Discussiones Mathematicae
General Algebra and Applications 44 (2024) 343–368
https://doi.org/10.7151/dmgaa.1467

THE SPACE OF MINIMAL PRIME D-FILTERS OF ALMOST DISTRIBUTIVE LATTICES

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Abstract

The concept of D-filters is introduced in an Almost Distributive Lattice (ADL) and studied their properties. An equivalency is established between the minimal prime D-filters of an ADL and its quotient ADL with respect to a congruence. Finally, some properties of prime D-filters and minimal prime D-filters of an ADL are studied topologically.

Keywords: Almost Distributive Lattice (ADL), prime filter, D-filter, D-normal ADL, congruence, compact, Hausdorff space, closure.

2020 Mathematics Subject Classification: 06D99, 06D15.

1. Introduction

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [9] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set of all principal ideals of an ADL forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. The concept of normal lattices was introduced by Cornish, in [2]. In [6] Rao and Ravi Kumar, introduced the concept of a minimal prime ideal belonging to an ideal of an ADL and studied their important properties. In [7], the concept of

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a normal ADL was given by Rao and Ravi Kumar. In that, authors have given equivalent conditions for an ADL to become normal in terms of its annulets. The notion of D-filters in lattices was introduced and studied their properties in [4] by Kumar *et al.* In that paper, a set of equivalent conditions was established for every proper D-filter of a lattice to become a prime D-filter. In this paper, the concepts of D-filters and prime D-filters are introduced in an ADL and studied their properties. A set of equivalent conditions is derived for every proper D-filter of an ADL to become a prime D-filter. We proved that every maximal D-filter of an ADL is a prime D-filter. Also, proved that for any prime D-filter M of an ADL R, $\mathcal{O}^D(M) = \{x \in R \mid x \in (a, D)$, for some $a \in R \setminus M\}$ is the intersection of all minimal prime D-filters contained in M. After that, we introduced the concept of D-normal ADL and it characterized in terms of relative annihilators with respect to a filter D. Derived an equivalency between the minimal prime D-filters of an ADL and its quotient ADL with respect to a congruence. Studied some topological properties of the space of all prime D-filters and the space of all minimal prime D-filters of an ADL.

2. Preliminaries

In this section, we recall certain definitions and important results from [5] and [9], those will be required in the text of the paper.

Definition [9]**.** An algebra $R = (R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(3) $(a \vee b) \wedge b = b$

(4) $(a \vee b) \wedge a = a$

(5) $a \vee (a \wedge b) = a$

(6) $0 \wedge a = 0$

(7) $a \vee 0 = a$, for all $a, b, c \in R$.

Example 1. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \{\, x \text{ if } x \neq x_0,\ y \text{ if } x = x_0; \quad x \wedge y = \{\, y \text{ if } x \neq x_0,\ x_0 \text{ if } x = x_0.$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

Theorem 2 [9]. *If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in R$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* R

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(7) $a \wedge (a \vee b) = a, \quad (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(8) $a \wedge a = a$ *and* $a \vee a = a$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

As in distributive lattices [1, 3], a non-empty subset I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset F of R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $\mathfrak{I}(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in \mathfrak{I}(R)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal(filter) P of R is called a prime ideal (filter) if, for any $x, y \in R$, $x \wedge y \in P (x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. A proper ideal(filter) M of R is said to be maximal if it is not properly contained in any proper ideal(filter) of R. It can be observed that every maximal ideal (filter) of R is a prime ideal (filter). Every proper ideal(filter) of R is contained in a maximal ideal (filter). For any subset S of R the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$ and such an ideal is called the principal ideal of R. Similarly, for any $S \subseteq R$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in R \text{ and } n \in \mathbb{N}\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$ and such a filter is called the principal filter of R.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \wedge (b] = (a \wedge b]$. Hence the set $(\mathfrak{I}^{PI}(R), \vee, \cap)$ of all principal ideals of R is a sublattice of the distributive lattice $(\mathfrak{I}(R), \vee, \cap)$ of all ideals of R. Also, we have that the set $(\mathfrak{F}(R), \vee, \cap)$ of all filters of R is a bounded distributive lattice.

Theorem 3 [6]**.** *Let R be an ADL with maximal elements. Then P is a prime ideal of R if and only if $R \setminus P$ is a prime filter of R.*

Definition [5]**.** An ADL R is said to be an associate ADL, if the operation $\vee$ is associative on R.

Definition [8]**.** For any nonempty subset A of an ADL R, define $A^* = \{x \in R \mid a \wedge x = 0 \text{ for all } a \in A\}$. Here A^* is called the annihilator of A in R.

For any $a \in R$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a. An element a of an ADL R is called dense element if $(a]^* = \{0\}$ and the set D of all dense elements in ADL is a filter if D is non-empty.

3. D-FILTERS OF ADLs

In this section, the concepts of D-filters and prime D-filters are introduced in an ADL and studied their properties. A set of equivalent conditions is derived for every proper D-filter of an ADL to become a prime D-filter. It is observed that every maximal D-filter of an ADL is a prime D-filter and also observed that $\mathcal{O}^D(M)$ is the intersection of all minimal prime D-filters contained in prime D-filter M. The concept of D-normal ADLs is introduced and it is characterized in terms of relative annihilators with respect to a filter D. An equivalency is derived between minimal prime D-filters of an ADL and its quotient ADL with respect to a congruence.

Definition. A filter G of R is said to be a D-filter of R if $D \subseteq G$.

Now we have the example of D-filter of an ADL.

Example 4. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee$, $\wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee, \wedge)$ is an ADL. Clearly, we have that $D = \{1, 2, 6\}$ and $G = \{1, 2, 3, 6\}$ are filters of R satisfying $D \subseteq G$. Therefore G is a D-filter of R.

It is easy to verify the proof of the following result.

Lemma 5. *For any non-empty subset A of an ADL R, $[A) \vee D$ is the smallest D-filter of R containing A.*

We denote $[A) \vee D$ by A^D, i.e., $A^D = [A) \vee D$. For, $A = \{a\}$, we denote simply $(a)^D$ for $\{a\}^D$. Clearly, we have that $(a)^D$ is the smallest D-filter containing a, which is known as the principal D-filter generated by a.

Lemma 6. *For any two elements x, y of an ADL R with maximal element m, we have the following:*

(1) $(0)^D = R$

(2) $(m)^D = D$

(3) $x \leq y$ *implies* $(y)^D \subseteq (x)^D$

(4) $(x \wedge y)^D = (x)^D \vee (y)^D$

(5) $(x \vee y)^D = (x)^D \cap (y)^D$

(6) $(x)^D = D$ *if and only if* $x \in D$.

Proof. (1) Now $(0)^D = [0) \vee D = R \vee D = R$.

(2) Now $(m)^D = [m) \vee D = \{m\} \vee D \subseteq D$. Clearly, we have $D \subseteq (m)^D$. Therefore $D = (m)^D$.

(3) Let $x \leq y$. Then $[y) \subseteq [x)$. Now $(y)^D = [y) \vee D \subseteq [x) \vee D = x^D$. Therefore $(y)^D \subseteq (x)^D$.

