Discussiones Mathematicae
General Algebra and Applications 44 (2024) 369–382
https://doi.org/10.7151/dmgaa.1468

IDEMPOTENCE AND REGULARITY OF GENERALIZED RELATIONAL HYPERSUBSTITUTIONS FOR ALGEBRAIC SYSTEMS

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Abstract

The concept of a generalized relational hypersubstitution for algebraic systems of type (τ, τ') is an extension of the concept of a generalized hypersubstitutions for universal algebra of type τ. The set of all generalized relational hypersubstitutions for algebraic systems of type (τ, τ') together with a binary operation defined on the set and its identity forms a monoid. The properties of this structure are expressed by terms and relational terms. In this paper, we study the semigroup properties of the monoid of type $((n), (m))$ for arbitrary natural numbers $n, m \geq 2$. In particular, we characterize the idempotent as well as regular elements in this submonoid.

Keywords: generalized hypersubstitutions, algebraic systems, idempotent elements, regular elements.

2020 Mathematics Subject Classification: 20M07, 08B15, 08B25.

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1. Introduction

To study algebraic systems, the first main approach is to produce new algebraic systems of the same type from given ones. The second main approach is to study the semigroup properties of the new algebraic system. Let S be a semigroup, an element $e \in S$ is called idempotent if $e^2 = ee = e$ and the set of all idempotent elements of a semigroup S is denoted by $E(S)$. Let a be an element of a semigroup S, then a is called regular if $a = axa$ for some $x \in S$. A semigroup S is said to be regular semigroup if every element of S is regular and a regular semigroup with identity is called a regular monoid. In universal algebra, identities are used to classify algebras into collections called *varieties* and *hyperidentities* are used to classify varieties into collections called *hypervarieties*. The tool which is used to study hyperidentities and hypervarieties is the concept of a hypersubstitution introduced by Taylor [9]. The notation of a hypersubstitution was originated by Denecke, Lau, Pöschel and Schweigert in 1991 [1]. The authors used this concept for the characterization of solid varieties of type τ. A solid variety is a variety which is closed under the following operation: Taking a universal algebra $(A, (f_i^A)_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ with the universe A and a family $(f_i^A)_{i\in I}$ of n_i-ary operations f_i^A on A for $i \in I$ of a variety, then we replace the operation f_i^A by any n_i-ary term operation $\sigma(f_i)^A$, for $i \in I$, and obtain a new universal algebra $(A, (\sigma(f_i)^A)_{i\in I})$, which also belongs to the variety. Hence, a hypersubstitution of a given type $\tau = (n_i)_{i\in I}$ is a mapping which maps every n_i-ary operation symbol f_i to an n_i-ary term of the same type, for $i \in I$. Moreover, the set $Hyp(\tau)$ of all hypersubstitutions of type τ together with an associative binary operation $\circ_h$ forms a monoid, see more details in [1, 10]. In 2000, Leeratanavalee and Denecke generalized the concept of a hypersubstitution to the concept of a generalized hypersubstitution [3]. Further, a binary operation $\circ_G$ on the set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ was introduced such that $(Hyp_G(\tau), \circ_G)$ is a monoid.

On the other hand, we can consider algebraic systems in the sense of Mal'cev [4]. An algebraic system of type (τ, τ') is a triple $(A, (f_i^A)_{i\in I}, (\gamma_j^A)_{j\in J})$ consisting of a nonempty set A, a sequence $(f_i^A)_{i\in I}$ of n_i-ary operations defined on A and a sequence $(\gamma_j^A)_{j\in J}$ of m_j-ary relations on A, where $\tau = (n_i)_{i\in I}$ is a sequence of the arity of each operation f_i^A and $\tau' = (m_j)_{j\in J}$ is a sequence of the arity of each relation γ_j^A. The pair (τ, τ') is called the *type* of an algebraic system, see more details in [5, 7].

In 2008 [2], Denecke and Phusanga introduced the concept of a hypersubstitution for algebraic systems which is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a formula which preserve the arity. The set of all hypersubstitutions for algebraic systems of type (τ, τ') is denoted by $Hyp(\tau, \tau')$. They defined an associative operation $\circ_r$ on this set and proved that $(Hyp(\tau,\tau^{'}),\circ_r,\sigma_{id})$ forms a monoid where σ_{id} is an identity hypersubstitution. In 2016 [6] Phusanga *et al.* extended this concept to generalized hypersubstitutions for algebraic systems of type $(\tau,\tau,)$. Later, D. Phusanga and J. Koppitz introduced the concept of a relational hypersubstitution for algebraic systems of type (τ,τ') and proved that the set of all relational hypersubstitutions for algebraic systems of type (τ,τ') together with an associative binary operation and the identity element forms a monoid [8]. There are several published papers on algebraic properties of this monoid. In the presen paper, we determine the set of all idempotent elements and regular elements of generalized relational hypersubstitutions for algebraic systems of type $(\tau,\tau')=((n),(m))$.

