ADL, we get $(a)^\square \vee (x)^\square = (a)^+ \vee (x)^+ = R$. Suppose $x \in L$. Then $0 = x \wedge y \in L$, which is a contradiction. Thus $x \notin L$ and hence L is a normal prime filter of R. ∎

Theorem 28. *The following assertions are equivalent in an ADL R:*

(1) $(a)^\square \vee (a)^{\square\square} = R$ *for all* $a \in R$;

(2) *every filter of the form* $F = F^{\square\square}$ *is strongly coaxial;*

(3) *for each* $a \in R, (a)^{\square\square}$ *is strongly coaxial.*

Proof. (1) $\Rightarrow$ (2): Assume condition (1). Let F be a filter of R such that $F = F^{\square\square}$. Clearly $\xi(F) \subseteq F$. Conversely, let $a \in F$. Clearly $(a)^{\square\square} \subseteq F^{\square\square}$. Hence $R = (a)^\square \vee (a)^{\square\square} \subseteq (a)^\square \vee F^{\square\square} = (a)^\square \vee F$. Thus $a \in \xi(F)$. Therefore F is a strongly coaxial filter of R.

(2) $\Rightarrow$ (3): It is obvious.

(3) $\Rightarrow$ (1): Assume condition (3). Then we get $\xi((a)^{\square\square}) = (a)^{\square\square}$. Since $a \in (a)^{\square\square}$, we get $(a)^{\square} \vee (a)^{\square\square} = R$. ∎

Definition. For any maximal filter P of an ADL R, define $\Omega(P) = \{a \in R \mid (a)^{\square} \nsubseteq P\}$.

For any maximal filter P of an ADL R, it can be easily observed that $\xi(P) = \Omega(P)$. Thus it can be easily seen that the set $\Omega(P)$ is a filter of R such that $\Omega(P) \subseteq P$. Let us denote that $Max_F R$ is the set of all maximal filter of an ADL R. For any filter F of an ADL R, let us consider that $\pi(F) = \{P \in Max_F R \mid F \subseteq P\}$.

Theorem 29. *Suppose $\pi(F)$ is finite for any filter F of an ADL R. Then $\xi(F) = \bigcap_{P\in\pi(F)} \Omega(P)$.*

Proof. Let $a \in \xi(F)$ and $F \subseteq P$ where $P \in Max_F R$. Then $R = (a)^{\square} \vee F \subseteq (a)^{\square} \vee P$. Suppose $(a)^{\square} \subseteq P$, then $P = R$, which is a contradiction. Hence $(a)^{\square} \nsubseteq P$. Thus $a \in \Omega(P)$ for all $P \in \pi(F)$. Therefore $\xi(F) \subseteq \bigcap_{P\in\pi(F)} \Omega(P)$. Conversely, let $a \in \bigcap_{P\in\pi(F)} \Omega(P)$. Then $a \in \Omega(P)$ for all $P \in \pi(F)$. Suppose $(a)^{\square} \vee F \neq R$. Then there exists a maximal filter P_0 such that $(a)^{\square} \vee F \subseteq P_0$. Hence $(a)^{\square} \subseteq P_0$ and $F \subseteq P$. Since $F \subseteq P_0$, by hypothesis, we get $a \in \Omega(P_0)$. Hence $(a)^{\square} \nsubseteq P_0$, which is a contradiction. Hence $(a)^{\square} \vee F = R$. Thus $a \in \xi(F)$. Therefore $\bigcap_{P\in\pi(F)} \Omega(P) \subseteq \xi(F)$. ∎

From the above theorem, it can be easily observed that $\xi(F) \subseteq \Omega(P)$ for every $P \in \pi(F)$. In the following, we derive a set of equivalent conditions for the class of all strongly coaxial filters of an ADL to become a sublattice of the filter lattice $\mathfrak{F}(R)$ of the ADL R.

Theorem 30. *Suppose $\pi(F)$ is finite for any filter F of an ADL R. Then the following assertions are equivalent:*

(1) *for any $P \in Max_F R$, $\Omega(P)$ is maximal;*

(2) *for any $F, S \in \mathfrak{F}(R)$, $F \vee S = R$ implies $\xi(F) \vee \xi(S) = R$;*

(3) *for any $F, S \in \mathfrak{F}(R)$, $\xi(F) \vee \xi(S) = \xi(F \vee S)$;*

(4) *for any two distinct maximal filters P and Q, $\Omega(P) \vee \Omega(Q) = R$;*

(5) *for any $P \in Max_F R$, P is the unique member of $Max_F R$ such that $\Omega(P) \subseteq P$.*

Proof. (1) $\Rightarrow$ (2) : Assume condition (1). Then clearly $\Omega(P) = P$ for all $P \in Max_F R$. Let $F, S \in \mathfrak{F}(R)$ be such that $F \vee S = R$. Suppose $\xi(F) \vee \xi(S) \neq R$.