(4) Clearly, we have that $[x \wedge y) = [x) \vee [y)$. Now, $(x \wedge y)^D = [x \wedge y) \vee D = [x) \vee [y) \vee D = ([x) \vee D) \vee ([y) \vee D)) = (x)^D \vee (y)^D$. Therefore $(x \wedge y)^D = (x)^D \vee (y)^D$.

(5) Since $x \leq x \vee y$ and $y \leq y \vee x$ and hence $[x \vee y) \subseteq [x)$ and $[y \vee x) \subseteq [y)$. Since $[x \vee y) = [y \vee x)$, we get that $[x \vee y) \subseteq [x) \cap [y)$. Let $t \in [x) \cap [y)$. Then $t \in [x)$ and $t \in [y)$. That implies $t \vee x = t$ and $t \vee y = t$. Now $t \wedge (x \vee y) = (t \wedge x) \vee (t \wedge y) = x \vee y$. That implies $t \vee (x \vee y) = t$ and hence $t \in [x \vee y)$. Therefore $[x) \cap [y) \subseteq [x \vee y)$. Thus $[x \vee y) = [x) \cap [y)$. Now $(x \vee y)^D = [x \vee y) \vee D = [[x) \cap [y)] \vee D = ([x) \vee D) \cap ([y) \vee D) = (x)^D \cap (y)^D$. Hence $(x \vee y)^D = (x)^D \cap (y)^D$.

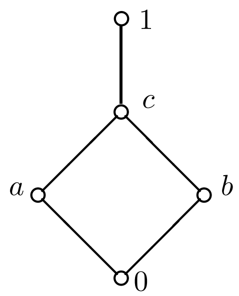
(6) Assume that $(x)^D = D$. Then $[x) \vee D = D$. That implies $[x) \subseteq D$ and hence $x \in D$. Conversely, assume that $x \in D$. Then $[x) \subseteq D$. This implies that $[x) \vee D \subseteq D$. Since $D \subseteq [x) \vee D$, we get that $D = [x) \vee D$. Therefore $(x)^D = D$. ■

We denote $\mathfrak{F}(R), \mathfrak{F}^D(R)$ and $\mathfrak{F}^{PDF}(R)$ as the set of all filters, D-filters and principal D-filters of an ADL R, respectively.

Theorem 7. *$\mathfrak{F}^D(R)$ forms a distributive lattice contained in $\mathfrak{F}(R)$, and $\mathfrak{F}^{PDF}(R)$ forms a sublattice of $\mathfrak{F}^D(R)$.*

Definition. An D-filter Q is said to be proper if $Q \subsetneq R$. A proper D-filter Q is said to be maximal if it is not properly contained in any proper D-filter of R. A proper D-filter Q of an ADL R is said to be a prime D-filter if Q is prime filter of R.

Example 8. Consider a distributive lattice $L = \{0, a, b, c, 1\}$ and discrete ADL $A = \{0', a'\}$.



Clearly, $R = A \times L = \{(0', 0), (0', a), (0', b), (0', c), (0', 1), (a', 0), (a', a), (a', b), (a', c), (a', 1)\}$ is an ADL with zero element $(0, 0')$. Clearly, the dense set $D = \{(a', c), (a', 1)\}$. Consider the D-filters

$F_1 = \{(0', a), (0', c), (0', 1), (a', a), (a', 1), (a', c)\}$
$F_2 = \{(0', b), (0', c), (0', 1), (a', 1), (a', b), (a', c)\}$
$F_3 = \{(a', a), (a', c), (a', 1)\}$
$F_4 = \{(a', b), (a', c), (a', 1)\}$
$F_5 = \{(0', c), (a', c), (a', 1), (0', 1)\}$.

Clearly, F_4 is a prime D-filter. But F_3 is not a prime D-filter, because $(0', a) \vee (a', b) = (a', c) \in D$, but $(0', a) \notin F_3$ and $(a', b) \notin F_3$.

Theorem 9. *For any D-filter Q of R, the following conditions are equivalent:*

(1) *Q is a prime D-filter*

(2) *for any two D-filters G, H of $R, G \cap H \subseteq Q \Rightarrow G \subseteq Q$ or $H \subseteq Q$*

(3) *for any $x, y \in R, (x)^D \cap (y)^D \subseteq Q \Rightarrow x \in Q$ or $y \in Q$.*

Proof. (1)⇒(2): Assume (1). Let G and H be two D-filters of R such that $G \cap H \subseteq Q$. We prove that $G \subseteq Q$ or $H \subseteq Q$. Suppose $G \nsubseteq Q$ and $H \nsubseteq Q$. Choose $x, y \in R$ such that $x \in G \setminus Q$ and $y \in H \setminus Q$. By our assumption we have that $x \vee y \notin Q$. Since $x \in G, y \in H$, which gives $x \vee y \in G \cap H \subseteq Q$. Therefore $x \vee y \in Q$, we get a contradiction. Thus $G \subseteq Q$ or $H \subseteq Q$.

(2)⇒(3): Assume (2). Let $x, y \in R$ with $(x)^D \cap (y)^D \subseteq Q$. Since $(x)^D$ and $(y)^D$ are D-filters of R, and by our assumption, we get that $(x)^D \subseteq Q$ or $(y)^D \subseteq Q$. Hence $x \in Q$ or $y \in Q$.

(3)⇒(1): Assume (3). Let $x, y \in R$ with $x \vee y \in Q$. Since Q is a D-filter, we have that $(x)^D \cap (y)^D = (x \vee y)^D \subseteq Q$. By our assumption, we get that $x \in Q$ or $y \in Q$. Hence Q is prime. ■

Theorem 10. *Every maximal D-filter of an ADL R is a prime D-filter.*

Proof. Let N be a maximal D-filter of R. Let $a, b \in R$ with $a \notin N$ and $b \notin N$. Then $N \vee (a)^D = R$ and $N \vee (b)^D = R$. That implies $R = N \vee ((a)^D \cap (b)^D) = N \vee (a \vee b)^D$. If $a \vee b \in N$ then $N = R$, we get a contradiction. Therefore $a \vee b \notin N$ and hence N is prime. ■

Corollary 11. *Let $N_1, N_2, N_3, \ldots, N_n$ and N be maximal D-filters of an ADL R with $\bigcap_{i=1}^{n} N_i \subseteq N$, then $N_j \subseteq N$, for some $j \in \{1, 2, 3, \ldots, n\}$.*

Theorem 12. *A proper D-filter Q of an ADL R is a prime D-filter if and only if $R \setminus Q$ is a prime ideal such that $(R \setminus Q) \cap D = \emptyset$.*

Proof. Assume that Q is a prime D-filter of R. Clearly, $R \setminus Q$ is a prime ideal of R. We prove that $(R \setminus Q) \cap D = \emptyset$. If $(R \setminus Q) \cap D \neq \emptyset$, choose $x \in (R \setminus Q) \cap D$. That implies $x \in D \subseteq Q$, which gives a contradiction. Hence $(R \setminus Q) \cap D = \emptyset$. Conversely, assume that $R \setminus Q$ is a prime ideal of R such that $(R \setminus Q) \cap D = \emptyset$. Clearly, Q is a prime filter of R and $D \subseteq R \setminus (R \setminus Q) = Q$. Therefore Q is a prime D-filter of R. ■