Next, we recall the concept of an n-ary term of type τ. Let $X := \{x_1,\ldots\}$ be a countably infinite set of symbols called variables. For each $n\geq 1$, let $X_n := \{x_1,\ldots,x_n\}$ be an n-element set which is called an n-element alphabet. Let $\{f_i : i\in I\}$ be the set of n_i-ary operation symbols indexed by the indexed set I, where $n_i\geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function $\tau=(n_i)_{i\in I}$ is called a *type*. An *n-ary term of type* τ is defined inductively as follows.

(i) Every variable $x_k\in X_n$ is an n-ary term of type τ.

(ii) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

We denote the set of all n-ary terms of type τ which contains $x_1,\ldots,x_n$ and is closed under finite application of (ii), by $W_\tau(X_n)$ and $W_\tau(X) := \bigcup_{n\in\mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

2. The monoid of generalized relational hypersubstitutions for algebraic systems

Any generalized relational hypersubstitution for algebraic systems is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a relational term which does not necessarily preserves the arity. Let (τ,τ') be a type. An n-ary relational term of type (τ,τ') and a generalized relational hypersubstitution for algebraic systems are defined as follows.

Definition [5]**.** Let I,J be indexed sets. If $i\in I, j\in J$ and $t_1,t_2,\ldots,t_{m_j}$ are n-ary terms of type $\tau=(n_i)_{i\in I}$ and γ_j is an m_j-ary relation symbol, then $\gamma_j(t_1,t_2,\ldots,t_{m_j})$ is an n-ary relational term of type $(\tau,\tau^{'})=((n_i)_{i\in I},(m_j)_{j\in J})$.

We denote the set of all n-ary relational terms of type $(\tau,\tau^{'})$ by $F^*_{\tau'}(W_\tau(X_n))$ and $F^*_{\tau'}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}} F^*_{\tau'}(W_\tau(X_n))$ be the set of all relational terms of type $(\tau,\tau^{'})$.

A generalized relational hypersubstitution for algebraic systems of type $(\tau, \tau^{'})$ is a mapping

$$\sigma : \{f_i | \ i \in I\} \cup \{\gamma_j | \ j \in J\} \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X)).$$

The set of all generalized relational hypersubstitutions for algebraic systems of type $(\tau, \tau^{'})$ is denoted by $Relhyp_G(\tau, \tau^{'})$. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S^n : W_\tau(X) \times (W_\tau(X))^n \rightarrow W_\tau(X)$ by the following steps. For any $t, t_1, \ldots, t_{n_i}, s_1, \ldots, s_n \in W_\tau(X)$,

(i) if $t = x_j$ for $1 \leq j \leq n$, then $S^n(t, s_1, \ldots, s_n) := s_j$,

(ii) if $t = x_j$ for $n < j$, then $S^n(t, s_1, \ldots, s_n) := x_j$,

(iii) if $t = f_i(t_1, \ldots, t_{n_i})$, then $S^n(t, s_1, \ldots, s_n) := f_i(S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_{n_i}, s_1, \ldots, s_n))$.

For any $F = \gamma_j(s_1, \ldots, s_{m_j}) \in F^*_{\tau'}(W_\tau(X))$, We define the superposition of relational terms $R^n : (W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))) \times (W_\tau(X))^n \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))$ by

(i) $R^n(t, t_1, \ldots, t_n) := S^n(t, t_1, \ldots, t_n)$,

(ii) $R^n(F, t_1, \ldots, t_n) := \gamma_j(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{m_j}, t_1, \ldots, t_n))$.

Every generalized relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \cup F^*_{\tau'}(W_\tau(X)) \rightarrow W_\tau(X) \cup F^*_{\tau'}(W_\tau(X))$ as follows:

(i) $\widehat{\sigma}[x_i] := x_i$ for $i \in \mathbb{N}$,

(ii) $\widehat{\sigma}[f_i(t_1 \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, where $i \in I$ and $t_1, \ldots, t_{n_i} \in W_\tau(X)$, i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1 \leq k \leq n_i$,

(iii) $\widehat{\sigma}[\gamma_j(s_1 \ldots, s_{m_j})] := R^{m_j}(\sigma(\gamma_j), \widehat{\sigma}[s_1], \ldots, \widehat{\sigma}[s_{m_j}])$, where $j \in J, s_1, \ldots, s_{m_j} \in W_\tau(X)$, i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1 \leq k \leq m_j$.

We define a binary operation $\circ_g$ on $Relhyp_G(\tau, \tau^{'})$ by $\sigma \circ_g \alpha := \hat{\sigma} \circ \alpha$ where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Relhyp_G(\tau, \tau^{'})$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$ and maps each m_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1, \ldots, x_{m_j})$.