Then there exists a maximal filter P such that $\xi(F)\vee\xi(S)\subseteq P$. Hence $\xi(F)\subseteq P$ and $\xi(S)\subseteq P$. Now

$$\begin{aligned}\xi(F)\subseteq P &\Rightarrow \bigcap_{Q\in\pi(F)}\Omega(Q)\subseteq P\\ &\Rightarrow \Omega(P_i)\subseteq P \quad \text{for some } P_i\in\pi(F) \text{ (since } P \text{ is prime)}\\ &\Rightarrow P_i\subseteq P \qquad \text{by condition (1)}\\ &\Rightarrow F\subseteq P \qquad \text{since } P_i\in\pi(F).\end{aligned}$$

Similarly, we can get $S\subseteq P$. Hence $R=F\vee S\subseteq P$, which is a contradiction. Therefore $\xi(F)\vee\xi(S)=R$.

(2) $\Rightarrow$ (3) : Assume condition (2). Let $F,S\in\mathfrak{F}(R)$. Clearly $\xi(F)\vee\xi(S)\subseteq\xi(F\vee S)$. Let $a\in\xi(F\vee S)$. Then $((a)^{\square}\vee F)\vee((a)^{\square}\vee S)=(a)^{\square}\vee F\vee S=R$. Hence by condition (2), we get $\xi((a)^{\square}\vee S)\vee\xi((a)^{\square}\vee S)=R$. Thus $a\in\xi((a)^{\square}\vee F)\vee\xi((a)^{\square}\vee S)$. Hence $a=r\wedge e$ for some $r\in\xi((a)^{\square}\vee F)$ and $e\in\xi((a)^{\square}\vee S)$. Now

$$\begin{aligned}r\in\xi((a)^{\square}\vee F) &\Rightarrow (r)^{\square}\vee(a)^{\square}\vee F=R\\ &\Rightarrow R=((r)^{\square}\vee(a)^{\square})\vee F\subseteq(r\vee a)^{\square}\vee F\\ &\Rightarrow (r\vee a)^{\square}\vee F=R\\ &\Rightarrow r\vee a\in\xi(F).\end{aligned}$$

Similarly, we can get $e\vee a\in\xi(S)$. Hence

$$\begin{aligned}a &= a\vee a\\ &= a\vee(r\wedge e)\\ &= (a\vee r)\wedge(a\vee e)\in\xi(F)\vee\xi(S).\end{aligned}$$

Hence $\xi(F\vee S)\subseteq\xi(F)\vee\xi(S)$. Therefore $\xi(F)\vee\xi(S)=\xi(F\vee S)$.

(3) $\Rightarrow$ (4) : Assume condition (3). Let P and Q be two distinct maximal filters of R. Choose $a\in P-Q$ and $b\in Q-P$. Since $a\notin Q$, there exists $a_1\in Q$ such that $a\wedge a_1=0$. Since $b\notin P$, there exists $b_1\in P$ such that $b\wedge b_1=0$. Hence $(a\wedge b_1)\wedge(b\wedge a_1)=(a\wedge a_1)\wedge(b\wedge b_1)=0$. Now

$$\begin{aligned}R &= \xi(R)\\ &= \xi([0))\\ &= \xi([(a\wedge b_1)\wedge(b\wedge a_1)))\\ &= \xi([a\wedge b_1)\vee[b\wedge a_1))\\ &= \xi([a\wedge b_1))\vee\xi([b\wedge a_1)) \qquad \text{By condition (4)}\\ &\subseteq \Omega(P)\vee\Omega(Q) \qquad \text{since } [a\wedge b_1)\subseteq P, [b\wedge a_1)\subseteq Q.\end{aligned}$$

Therefore $\Omega(P) \vee \Omega(Q) = R$.

$(4) \Rightarrow (5)$: Assume condition (4). Let $P \in Max_F R$. Suppose $Q \in Max_F R$ such that $Q \neq P$ and $\Omega(Q) \subseteq P$. Since $\Omega(P) \subseteq P$, by hypothesis, we get $R = \Omega(P) \vee \Omega(Q) = P$, which is a contradiction. Therefore P is the unique maximal filter such that $\Omega(P)$ is contained in P.

$(5) \Rightarrow (1)$: Let $P \in Max_F R$. Suppose $\Omega(P)$ is not maximal. Let P_0 be a maximal filter of R such that $\Omega(P) \subseteq P_0$. We have always $\Omega(P_0) \subseteq P_0$, which is a contradiction to the hypothesis. Therefore $\Omega(P)$ is maximal. ∎

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