Theorem 13. *Let G be a D-filter of an ADL R, and K be any non-empty subset of R, which is closed under the operation $\vee$ such that $G \cap K = \emptyset$. Then there exists a prime D-filter Q of R containing G such that $Q \cap K = \emptyset$.*

Proof. Let K be a non-empty subset of R, which is closed under the operation $\vee$ such that $G \cap K = \emptyset$. Consider $\mathfrak{F} = \{H \mid H$ is an D-filter of $R, G \subseteq H$ and $H \cap K = \emptyset\}$. Clearly, it satisfies the hypothesis of the Zorn's lemma and hence $\mathfrak{F}$ has a maximal element say Q. That is, Q is a D-filter of R such that $G \subseteq Q$ and $Q \cap K = \emptyset$. Let $x, y \in R$ be such that $x \vee y \in Q$. We prove that $x \in Q$ or $y \in Q$. Suppose that $x \notin Q$ and $y \notin Q$. Then clearly $Q \vee (x)^D$ and $Q \vee (y)^D$ are D-filters of R such that $Q \subsetneq Q \vee (x)^D$ and $Q \subsetneq Q \vee (y)^D$. Since Q is maximal in $\mathfrak{F}$, we get that $(Q \vee (x)^D) \cap K \neq \emptyset$ and $(Q \vee (y)^D) \cap K \neq \emptyset$. Choose $s \in (Q \vee (x)^D) \cap K$ and $t \in (Q \vee (y)^D) \cap K$. Then $s \in (Q \vee (x)^D)$, $t \in (Q \vee (y)^D)$ and $s, t \in K$. Since K is closed under $\vee$, we get $s \vee t \in K$. Now $s \vee t = \{Q \vee (x)^D\} \cap \{Q \vee (y)^D\} = Q \vee \{(x)^D \cap (y)^D\} = Q \vee (x \vee y)^D$. Since $x \vee y \in Q$, we get that $s \vee t \in Q$. Since $s \vee t \in K$, we get that $s \vee t \in Q \cap K$, which is a contradiction to $Q \cap K = \emptyset$. Therefore either $x \in Q$ or $y \in Q$. Thus Q is a prime D-filter of R. ■

Corollary 14. *For any D-filter G of an ADL R with $x \notin G$, there exists a prime D-filter Q of R such that $G \subseteq Q$ and $x \notin Q$.*

Corollary 15. *For any D-filter G of an ADL R,* $G = \bigcap\{Q \mid Q$ *is a prime D-filter of R and* $G \subseteq Q\}$.

Corollary 16. *D is the intersection of all prime D-filters of R.*

Proof. Let Q be any prime D-filter of R. Clearly, we have that $D \subseteq \bigcap Q$. Let Q be any prime D-filter of an ADL R and $x \in \bigcap Q$. Suppose $x \notin D$. Then there exists prime ideal N such that $x \in N$ and $N \cap D = \emptyset$. That implies $x \notin R \setminus N$ and $D \subseteq R \setminus N$. Therefore $R \setminus N$ is a prime D-filer of R and $x \notin R \setminus N$, which is a contradiction. Therefore $x \in D$ and hence $\bigcap Q \subseteq D$. Thus $D = \bigcap Q$. ■

Theorem 17. *In an ADL the following are equivalent:*

(1) *Every proper D-filter is prime*

(2) $\mathfrak{F}^D(R)$ *is a chain*

(3) $\mathfrak{F}^{PDF}(R)$ *is a chain.*

Proof. (1)⇒(2): Assume (1). Clearly $(\mathfrak{F}^D(R), \subseteq)$ is a poset. Let S and T be two proper D-filters of R. By (1), we have that $S \cap T$ is a prime D-filter of R. Since $S \cap T \subseteq S \cap T$, we get $S \subseteq S \cap T \subseteq T$ or $T \subseteq S \cap T \subseteq S$. Hence $\mathfrak{F}^D(R)$ is a chain.

(2)⇒(3): It is obvious.

(3)⇒(1): Assume that (3). Let G be a proper D-filter of R. We prove that G is prime. Let $x, y \in R$ such that $(x)^D \cap (y)^D \subseteq G$. By our assumption, we get that $(x)^D \subseteq (y)^D$ or $(y)^D \subseteq (x)^D$. That implies $x \in (x)^D = (x)^D \cap (y)^D \subseteq G$ or $y \in (y)^D = (x)^D \cap (y)^D \subseteq G$. Therefore G is a prime D-filter of R. ■

Now we introduce the concept of a relative annihilator in the following definition.

Definition. For any nonempty subset S of R, define $(S, D) = \{a \in R \mid s \vee a \in D$, for all $s \in S\}$. We call this set as relative annihilator of S with respect to the filter D.

For $S = \{s\}$, we denote $(\{s\}, D)$ by (s, D).

Lemma 18. *If S, T are nonempty subsets of an ADL R, then we have the following:*

(1) $(R, D) = D = (\{0\}, D)$

(2) $(D, D) = R$

(3) $D \subseteq (S, D)$

(4) (S, D) *is a D-filter of R*

(5) $S \subseteq D$ *iff* $(S, D) = R$

(6) $S \subseteq T$ *implies* $(T, D) \subseteq (S, D)$ *and* $((S, D), D) \subseteq ((T, D), D)$

(7) $S \subseteq ((S, D), D)$

(8) $(((S, D), D), D) = (S, D)$

(9) $(S, D) = ([S), D)$

(10) $\bigcap_{i \in \triangle}(S_i, D) = (\bigcup_{i \in \triangle} S_i, D)$

(11) $(S, D) \subseteq (S \cap T, (T, D))$

(12) *If* $S \subseteq T$ *then* $(S, (T, D)) = (S, D)$

(13) $(S \cup T, D) \subseteq (S, (T, D)) \subseteq (S \cap T, D)$

(14) $(S, (S, D)) = (S, D)$.

Proof. (1) Let $x \in (R, D)$. Then $a \vee x \in D$, for all $a \in R$. That implies $x \vee x \in D$. So that $x \in D$. Hence $(R, D) \subseteq D$. Let $x \in D$. Then $a \vee x \in D$, for all $a \in R$. Thus $x \in (R, D)$. Therefore $D \subseteq (R, D)$ and hence $(R, D) = D$. Clearly, we have that $(\{0\}, D) = D$.

(2) Let $x \in D$. Then $x \vee a \in D$, for all $a \in R$. Since $x \vee a \in D$, for all $x \in D$, we get that $a \in (D, D)$, for all $a \in R$. Therefore $R \subseteq (D, D)$ and hence $R = (D, D)$.

(3) Let $x \in D$. Then $y \vee x \in D$, for all $y \in R$. Then $a \vee x \in D$, for all $a \in S \subseteq R$. That implies $x \in (S, D)$. Therefore $D \subseteq (S, D)$.

(4) Let $a, b \in (S, D)$. Then $s \vee a, s \vee b \in D$, for all $s \in S$. This implies $(s \vee a) \wedge (s \vee b) \in D$. Therefore $s \vee (a \wedge b) \in D$. Hence $a \wedge b \in (S, D)$. Let $a \in (S, D)$ and $b \in R$ with $a \leq b$. Then $s \vee a \in D$ and $s \vee a \leq s \vee b$, for all $s \in S$. Since $s \vee a \in D$ and D is a filter, we get $s \vee b \in D$. Hence $b \in (S, D)$, for all $s \in S$. Thus (S, D) is a filter of R. Since $D \subseteq (S, D)$, we get that (S, D) is a D-filter of R.