Throughout this paper, we focus the algebraic systems of type $((n), (m))$. Let f be an n-ary operation symbol and γ be an m-ary relation symbol. We denote the generalized relational hypersubstitution for algebraic systems of type $((n), (m))$ which maps f to a term $t \in W_{(n)}(X)$ and maps γ to a relational term

$F \in F^*_{(m)}(W_{(n)}(X))$ by $\sigma_{t,F}$.

For $t \in W_{(m)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$, we introduce the following notation:

$var(t)$:= the set of all variables occurring in the term t.

$var(F)$:= the set of all variables occurring in the relational term F.

In 2015, Wongpinit and Leeratanavalee [11] introduced the concept of the $i - most$ of terms as follows.

Definition [11]**.** Let $\tau = (n)$ be a type with an n-ary operation symbol f, $t \in W_{(n)}(X)$ and $1 \leq i \leq n$. An $i - most(t)$ is defined inductively as follows:

(i) if t is a variable, then $i - most(t) = t$,

(ii) if $t = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n \in W_{(n)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 1. Let $\tau = (3)$ with ternary operation symbol f. Let $t = f(f(x_5, x_1, x_2), x_3, f(x_8, x_1, x_9))$. Then $1 - most(t) = 1 - most(f(x_5, x_1, x_2)) = x_5, 2 - most(t) = 2 - most(x_3) = x_3$ and $3 - most(t) = 3 - most(f(x_8, x_1, x_9)) = x_9$.

Lemma 2 [11]**.** *Let $s, t \in W_{(n)}(X)$. If $j - most(t) = x_k \in X_n$ and $k - most(s) = x_i$, then $j - most(\hat{\sigma}_t[s]) = x_i$.*

Example 3. Let $\tau = (3)$ be a type. Let $s = f(x_2, x_7, f(x_4, x_3, x_1)) \in W_{(3)}(X)$ and a hypersubstitution σ_t with $t = f(x_3, f(x_5, x_3, x_2), x_8) \in W_{(3)}(X)$. Then we have $2 - most(t) = 2 - most(f(x_5, x_3, x_2)) = x_3$ and $3 - most(s) = 3 - most(f(x_4, x_3, x_1)) = x_1$. Consider

$$\begin{aligned}
\widehat{\sigma}_t[s] &= \widehat{\sigma}_t[f(x_2, x_7, f(x_4, x_3, x_1))] \\
&= S^3(\sigma_t(f), x_2, x_7, S^3(\sigma_t(f), x_4, x_3, x_1)) \\
&= S^3(t, x_2, x_7, f(x_1, f(x_5, x_1, x_3), x_8)) \\
&= f(f(x_1, f(x_5, x_1, x_3), x_8), f(x_5, f(x_1, f(x_5, x_1, x_3), x_8), x_7), x_8) \\
&= f(t_1, t_2, t_3)
\end{aligned}$$

where $t_1 = f(x_1, f(x_5, x_1, x_3), x_8)$, $t_2 = f(x_5, f(x_1, f(x_5, x_1, x_3), x_8), x_7)$ and $t_3 = x_8$. Then $2 - most(\widehat{\sigma}_t[s]) = 2 - most(f(x_5, f(x_1, f(x_5, x_1, x_3), x_8), x_7)) = 2 - most(f(x_1, f(x_5, x_1, x_3), x_8)) = 2 - most(f(x_5, x_1, x_3)) = x_1$. So $3 - most(s) = x_1 = 2 - most(\widehat{\sigma}_t[s])$. Hence, we can see that if $2 - most(t) = x_3$, then $3 - most(s) = 2 - most(\widehat{\sigma}_t[s])$.

The above lemma can be applied to any generalized relational hypersubstitution for algebraic systems of type $((n), (m))$, such as the following: Let $s, t \in W_{(n)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$. If $i - most(t) = x_j$, then $i - most(\widehat{\sigma}_{t,F}[s]) = j - most(s)$.

3. ALL IDEMPOTENT ELEMENTS IN $Relhyp_G((n),(m))$

In this section, we especially focus on idempotent elements of $Relhyp_G((n),(m))$, for natural numbers $n, m \geq 2$. For any $\sigma_{t,F} \in Relhyp_G((n),(m))$, where $t \in W_{(n)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$, $\sigma_{t,F}$ is called idempotent if and only if $\sigma_{t,F} \circ_g \sigma_{t,F} = \sigma_{t,F}$. Then if $\sigma_{t,F}$ is idempotent, we have

$$(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[t] = t \text{ and}$$
$$(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[F] = F.$$

Theorem 4. *Let $t = x_i \in X_n$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is idempotent if and only if one of the following conditions are satisfied:*