(5) Suppose $(S, D) = R$. Then $0 \in (S, D)$. That implies $a = a \vee 0 \in D$, for all $a \in S$. Hence $a \in D$, for all $a \in S$. Therefore $S \subseteq D$. Conversely, assume that $S \subseteq D$. Let $x \in R$. Since D is a filter, we get $a \vee x \in D$, for all $a \in S \subseteq D$. Hence $x \in (S, D)$. Therefore $(S, D) = R$.

(6) Suppose $S \subseteq T$. Let $a \in (T, D)$. Then $t \vee a \in D$, for all $t \in T$. Since $S \subseteq T$, we get that $s \vee a \in D$, for all $s \in S$. That implies $a \in (S, D)$. Therefore $(T, D) \subseteq (S, D)$ and hence $((S, D), D) \subseteq ((T, D), D)$.

(7) Let $x \in (S, D)$. Then $s \vee x \in D$, for all $s \in S$. That implies $x \vee s \in D$, for all $x \in (S, D)$. That implies $s \in ((S, D), D)$, for all $s \in S$. Thus $S \subseteq ((S, D), D)$.

(8) By (7), we have that $(((S, D), D), D) \subseteq (S, D)$. Let $x \notin (((S, D), D), D)$. Then there exists an element $a \notin ((S, D), D)$ such that $a \vee x \notin D$. Since $S \subseteq ((S, D), D)$, we have that $a \notin S$. So that $a \vee x \notin D$ and $s \notin S$. Therefore $x \notin (S, D)$, it concludes that $(S, D) \subseteq (((S, D), D), D)$. Thus $(((S, D), D), D) = (S, D)$.

(9) Since $S \subseteq [S)$, we get that $([S), D) \subseteq (S, D)$. Let $x \in (S, D)$. Then $a \vee x \in$

D, for all $a \in S \subseteq [S)$. That implies $x \in ([S), D)$. Therefore $(S, D) \subseteq ([S), D)$. Therefore $(S, D) \subseteq ([S), D)$. Hence $(S, D) = ([S), D)$.

(10) Since $S_i \subseteq \bigcup_{i\in\triangle} S_i$, for all $i \in \triangle$, we get that $(\bigcup_{i\in\triangle} S_i, D) \subseteq (S_i, D)$, for all $i \in \triangle$. That implies $(\bigcup_{i\in\triangle} S_i, D) \subseteq \bigcap_{i\in\triangle}(S_i, D)$. Let $x \in \bigcap_{i\in\triangle}(S_i, D)$. Then $x \in (S_i, D)$, for all $i \in \triangle$. That implies $a \vee x \in D$, for all $a \in S_i \subseteq \bigcup S_i$. That implies $\bigcap_{i\in\triangle}(S_i, D) \subseteq (\bigcup_{i\in\triangle} S_i, D)$. Therefore $\bigcap_{i\in\triangle}(S_i, D) = (\bigcup_{i\in\triangle} S_i, D)$.

(11) Since D is a filter in R, we have that $D \subseteq (T, D)$ and hence we get that $(S, D) \subseteq (S, (T, D))$. Since $S \cap T \subseteq S$, we get that $(S, (T, D)) \subseteq (S \cap T, (T, D))$. Therefore $(S, D) \subseteq (S \cap T, (T, D))$.

(12) Let S, T be two non empty subsets of R such that $S \subseteq T$. Since $D \subseteq (T, D)$, we have that $(S, D) \subseteq (S, (T, D))$. Let $x \in (S, (T, D))$. Then $a \vee x \in (T, D)$, for all $a \in S$. That implies $a \vee x \in (S, D)$, for all $a \in S$. Since $a \vee x \in (S, D)$, we get that $s \vee (a \vee x) \in D$, for all $s \in S$ and hence $a \vee x \in D$, for all $a \in S$. Therefore $x \in (S, D)$ and hence $(S, (T, D)) \subseteq (S, D)$. Thus $(S, (T, D)) = (S, D)$.

(13) Clearly, we have that $(S \cup T, D) \subseteq (S, D)$ and $D \subseteq (T, D)$. So that $(S, D) \subseteq (S, (T, D))$. Also $S \cap T \subseteq S$. It follows that $(S, (T, D)) \subseteq (S \cap T, D)$. Therefore $(S \cup T, D) \subseteq (S, (T, D)) \subseteq (S \cap T, D)$.

(14) It is clear by (12). ∎

Proposition 19. *Let S and T be any two filters of and ADL R. Then we have the following:*

(1) $(S, D) \cap ((S, D), D) = D$

(2) $(S \vee T, D) = (S, D) \cap (T, D)$

(3) $((S \cap T, D), D) \subseteq ((S, D), D) \cap ((T, D), D)$.

Proof. (1) We have that $D \subseteq (S, D) \cap ((S, D), D)$. Let $x \in (S, D) \cap ((S, D), D)$. Then $x \in (S, D)$ and $x \in ((S, D), D)$. Since $x \in ((S, D), D))$, we have that $a \vee x \in D$, for all $a \in (S, D)$. Since $x \in (S, D)$, we get that $x \in D$ and hence $(S, D) \cap ((S, D), D) \subseteq D$. Thus $(S, D) \cap ((S, D), D) = D$.

(2) Clearly, $S \subseteq S \vee T$ and $T \subseteq S \vee T$. Then $((S \vee T), D) \subseteq (S, D)$ and $((S \vee T), D) \subseteq (T, D)$. That implies $((S \vee T), D) \subseteq (S, D) \cap (T, D)$. Let $x \in (S, D) \cap (T, D)$. Then $x \in (S, D)$ and $x \in (T, D)$. That implies $s \vee x \in D$, for all $s \in S$ and $t \vee x \in D$, for all $t \in T$. That implies $(s \vee x) \wedge (t \vee x) \in D$ and have $(s \wedge t) \vee x \in D$. Since $s \in S$ and $t \in T$, we get $s \wedge t \in S \vee T$. Therefore $(s \wedge t) \vee x \in D$, for all $s \wedge t \in S \vee T$. That implies $x \in (S \vee T, D)$. Therefore $(S, D) \cap (T, D) \subseteq (S \vee T, D)$. Hence $(S, D) \cap (T, D) = (S \vee T, D)$.

(3) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, we get that $(S, D) \subseteq (S \cap T, D)$ and $(T, D) \subseteq (S \cap T, D)$. That implies $((S \cap T, D), D) \subseteq ((S, D), D)$ and $((S \cap T, D), D) \subseteq ((T, D), D)$. Hence $((S \cap T, D), D) \subseteq ((S, D), D) \cap ((T, D), D)$. ∎

Theorem 20. *For any non-empty subset S of an ADL $R, (S, D) = \bigcap_{s\in S}([s), D)$.*

Proof. Let $x \in \bigcap_{s\in S}([s), D)$. Then $x \in ([s), D)$, for all $s \in S$. That implies $t \vee x \in D$, for all $t \in [s)$ and for all $s \in S$. It follows that $s \vee x \in D$ for all $s \in S$. Therefore $x \in (S, D)$. Hence $x \in \bigcap_{s\in S}([s), D) \subseteq (S, D)$. Let s be any element of S. Take $t \in [s)$. Then $t \vee s = t$. Now, $x \in (S, D)$. That implies $s \vee x \in D$, for all $s \in S$. So that $t \vee s \vee x \in D$, for all $t \in [s)$ and for all $s \in S$. That implies $t \vee x \in D$, for all $t \in [s)$ and for all $s \in S$. So that $[s) \vee x \subseteq D$, for all $s \in S$. That implies $x \in ([s), D)$, for all $s \in S$. Therefore $x \in \bigcap_{s\in S}([s), D)$ and hence $(S, D) \subseteq \bigcap_{s\in S}([s), D)$. Thus $(S, D) = \bigcap_{s\in S}([s), D)$. ■

Corollary 21. *Let $x \in R$ and S be arbitrary subset of R. Then $(S, [x)) = \bigcap_{a\in S}(a, [x))$.*

Corollary 22. *For any $x, y \in R$ we have the following:*

(1) $([x), D) = (x, D)$

(2) $x \leq y \Rightarrow (x, D) \subseteq (y, D)$

(3) $(x \wedge y, D) = (x, D) \cap (y, D)$

(4) $((x \vee y, D), D) = ((x, D), D) \cap ((y, D), D)$

(5) $(x, D) = R \Leftrightarrow x \in D$.

Theorem 23. *Let G be a D-filter of an ADL L. Then we have*

(1) $G \cap (G, D) = D$

(2) $((G \vee (G, D)), D) = D$.

Proof. (1) It is clear.

(2) Clearly, $((G \vee (G, D)), D) \subseteq (G, D) \cap ((G, D), D)$. Let $a \in (G, D) \cap ((G, D), D)$. Let $b \in G \vee (G, D)$. Then $b = c \wedge d$, for some $c \in G$ and $d \in (G, D)$. That implies $a \vee c \in D$ and $a \vee d \in D$. Now $a \vee b = a \vee (c \wedge d) = (a \vee c) \wedge (a \vee d) \in D$, for all $b \in G \vee (G, D)$. Therefore $a \in ((G \vee (G, D)), D)$ and hence $(G, D) \cap ((G, D), D) \subseteq ((G \vee (G, D)), D)$. Thus $D = (G, D) \cap ((G, D), D) = ((G \vee (G, D)), D)$. ■

Lemma 24. *Let R_1 and R_2 be two ADLs with zero elements 0 and $0'$ respectively. Then for any $(x, y) \in R_1 \times R_2$, we have the following:*

(1) $(x, y)^* = (a)^* \times (y)^*$

(2) $(x, y)^* = (0, 0')$ *iff* $(x)^* = \{0\}$ *and* $(y)^* = \{0'\}$

(3) $((x, y), D) = (a, D) \times (y, D)$.

Let D_1 and D_2 be dense sets of R_1 and R_2 respectively. Them from the above result, it can be conclude that $D_1 \times D_2$ is a dense set of $R_1 \times R_2$. Further, every dense set of $R_1 \times R_2$ is form the form $D_1 \times D_2$.

Theorem 25. *Let M_i be a prime D_i-filter of ADLs R_i for $i = 1, 2$ and $D = D_1 \times D_2$. Then $M_1 \times R_2$ and $R_1 \times M_2$ are prime D-filters of $R_1 \times R_2$.*

Proof. Since $D_1 \subseteq M_1$ and $D_2 \subseteq M_2$, we get $D_1 \times D_2 \subseteq M_1 \times R_2$ and $D_1 \times D_2 \subseteq R_1 \times M_2$. That implies $M_1 \times R_2$ and $R_1 \times M_2$ are D-filters of $R_1 \times R_2$. Let $(a, b), (c, d) \in R_1 \times R_2$ with $(a, b) \vee (c, d) \in M_1 \times R_2$. Then $a \vee c \in M_1$. Since M_1 is a prime D-filter of R_1, we get $a \in M_1$ or $c \in M_1$. Thus $(a, b) \in M_1 \times R_2$ or $(c, d) \in M_1 \times R_2$. Therefore $M_1 \times R_2$ is a prime D-filter of $R_1 \times R_2$. Similarly, we can prove that $R_1 \times M_2$ is also a prime D-filter of $R_1 \times R_2$. ■

Theorem 26. *Let R_1 and R_2 be two ADLs with maximal elements m and m', respectively. For any prime D-filter P of $R_1 \times R_2$, P is of the form $P_1 \times R_2$ or $R_1 \times P_2$, where $D = D_1 \times D_2$ and P_i is a prime D-filter of R_i, for $i = 1, 2$.*

Proof. Let P be a prime D-filter of $R_1 \times R_2$. Consider $P_1 = \pi_1(P) = \{x_1 \in R_1 \mid (x_1, x_2) \in P$, for some $x_2 \in R_2\}$ and $P_2 = \pi_2(P) = \{x_2 \in R_2 \mid (x_1, x_2) \in P$, for some $x_1 \in R_1\}$. It is easy to verify that P_1 and P_2 are D-filters of R_1 and R_2 respectively. We first show that P_1 and P_2 are prime D-filters of R_1 and R_2 respectively. Suppose $P_1 = R_1$ and $P_2 = R_2$. Let $(a, b) \in R_1 \times R_2$. Then there exist $x \in R_1$ and $y \in R_2$ such that $(a, y) \in P$ and $(x, b) \in P$. Since $(a, m') \vee (a, y) \in P$ and $(m, b) \vee (x, b) \in P$, we get $(a, m') \in P$ and $(m, b) \in P$. Therefore $(a, b) = (a, m') \wedge (m, b) \in P$. Hence $P = R_1 \times R_2$, which is a contradiction to that P is proper. Next suppose that $P_1 \neq R_1$ and $P_2 \neq R_2$. Choose $a \in R_1 \setminus P_1$ and $b \in R_2 \setminus P_2$. Then $(a, y) \notin P$ for all $y \in R_2$ and $(x, b) \notin P_1$ for all $x \in R_1$. In particular, $(a, m') \notin P$ and $(m, b) \notin P$. Since P is prime, we get $(m, m') \notin P$, which is a contradiction. From the above observations, we get that either $P_1 = R_1$ and $P_2 \neq R_2$ or $P_1 \neq R_1$ and $P_2 = R_2$.

Case (i). Suppose $P_1 = R_1$ and $P_2 \neq R_2$. Let $x_2, y_2 \in R_2$ be such that $x_2 \vee y_2 \in P_2$. Then there exists $a \in R_1 = P_1$ such that $(a, x_2 \vee y_2) \in P$. Therefore $(a, x_2) \vee (a, y_2) = (a \vee a, (x_2 \vee y_2)) = (a, x_2 \vee y_2) \in P$. Since P is prime, we get $(a, x_2) \in P$ or $(a, y_2) \in P$. Hence $x_2 \in P_2$ or $y_2 \in P_2$. Therefore P_2 is a prime D-filter of R_2. We now show that $P = R_1 \times P_2$. Clearly $P \subseteq R_1 \times P_2$. On the other hand, suppose $(a, y) \in R_1 \times P_2$. Since $P_1 = R_1$, there exists $b \in R_2$ such that $(a, b) \in P$ and there exists $x \in R_1$ such that $(x, y) \in P$. Since $(a, m') \vee (a, b) = (a, m')$ and $(m, y) \vee (x, y) = (m, y)$, we get $(a, m') \in P$ and $(m, y) \in P$. Since P is a filter, it gives $(a, y) = (a, m') \wedge (m, y) \in P$. Hence $R_1 \times P_2 \subseteq P$. Therefore $P = R_1 \times P_2$.