(i) *$var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $i-most(s_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$,*
(ii) *$var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ is idempotent. Then $t = \widehat{\sigma}_{t,F}[t]$ and $F = \widehat{\sigma}_{t,F}[F] = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. Assume that $var(F) \cap X_m \neq \emptyset$, let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ we will show that $i - most(s_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$. Consider

$$\begin{aligned} R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m]) &= \gamma(S^n(s_1, \widehat{\sigma}_{x_i,F}[s_1], \ldots, \widehat{\sigma}_{x_i,F}[s_m]), \ldots, \\ &\quad S^n(s_m, \widehat{\sigma}_{x_i,F}[s_1], \ldots, \widehat{\sigma}_{x_i,F}[s_m])) \\ &= \gamma(S^n(s_1, i - most(s_1), \ldots, i - most(s_m)), \ldots, \\ &\quad S^n(s_m, i - most(s_1), \ldots, i - most(s_m))). \end{aligned}$$

Since $var(F) \cap X_m \neq \emptyset$ and $F = \widehat{\sigma}_{t,F}[F]$, we know that $x_{b_l} \in var(F)$ for all $l = 1, \ldots, j$ is replaced by $\widehat{\sigma}_{t,F}[s_{b_l}]$. So $i - most(s_{b_l}) = \widehat{\sigma}_{t,F}[s_{b_l}] = x_{b_l}$.

Conversely, if $i - most(s_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$, then

$$\begin{aligned} (\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}[F]] \\ &= \gamma(S^n(s_1, i - most(s_1), \ldots, i - most(s_m)), \ldots, \\ &\quad S^n(s_m, i - most(s_1), \ldots, i - most(s_m))). \end{aligned}$$

We substitue x_{b_l} in the relational term F by $i-most(s_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$. So $F = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. If $var(F) \cap X_n = \emptyset$, then it is easily seen that $(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = F = \sigma_{t,F}(\gamma)$. By straightforward calculations, we obtain $(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = t = \sigma_{t,F}(f)$. Therefore, $\sigma_{t,F}$ is idempotent. ■

Theorem 5. *Let $t \in W_{(n)}(X \backslash X_n)$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is idempotent if and only if one of the following conditions are satisfied:*

(i) *$var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{b_l}$ for all $l = 1, \ldots, j$,*
(ii) *$var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ is idempotent. Then $t = \widehat{\sigma}_{t,F}[t]$ and $F =! \widehat{\sigma}_{t,F}[F] = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. Assume that $var(F) \cap X_m \neq \emptyset$, let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ and $s_{b_l} \neq x_{b_l}$, say $s_{b_l} = x_{b_k}$. Consider

$$\begin{aligned} R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m]) &= R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_{b_l}], \ldots, \widehat{\sigma}_{t,F}[s_m]) \\ &= R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, x_{b_k}, \ldots, \widehat{\sigma}_{t,F}[s_m]). \end{aligned}$$

Since $var(F) \cap X_m \neq \emptyset$ and $F = \widehat{\sigma}_{t,f}[F]$, we know that $x_{b_l} \in var(F)$ for all $l = 1, \ldots, j$ is replaced by $\widehat{\sigma}_{t,F}[s_{b_l}]$. So $F \neq R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$, which is a contradiction.

Conversely, if $s_{b_l} = x_{b_l}$ for all $l = 1, \ldots, j$, then

$$\begin{aligned} (\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\sigma_{t,F}[F]] \\ &= R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_{b_l}], \ldots, \widehat{\sigma}_{t,F}[s_m]) \\ &= R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, x_{b_k}, \ldots, \widehat{\sigma}_{t,F}[s_m]). \end{aligned}$$

Since we substitute x_{b_l} in the relational term F by $\widehat{\sigma}_{t,F}[s_{b_l}] = x_{b_l}$ for all $l = 1, \ldots, j$. So $F = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. If $var(F) \cap X_m = \emptyset$, it is easy to check that $F = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. By straightforward calculations, we obtain $(\sigma_{t,F} \circ_g \sigma_{t,F})(f) = t = \sigma_{t,F}(f)$. Therefore, $\sigma_{t,F}$ is idempotent. ∎

Theorem 6. *Let $t \in W_{(n)}(X) \backslash X$ such that $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ where $t_{a_k} = x_{a_k}$ for all $k = 1, \ldots, i$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is idempotent if and only if one of the following conditions are satisfied:*

(i) *$var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $s_{b_l} = x_{b_l}$ for all $l = 1, \ldots, j$,*

(ii) *$var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ is idempotent. Then $t = S^n(t, \widehat{\sigma}_{t,F}[t_1], \ldots, \widehat{\sigma}_{t,F}[t_n])$ and $F = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. Assume that $var(F) \cap X_m \neq \emptyset$, let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ we will show that $s_{b_l} = x_{b_l}$ for all $l = 1, \ldots, j$. Let $s_{b_l} \in W_{(n)}(X_n \backslash X)$. Consider

$$\begin{aligned} x_{b_l} &= \widehat{\sigma}_{t,F}[s_{b_l}] \\ &= R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m]) \\ &= R^m(f(s_1, \ldots, s_m), \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m]) \\ &= \gamma(S^m(s_1, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m]), \ldots, S^m(s_m, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])) \notin X_m. \end{aligned}$$

So $s_{b_l} \in X_m$ for all $l = 1, \ldots, j$. If $s_{b_l} = x_q \in X_m$, then $x_{b_l} = \widehat{\sigma}_{t,F}[s_{b_l}] = \widehat{\sigma}_{t,F}[x_q] = x_q$. Hence $s_{b_l} = x_{b_l}$.

Conversely, let $t_{a_k} = x_{a_k}$ for all $k = 1, \ldots, i$ and $s_{b_l} = x_{b_l}$ for all $l = 1, \ldots, j$, then

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[t] \\ &= S^n(t, \widehat{\sigma}_{t,F}[t_1], \ldots, \widehat{\sigma}_{t,F}[t_{a_k}], \ldots, \widehat{\sigma}_{t,F}[t_n]) \\ &= S^n(t, \widehat{\sigma}_{t,F}[t_1], \ldots, x_{a_k}, \ldots, \widehat{\sigma}_{t,F}[t_n]).\end{aligned}$$

Since we substitute x_{a_k} in the term t by $\widehat{\sigma}_{t,F}[t_{a_k}] = x_{a_k}$ for all $k = 1, \ldots, i$. So $t = S^n(t, \widehat{\sigma}_{t,F}[t_1], \ldots, \widehat{\sigma}_{t,F}[t_n])$. Similarly, we have $(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = F = \sigma_{t,F}(\gamma)$. If $var(F) \cap X_m = \emptyset$, it is easy to check that $F = R^m(F, \widehat{\sigma}_{t,F}[s_1], \ldots, \widehat{\sigma}_{t,F}[s_m])$. Hence $(\sigma_{t,F} \circ_g \sigma_{t,F})(\gamma) = F = \sigma_{t,F}(\gamma)$. Therefore, $\sigma_{t,F}$ is idempotent. ■

4. All regular elements in $Relhyp_G((n),(m))$

In this section, we determine all regular elements of $Relhyp_G((n),(m))$, for natural numbers $n, m \geq 2$. For any $\sigma_{t,F} \in Relhyp_G((n),(m))$, where $t \in W_{(n)}(X)$ and $F \in F^*_{(m)}(W_{(n)}(X))$, $\sigma_{t,F}$ is call regular if there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that $\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F} = \sigma_{t,F}$. Then if $\sigma_{t,F}$ is regular, $\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] = t$ and $\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]] = F$.

Lemma 7. *Let $t, u \in W_{(n)}(X)$ and $F, H \in F^*_{(m)}(W_{(n)}(X))$ such that $F = \widehat{\sigma}_{t,F}[H]$ with $x_j \in var(F)$. Then we have:*

(i) *if $t = x_k \in X_n$, then $k - most(h_j) = x_j$;*

(ii) *if $t \in W_{(n)}(X) \backslash X$, then $h_j = x_j$.*

Proof. Let $F = \widehat{\sigma}_{t,F}[H]$ with $x_j \in var(F)$. We will show that (i), (ii) hold.

(i) Let $t = x_k \in X_n$ and $F = R^m(F, \widehat{\sigma}_{x_k,F}[h_1], \ldots, \widehat{\sigma}_{x_k,F}[h_m]) = R^m(F, k - most(h_1), \ldots, k - most(h_m))$. Since $x_j \in var(F)$ and $F = \widehat{\sigma}_{t,F}[H]$, we have to replace a variable x_j in the relational term F by $k - most(h_j)$. So $k - most(h_j) = x_j$.

(ii) Let $t \in W_{(n)}(X) \backslash X$ and $F = R^m(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_m])$. Since $x_j \in var(F)$ and $F = \widehat{\sigma}_{t,F}[H]$, we have to replace a variable x_j in the relational term F by $\widehat{\sigma}_{t,F}[h_j]$. So $h_j = x_j$. Therefore the proof is complete. ■

Example 8. Let $(\tau, \tau') = ((3),(3))$ with a ternary operation symbol f and a ternay relation symbol γ.