Case (ii). Suppose $P_1 \neq R_1$ and $P_2 = R_2$. Similarly, we can prove that P_1 is prime D-filter of R_1 and $P = P_1 \times R_2$. ■

Theorem 27. *Let S be a sub ADL of an ADL R and P is a prime D-filter of S. Then there exists a prime D-filter Q of R such that $Q \cap S = P$.*

Proof. Let P be a prime D-filter of S. Then $S \setminus P$ is a prime ideal of S. Consider $F = [P)$. Then $P \subseteq F \cap S$. Suppose $F \cap (S \setminus P) \neq \emptyset$. Choose $x \in F \cap (S \setminus P)$. Then $x \in F$ and $x \in (S \setminus P)$. Since $x \in F = [P)$, there exists $a_1 \wedge a_2 \wedge \cdots \wedge a_n \in P$ such that $x = y \vee (a_1 \wedge a_2 \wedge \cdots \wedge a_n)$. Since P is a filter of S, we get $a_1 \wedge a_2 \wedge \cdots \wedge a_n \in P$ and hence $x \in P$. Since $x \in (S \setminus P)$, we get a contradiction. Hence $F \cap (S \setminus P) = \emptyset$. Then there exists a prime D-filter Q of R such that $F \subseteq Q$ and $Q \cap (S \setminus P) = \emptyset$. Since $I \subseteq Q$, we get $I \cap S \subseteq Q \cap S$. Since $Q \cap (S \setminus P) = \emptyset$, we get $Q \subseteq P$. Hence, both observations lead to $P \subseteq I \cap S \subseteq Q \cap S \subseteq P \cap S \subseteq P$. Therefore $P = Q \cap S$. ∎

Now, we have the following definition

Definition. A prime D-filter M of an ADL R containing a D-filter G is said to be a minimal prime D-filter belonging to G if there exists no prime D-filter N such that $G \subseteq N \subseteq M$.

Note that if we take $D = G$ in the above definition then we say that M is a minimal prime D-filter.

Example 28. From the Example 8, we have that F_2 is a prime D-filter and F_4 is a D-filter of R. Clearly $F_4 \subseteq F_2$. Clearly there is no D-filter N of R such that $F_4 \subseteq N \subseteq F_2$. Hence F_2 is a minimal prime D-filter belonging to F_4.

Proposition 29. *Let G be a D-filter and M, a prime D-filter of R with $G \subseteq M$. Then M is a minimal prime D-filter belonging to G if and only if $R \setminus M$ is a maximal ideal with $(R \setminus M) \cap G = \emptyset$.*

Proof. Clearly, $R \setminus M$ is a proper ideal and we have $(R \setminus M) \cap G = \emptyset$. We prove that $R \setminus M$ is maximal. Let N be any proper ideal of R such that $N \cap G = \emptyset$ and $R \setminus M \subseteq N$. Then $G \subseteq R \setminus N \subseteq M$. By the minimality of M, we get $R \setminus N = M$. Therefore $R \setminus M$ is a maximal ideal with respect to the property $(R \setminus M) \cap G = \emptyset$. Conversely, assume that $R \setminus M$ be a maximal ideal with respect to the property $(R \setminus M) \cap G = \emptyset$. We prove that M is minimal. If N is any prime D-filter of R such that $D \subseteq G \subseteq N \subseteq M$. Clearly, $R \setminus N$ is an ideal such that $R \setminus M \subseteq R \setminus N$ and $(R \setminus N) \cap G = \emptyset$, which is a contradiction. Therefore M is a minimal prime D-filter belonging to G. ∎

Theorem 30. *Let G be a D-filter and M, a prime D-filter of R with $G \subseteq M$. Then M is a minimal prime D-filter belonging to G if and only if for any $a \in M$, there exists $b \notin M$ such that $a \vee b \in G$.*

Proof. Assume that M is a minimal prime D-filter belonging to G. Then $R \setminus M$ is a maximal ideal with respect to the property that $(R \setminus M) \cap G = \emptyset$. Let $a \in M$. Then $a \notin R \setminus M$. That implies $R \setminus M \subset (R \setminus M) \vee (a]$. By the maximality of

$R \setminus M$, we get that $((R \setminus M) \vee (a]) \cap G \neq \emptyset$. Choose $s \in ((R \setminus M) \vee (a]) \cap G$. Then there exists $b \in R \setminus M$ and $s \in G$ such that $s = b \vee a$. Therefore $b \vee a \in G$. Conversely, assume that for any $a \in M$, there exists $b \notin M$ such that $a \vee b \in G$. Suppose M is not a minimal prime D-filter belonging to G. Then there exists a prime D-filter N of R such that $D \subseteq G \subseteq N \subseteq M$. Choose $a \in M \setminus N$. Then, by the our assumption, there exists $b \notin M$ such that $a \vee b \in G \subseteq N$. Since $a \notin N$, we get that $b \in N \subseteq M$, which is a contradiction. Therefore M is a minimal prime D-filter belonging to G. ■

Corollary 31. *A prime D-filter M of an ADL R is minimal if and only if for any $a \in M$ there exists $b \notin M$ such that $a \vee b \in D$.*

Definition. For any prime D-filter M of R, define the set $\mathcal{O}^D(M)$ as follows:

$$\mathcal{O}^D(M) = \{x \in R \mid x \in (y, D), \text{ for some } y \notin M\}.$$

Clearly, observe that $\mathcal{O}^D(M) = \bigcup_{y \notin M}(y, D)$.

Lemma 32. *Let M be prime D-filter of an ADL R. Then $\mathcal{O}^D(M)$ is a D-filter such that $\mathcal{O}^D(M)$ is contained in M.*

Proof. Let $a, b \in \mathcal{O}^D(M)$. There exist elements $s \notin M$ and $t \notin M$ such that $a \in (s, D)$ and $b \in (t, D)$. That implies $((s, D), D) \subseteq (a, D)$ and $((t, D), D) \subseteq (b, D)$. So that $((s \vee t, D), D) = ((s, D), D) \cap ((t, D), D) \subseteq (a, D) \cap (b, D) = (a \wedge b, D)$. Hence $a \wedge b \in ((a \wedge b, D), D) \subseteq (((s \vee t, D), D), D) = (s \vee t, D)$. Since $s \vee t \notin M$, we get that $a \wedge b \in \mathcal{O}^D(M)$. Let $a \in \mathcal{O}^D(M)$ and $a \leq b$. There exists $s \notin M$ such that $a \in (s, D)$. Since (s, D) is a filter, we get that $b \in (s, D)$. Therefore $b \in \mathcal{O}^D(M)$ and hence $\mathcal{O}^D(M)$ is a filter of R. Clearly, we have that $D \subseteq \mathcal{O}^D(M)$. Thus $\mathcal{O}^D(M)$ is a D-filter of R. Let $a \in \mathcal{O}^D(M)$. Then there exists $s \notin M$ such that $a \in (s, D)$. That implies $a \vee s \in D \subseteq M$. Since M is prime, we get that $a \in M$. Hence $\mathcal{O}^D(M) \subseteq M$. ■