Case (i). Let $t = x_2$, $F = \gamma(x_1, f(x_3, x_1, x_5), x_7)$ and $H = \gamma(f(x_5, x_1, x_8),$

$x_1, x_3)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{t,F}[H] \\ &= R^3(F, \widehat{\sigma}_{t,F}[f(x_5, x_1, x_8)], \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[x_3]) \\ &= R^3(F, S^3(t, x_5, x_1, x_8), x_1, x_3) \\ &= R^3(F, x_1, x_1, x_3) \\ &= \gamma(x_1, f(x_3, x_1, x_5), x_7)\end{aligned}$$

Case (ii). Let $t = f(x_3, x_5, x_7)$, $F = \gamma(x_1, f(x_3, x_1, x_5), x_7)$ and $H = \gamma(x_1, x_9, x_3)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{t,F}[H] \\ &= R^3(F, \widehat{\sigma}_{t,F}[x_1], \widehat{\sigma}_{t,F}[x_9], \widehat{\sigma}_{t,F}[x_3]) \\ &= R^3(F, x_1, x_9, x_3) \\ &= \gamma(x_1, f(x_3, x_1, x_5), x_7)\end{aligned}$$

Theorem 9. *Let $t = x_i \in X_n$ and $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if one of the following conditions are satisfied:*

(i) *$var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ such that $k-most(s_{b'_l}) = x_{b_l}$ where $\{b'_1, \ldots, b'_j\} \subseteq \{1, \ldots, m\}$,*

(ii) *$var(F) \cap X_m = \emptyset$.*

Proof. Let $\sigma_{t,F}$ is regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n), (m))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}$. Assume that $var(F) \cap X_m \neq \emptyset$, let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ we will show that $k - most(s_{b'_l}) = x_{b_l}$ for all $l = 1, \ldots, j$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{u,H}[F] \\ &= R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]) \\ &= \gamma(S^m(h_1, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]), \ldots, \\ &\quad S^m(h_m, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])) \\ &= \gamma(a_1, \ldots, a_m),\end{aligned}$$

where $a_i = S^m(h_i, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])$ for all $i = 1, \ldots, m$. Since $F = \widehat{\sigma}_{x_i,F}[\widehat{\sigma}_{u,H}[F]] = \widehat{\sigma}_{x_i,F}[\gamma(a_1, \ldots, a_m)]$ and $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$, by Lemma 7(i) we have $i - most(a_{b_l}) = x_{b_l}$ for all $l = 1, \ldots, j$. So

$$\begin{aligned}x_{b_l} &= i - most(a_{b_l}) \\ &= i - most(S^m(h_{b_l}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])) \\ &= S^m(i - most(h_{b_l}), i - most(\widehat{\sigma}_{u,H}[s_1]), \ldots, i - most(\widehat{\sigma}_{u,H}[s_m])).\end{aligned}$$

Let $i-most(h_{b_l})=x_{b'_l}$ for some $b'_l=b_1,\ldots,b_j$ and $i-most(u)=x_k\in X_m$. By Lemma 2, we have $x_{b_l}=S^m(x_{b'_l},i-most(\widehat{\sigma}_{u,H}[s_1]),\ldots,i-most(\widehat{\sigma}_{u,H}[s_n]))=i-most(\widehat{\sigma}_{u,H}[s_{b'_l}])=k-most(s_{b'_l})$.

Conversly, choose $\sigma_{u,H}\in Relhyp_G((n),(m))$ such that $u=x_k, H=\gamma(h_1,\ldots,h_m)$ where $h_{b_l}=x_{b'_l}$ for all $l=1,\ldots,j$. Consider

$$\begin{aligned}(\sigma_{t,F}\circ_g\sigma_{u,H}\circ_g\sigma_{t,F})(\gamma)&=\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]]\\&=\widehat{\sigma}_{t,F}[R^m(H,\widehat{\sigma}_{x_k,H}[s_1],\ldots,\widehat{\sigma}_{x_k,H}[s_m])]\\&=\widehat{\sigma}_{t,F}[R^m(\gamma(h_1,\ldots,h_m),k-most(s_1),\ldots,\\&\quad k-most(s_m))]\\&=\widehat{\sigma}_{t,F}[\gamma(S^m(h_1,k-most(s_1),\ldots,k-most(s_m)),\ldots,\\&\quad S^m(h_m,k-most(s_1),\ldots,k-most(s_m)))]\\&=\widehat{\sigma}_{t,F}[\gamma(a_1,\ldots,a_m)]\end{aligned}$$

where $a_i=S^m(h_i,k-most(s_1),\ldots,k-most(s_m))$ for all $i=1,\ldots,m$.

Case I. $k-most(s_{b'_l})=x_{b_l}$. So

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(a_1,\ldots,a_m)]&=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[a_m])\\&=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[S^m(h_{b_l},k-most(s_1),\ldots,\\&\quad k-most(s_m))],\ldots,\widehat{\sigma}_{t,F}[a_m])\\&=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[S^m(x_{b'_l},k-most(s_1),\ldots,\\&\quad k-most(s_m))],\ldots,\widehat{\sigma}_{t,F}[a_m])\\&=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[k-most(s_{b'_l})],\ldots,\widehat{\sigma}_{t,F}[a_m])\\&=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[x_{b_l}],\ldots,\widehat{\sigma}_{t,F}[a_m])\\&=F;\ \text{since } x_{b_l}\in var(F) \text{ must be replaced by } \widehat{\sigma}_{t,F}[a_{b_l}]=x_{b_l}.\end{aligned}$$

Case II. $var(F)\cap X_m=\emptyset$. So

$$\widehat{\sigma}_{t,F}[\gamma(a_1,\ldots,a_m)]=R^m(F,\widehat{\sigma}_{t,F}[a_1],\ldots,\widehat{\sigma}_{t,F}[a_m])=F.$$

It is esely to calculate that $(\sigma_{t,F}\circ_g\sigma_{u,H}\circ_g\sigma_{t,F})(f)=\widehat{\sigma}_{t,F}(f)$. Therefore $\sigma_{t,F}$ is regular. ∎

Theorem 10. *Let $t\in W_{(n)}(X\backslash X_n)$ and $F=\gamma(s_1,\ldots,s_m)\in F^*_{(m)}(W_{(n)}(X))$. Then $\sigma_{t,F}$ is regular if and only if one of the following conditions are satisfied:*

(i) *$var(F)\cap X_m=\{x_{b_1},\ldots,x_{b_j}\}$ such that $s_{b'_l}=x_{b_l}$ where $\{b'_1,\ldots,b'_j\}\subseteq\{1,\ldots,m\}$,*

(ii) *$var(F)\cap X_m=\emptyset$.*

Proof. Let $\sigma_{t,F}$ is regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}$. Assume $var(F) \cap X_m \neq \emptyset$, let $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ and $s_{b'_l} \neq x_{x_l}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]] \\ &= \widehat{\sigma}_{t,F}[R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[\gamma(S^m(h_1, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]), \ldots, \\ &\quad S^m(h_m, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]))] \\ &= \widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)] \\ &= R^m(F, \sigma_{t,F}[a_1], \ldots, \sigma_{t,F}[a_m]) \\ &= R^m(\gamma(s_1, \ldots, s_m), \sigma_{t,F}[a_1], \ldots, \sigma_{t,F}[a_m]) \\ &= \gamma(S^m(s_1, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m]), \ldots, \\ &\quad S^m(s_m, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m])).\end{aligned}$$

By Lemma 7(ii), we have $\widehat{\sigma}_{t,F}[a_{b_l}] = x_{b_l}$. Since $x_{b_l} \in var(F)$, we have to replace a variable x_{b_l} in the relational term F by $\widehat{\sigma}_{t,F}[a_{b_l}]$. Consider $\widehat{\sigma}_{t,F}[a_{b_l}] = \widehat{\sigma}_{t,F}[S^m(h_{b_l}, \sigma_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])]$ implies that $h_{b_l} \in X_m$ for all $l = 1, \ldots, j$. Therefore there exists $b'_l \in \{1, \ldots, m\}$ such that $h_{b_l} = x_{b'_l}$. Then

$$\begin{aligned}\widehat{\sigma}_{t,F}[a_{b_l}] &= \widehat{\sigma}_{t,F}[S^n(x_{b'_l}, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_{b'_l}], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[s_{b'_l}]] \\ &\neq x_{b_l}.\end{aligned}$$

So $(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{u,H})(\gamma) = \widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m) \neq F$. Conversly, choose $\sigma_{u,H} \in Relhyp_G((n),(m))$ with $u = f(u_1, \ldots, u_n)$ and $H = \gamma(h_1, \ldots, h_m)$ such that $h_{b_l} = x_{b'_l}$ for all $l = 1, \ldots, j$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[F]] \\ &= \widehat{\sigma}_{t,F}[R^m(H, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[R^m(\gamma(h_1, \ldots, h_m), \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])] \\ &= \widehat{\sigma}_{t,F}[\gamma(S^m(h_1, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]), \ldots, \\ &\quad S^m(h_m, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m]))] \\ &= \widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)]\end{aligned}$$

where $a_i = S^m(h_i, \widehat{\sigma}_{u,H}[s_1], \ldots, \widehat{\sigma}_{u,H}[s_m])$ for all $i = 1, \ldots, m$.

Case I. $s_{b'_l} = x_{b_l}$ for all $l = 1, \ldots, j$. So

$$\begin{aligned}
\widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)] &= R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m]) \\
&= R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[S^m(h_{b_l}, \widehat{\sigma}_{u,H}[s_1], \\
&\quad \ldots, \widehat{\sigma}_{u,H}[s_m])], \ldots, \widehat{\sigma}_{t,F}[a_m]) \\
&= R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[S^m(x_{b'_l}, \widehat{\sigma}_{u,H}[s_1], \\
&\quad \ldots, \widehat{\sigma}_{u,H}[s_m])], \ldots, \widehat{\sigma}_{t,F}[a_m]) \\
&= R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[s_{b'_l}]], \ldots, \widehat{\sigma}_{t,F}[a_m]) \\
&= R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[x_{b_l}], \ldots, \widehat{\sigma}_{t,F}[a_m]) \\
&= F; \text{ since } x_{b_l} \in var(F) \text{ must be replaced by } \widehat{\sigma}_{t,F}[a_{b_l}] = x_{b_l}.
\end{aligned}$$