Corollary 33. *For any prime D-filter M of R, M is minimal if and only if $\mathcal{O}^D(M) = M$.*

Theorem 34. *Every minimal prime D-filter M of R belonging to $\mathcal{O}^D(M)$ is contained in M.*

Proof. Let N be any minimal prime D-filter belonging to $\mathcal{O}^D(M)$. We prove that $N \subseteq M$. Suppose $N \nsubseteq M$. Choose $a \in N \setminus M$. Then there exists $b \notin N$ such that $a \vee b \in \mathcal{O}^D(M)$. Hence $a \vee b \in (s, D)$, for some $s \notin M$. That implies $b \vee (a \vee s) \in D \subseteq M$. Since $a \notin M, s \notin M$, and M is prime, we get $a \vee s \notin M$. Therefore $b \in \mathcal{O}^D(M) \subseteq N$, which is a contradiction. Hence $N \subseteq M$. ■

Theorem 35. *For any prime D-filter M of an ADL R, $\mathcal{O}^D(M)$ is the intersection of all minimal prime D-filters contained in M.*

Proof. Let M be a prime D-filter of R. By Zorn's lemma, M contains a minimal prime D-filter. Let $\{S_\alpha\}_{\alpha\in\triangle}$ be the set of all minimal prime D-filters contained in M. Let $x \in \mathcal{O}^D(M)$. Then $x \in (a, D)$, for some $a \notin M$. Since each $S_\alpha \subseteq M$, we have that $a \notin S_\alpha$, for all $\alpha \in\triangle$. Since $x \vee a \in D \subseteq S_\alpha$ and $a \notin S_\alpha$, for all $\alpha \in\triangle$, we get $x \in S_\alpha$ for all $\alpha \in\triangle$. Hence $x \in \bigcap_{\alpha\in\triangle} S_\alpha$. Therefore $\mathcal{O}^D(M) \subseteq \bigcap_{\alpha\in\triangle} S_\alpha$. Let $x \notin \mathcal{O}^D(M)$. Consider $S = (R \setminus M) \vee [x)$. Suppose $D \cap S \neq \emptyset$. Choose $a \in D \cap S$. Since $a \in S$, we get $a = t \vee x$, for some $t \in R \setminus M$. Since $a \in D$, we get that $t \vee x \in D$. Hence $x \in (t, D)$, where $t \notin M$. Thus $x \in \mathcal{O}^D(M)$, which is a contradiction. Therefore $S \cap D = \emptyset$. Let M be a maximal ideal such that $S \subseteq M$ and $M \cap D = \emptyset$. Then $R \setminus M$ is a minimal prime D-filter such that $R \setminus M \subseteq M$ and $x \notin R \setminus M$, since $x \in S \subseteq M$. Hence $x \notin \bigcap_{\alpha\in\triangle} S_\alpha$. Therefore $\bigcap_{\alpha\in\triangle} S_\alpha \subseteq \mathcal{O}^D(M)$. ∎

Proposition 36. *Let M_1 and M_2 be two prime D-filters in an ADL R with $M_1 \subseteq M_2$. Then $\mathcal{O}^D(M_2) \subseteq \mathcal{O}^D(M_1)$.*

Proof. Let $x \in \mathcal{O}^D(M_2)$. Then there exists an element $a \notin M_2$ such that $x \in (a, D)$. That implies $x \in (a, D)$ and $a \notin M_1$. So that $x \in \mathcal{O}^D(M_1)$. Therefore $\mathcal{O}^D(M_2) \subseteq \mathcal{O}^D(M_1)$. ∎

Proposition 37. *For any non maximal element $a \in R$ with $a \notin D$, there is a minimal prime D-filter not containing a.*

Proof. Let a be any non maximal element of R with $a \notin D$. By Corollary 14, there exists a prime D-filter P of R such that $a \notin P$. Consider $\mathfrak{F} = \{Q \mid Q$ is a prime D-filter of $R, a \notin Q$ and $Q \subseteq P\}$. It satisfies the hypothesis of Zorn's Lemma. So that $\mathfrak{F}$ has a minimal element say M, i.e., M is minimal and $a \notin M$. ∎

Theorem 38. *For any any prime D-filter M of an ADL R, the following are equivalent:*

(1) *M is minimal prime D-filter*

(2) $M = \mathcal{O}^D(M)$

(3) *for any $x \in R, M$ contains precisely one of x or (x, D).*

Proof. (1)⇒(2): Assume (1). Let $x \in M$. Then there exists $y \notin M$ such that $x \vee y \in D$. This implies that $x \in \mathcal{O}^D(M)$. So that $M \subseteq \mathcal{O}^D(M)$. Since $\mathcal{O}^D(M) \subseteq M$, we get that $M = \mathcal{O}^D(M)$.

(2)⇒(3): Assume (2). Let $x \in R$. Suppose $x \notin M$. Let $a \in (x, D)$. Then $a \vee x \in D$. That implies $a \vee x \in M$. So that $a \in M$. Since $x \notin M$. Therefore $(x, D) \subseteq M$.

(3)⇒(1): Let Q be any prime D-filter of R with $Q \subsetneq M$. Then choose $x \in M$ such that $x \notin Q$. That implies $(x, D) \subseteq Q \nsubseteq M$. So that $(x, D) \nsubseteq M$, which is a contradiction. ∎

Corollary 39. *Let P be a minimal prime D-filter of an ADL R and $a \in R$. Then $a \in P$ if and only if $((a, D), D) \subseteq P$.*

Proof. Assume that $a \in P$. Then $(a, D) \nsubseteq P$. Let $t \in ((a, D), D)$. Then $(a, D) \subseteq (t, D)$. Suppose $t \notin P$. Then $(a, D) \subseteq (t, D) \subseteq P$, which is a contradiction. That implies $t \in P$, which gives $((a, D), D) \subseteq P$. The converse follows from the fact that $a \in ((a, D), D)$. ■

Definition. An ADL R with maximal elements is called an D- semi complemented if for each non zero element $x \in R$, there exists a non maximal element $y \notin D$ such that $x \vee y \in D$.

Example 40. Let $L_1 = \{0, a\}$ and $L_2 = \{0, b_1, b_2\}$ be two discrete ADLs. Then $R = L_1 \times L_2 = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. Then $(R, \wedge, \vee, 0)$ is an ADL, but not a lattice, because $(a, b_1) \wedge (a, b_2) = (a, b_2) \neq (a, b_1) = (a, b_2) \wedge (a, b_1)$. Clearly, $D = \{(a, b_1), (a, b_2)\}$ is a dense set of R. We have that for any non zero element $x \in R$, there exists a non maximal element $a \notin D$ such that $x \vee a \in D$. Hence R is an D-semi complemented ADL.