Case II. $var(F) \cap X_n = \emptyset$. So

$$\widehat{\sigma}_{t,F}[\gamma(a_1, \ldots, a_m)] = R^m(F, \widehat{\sigma}_{t,F}[a_1], \ldots, \widehat{\sigma}_{t,F}[a_m]) = F.$$

It is esely to calculate that $(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}(f)$. Therefore $\sigma_{t,F}$ is regular. ∎

Theorem 11. *Let* $t \in W_{(n)}(X) \backslash X$ *such that* $var(t) \cap X_n = \{x_{a_1}, \ldots, x_{a_i}\}$ *where* $t_{a'_k} = x_{a_k}$ *where* $\{a'_1, \ldots, a'_i\} \subseteq \{1, \ldots, n\}$ *and* $F = \gamma(s_1, \ldots, s_m) \in F^*_{(m)}(W_{(n)}(X))$. *Then* $\sigma_{t,F}$ *is regular if and only if one of the following conditions are satisfied:*

(i) $var(F) \cap X_m = \{x_{b_1}, \ldots, x_{b_j}\}$ *such that* $s_{b'_l} = x_{b_l}$ *where* $\{b'_1, \ldots, b'_j\} \subseteq \{1, \ldots, m\}$,

(ii) $var(F) \cap X_m = \emptyset$.

Proof. Let $\sigma_{t,F}$ is regular. Then there exists $\sigma_{u,H} \in Relhyp_G((n),(m))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F}$. The proof is similar to Theorem 10. Conversely, choose $\sigma_{u,H} \in Relhyp_G((n),(m))$ with $u = f(u_1, \ldots, u_n)$ such that $u_{a_k} = x_{a'_k}$ for all $k = 1, \ldots, i$ and $H = \gamma(h_1, \ldots, h_m)$ such that $h_{b_l} = x_{b'_l}$ for all $l = 1, \ldots, j$. Consider

$$\begin{aligned}
(\sigma_{t,F} \circ_g \sigma_{u,H} \circ_g \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t]] \\
&= \widehat{\sigma}_{t,F}[S^n(f(u_1, \ldots, u_n), \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])] \\
&= \widehat{\sigma}_{t,F}[f(S^n(u_1, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]), \ldots, \\
&\quad S^n(u_n, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n]))] \\
&= \widehat{\sigma}_{t,F}[f(w_1, \ldots, w_n)]
\end{aligned}$$

where $w_j = S^n(u_j, \widehat{\sigma}_{u,H}[t_1], \ldots, \widehat{\sigma}_{u,H}[t_n])$ for all $j = 1, \ldots, n$. Since $u_{a_k} = x_{a'_k}$ for all $k = 1, \ldots, i$. So

$$
\begin{aligned}
\widehat{\sigma}_{t,F}[f(w_1,\ldots,w_n)] &= S^n(t,\widehat{\sigma}_{t,F}[w_1],\ldots,\widehat{\sigma}_{t,F}[w_n])\\
&= S^n(t,\widehat{\sigma}_{t,F}[w_1],\ldots,\widehat{\sigma}_{t,F}[S^n(u_{a_k},\widehat{\sigma}_{u,H}[t_1],\\
&\quad \ldots,\widehat{\sigma}_{u,H}[t_n])],\ldots,\widehat{\sigma}_{t,F}[w_n])\\
&= S^n(t,\widehat{\sigma}_{t,F}[w_1],\ldots,\widehat{\sigma}_{t,F}[S^n(x_{a'_k},\widehat{\sigma}_{u,H}[t_1],\\
&\quad \ldots,\widehat{\sigma}_{u,H}[t_n])],\ldots,\widehat{\sigma}_{t,F}[w_n])\\
&= S^n(t,\widehat{\sigma}_{t,F}[w_1],\ldots,\widehat{\sigma}_{t,F}[\widehat{\sigma}_{u,H}[t_{a'_k}]],\ldots,\widehat{\sigma}_{t,F}[w_n])\\
&= S^n(t,\widehat{\sigma}_{t,F}[w_1],\ldots,\widehat{\sigma}_{t,F}[x_{a_k}],\ldots,\widehat{\sigma}_{t,F}[w_n])\\
&= t;\ \text{since } x_{a_k}\in var(t) \text{ must be replaced by } \widehat{\sigma}_{t,F}[w_{a_k}] = x_{a_k}.
\end{aligned}
$$

Similarly, we have $(\sigma_{t,F}\circ_g\sigma_{u,H}\circ_g\sigma_{u,H})(\gamma) = \widehat{\sigma}_{t,F}(\gamma)$. Therefore $\sigma_{t,F}$ is regular. ∎

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Received 7 March 2023
Revised 9 June 2023
Accepted 10 June 2023