Theorem 41. *Let R be an ADL with maximal elements. Then R is D- semi complemented if and only if the intersection of all maximal ideals disjoint with D is $\{0\}$.*

Proof. Assume that R is D-semi complemented. Consider

$$K = \bigcap \{M \mid M \text{ is a maximal ideal of } R \text{ and } M \cap D = \emptyset\}.$$

We have to prove that $K = \{0\}$. Let $x \in K$ with $x \neq 0$. Then $x \in M$, for all maximal ideal M disjoint with D. Then $x \notin D$. Since $x \neq 0$ and R is D-semi complemented, there exists a non maximal element $y \notin D$ such that $x \vee y \in D$. Then $x \vee y \notin M$. That implies $M \vee (x \vee y] = R$. Since y is non maximal element of D, there exists a minimal prime D-filter N of R such that $y \notin N$. That implies $y \in R \setminus N$ and $(R \setminus N) \cap D = \emptyset$, where $R \setminus N$ is maximal ideal of R. So that $x, y \in R \setminus N$. We have $x \vee y \in R \setminus N$. Therefore $(R \setminus N) \cap D \neq \emptyset$, which is a contradiction. Therefore $x = 0$. Hence $K = \{0\}$. Conversely, assume that $\bigcap\{M \mid M \text{ is a maximal ideal of } R \text{ and } M \cap D = \emptyset\} = \{0\}$. Let x be any non zero element of R. Then there exists a maximal ideal M such that $x \notin M$ and $M \cap D = \emptyset$. That implies $M \vee (x] = R$. So that $a \vee x$ is maximal, for some $a \in M$. Since $a \in M$ and $M \cap D = \emptyset$, we get $a \notin D$. Clearly, $a \vee x \in D$. That is, for any non zero element x of R, there exists a non maximal element $a \notin D$ such that $a \vee x \in D$. Hence R is D- semi complemented. ■

Definition. An ADL R is said to be D-normal if for any $a, b \in R$ such that $a \vee b \in D$, there exists $x \in (a, D)$ and $y \in (b, D)$ such that $x \wedge y = 0$.

From the Example 8, clearly we have that R is a D-normal ADL. The following result is a direct consequence of the above definition.

Theorem 42. *R is D-normal if and only if $(a, D) \vee (b, D) = R$, for any $a, b \in R$, with $a \vee b \in D$.*

Definition. Two D-filters G_1 and G_2 of R are said to be co-maximal if $G_1 \vee G_2 = R$.

Example 43. From the Example 8, we have that F_2 and F_3 are D-filters of R. Clearly, $F_2 \vee F_3 = R$. Therefore F_2 and F_3 are co-maximal. Also, we have F_4 and F_5 are D-filters of R, but not co-maximal.

Theorem 44. *In an ADL R, the following are equivalent.*

(1) *For any $a, b \in R$ with $a \vee b \in D, (a, D) \vee (b, D) = R$.*

(2) *For any $a, b \in R, (a, D) \vee (b, D) = (a \vee b, D)$.*

(3) *Any two distinct minimal prime D-filters are co-maximal.*

(4) *Every prime D-filter contains a unique minimal prime D-filter.*

(5) *For any prime D-filter $P, \mathcal{O}^D(P)$ is prime.*

Proof. (1)⇒(2): Assume (1). Let $a, b \in R$ with $x \in (a \vee b, D)$. Then $x \vee (a \vee b) \in D$ and hence $(x \vee a) \vee (x \vee b) \in D$. By (1), we have that $(x \vee a, D) \vee (x \vee b, D) = R$. That implies $x \in (x \vee a, D) \vee (x \vee b, D)$. Then there exists $r \in (x \vee a, D)$ and $s \in (x \vee b, D)$ such that $x = r \wedge s$. Since $r \in (x \vee a, D)$, $s \in (x \vee b, D)$ we get that $r \vee x \in (a, D)$ and $s \vee x \in (b, D)$. That implies $(x \vee r) \wedge (x \vee s) \in (a, D) \vee (b, D)$ and hence $x \vee (r \wedge s) \in (a, D) \vee (b, D)$. Since $x = r \wedge s$, we get that $x \in (a, D) \vee (b, D)$. Therefore $(a \vee b, D) \subseteq (a, D) \vee (b, D)$. Since $(a, D) \vee (b, D) \subseteq (a \vee b, D)$, we get that $(a, D) \vee (b, D) = (a \vee b, D)$, for all $a, b \in R$.

(2)⇒(3): Assume (2). Let M and N be two distinct minimal prime D-filters of R. Choose elements $x, y \in R$ such that $x \in M \setminus N$ and $y \in N \setminus M$. Since M and N are minimal, $x \vee a \in D,\ y \vee b \in D$, for some $a \notin M,\ b \notin N$. That implies $x \vee a \vee y \vee b \in D$ and hence $R = (x \vee a \vee y \vee b, D)$. By (2), we get that $(x \vee b, D) \vee (a \vee y, D) = R$. Since $a \notin M$ and $y \notin M$, we get that $a \vee y \notin M$. That implies $(a \vee y, D) \subseteq M$. Similarly, we have that $(x \vee b, D) \subseteq N$. That implies $((x \vee b) \vee (a \vee y), D) \subseteq M \vee N$ and hence $R = M \vee N$. Therefore M and N are co-maximal.

(3)⇒(4): Assume (3). Let M be a prime D-filter of R. Suppose M contains two distinct minimal prime D-filters, say N_1 and N_2. By (3), we get that $R = N_1 \vee N_2 \subseteq M$, we get a contradiction. Therefore every prime D-filter contains a unique minimal prime D-filter.

(4)⇒(5): Assume that every prime D-filter P of R contains a unique minimal prime D-filter. Then by Corollary 33, we get that $\mathcal{O}^D(P)$ is a prime D-filter.

(5)⇒(1): Assume (5). Let $a, b \in R$ be such that $a \vee b \in D$. Suppose $(a, D) \vee (b, D) \neq R$. Then there exists a maximal D-filter M such that $(a, D) \vee (b, D) \subseteq M$. That implies $(a, D) \subseteq M$ and $(b, D) \subseteq M$ That implies $a \notin \mathcal{O}^D(M)$ and $b \notin \mathcal{O}^D(M)$. Since $\mathcal{O}^D(M)$ is prime, we get $a \vee b \notin \mathcal{O}^D(M)$. So that $D \nsubseteq \mathcal{O}^D(M)$, which is a contradiction. Therefore $(a, D) \vee (b, D) = R$. ■

Theorem 45. *In an ADL R, the following conditions are equivalent:*

(1) *R is D-normal.*

(2) *For any two distinct maximal ideal G_1 and G_2 of R with $G_1 \cap D = \emptyset$, $G_2 \cap D = \emptyset$ there exist $a \notin G_1$ and $b \notin G_2$ such that $a \wedge b = 0$.*

(3) *For any maximal ideal G with $G \cap D = \emptyset$, G is the unique maximal ideal containing $R \setminus \mathcal{O}^D(P)$.*

Proof. (1)⇒(2): Assume that R is D-normal. Let G_1 and G_2 be two distinct maximal ideals of R with $G_1 \cap D = \emptyset, G_2 \cap D = \emptyset$. Then $R \setminus G_1$ and $R \setminus G_2$ are distinct minimal prime D-filters of R. By our assumption, we get $R \setminus G_1$ and $R \setminus G_2$ are co-maximal. That is, $(R \setminus G_1) \vee (R \setminus G_2) = R$. Since $0 \in R$, there exist $a \in R \setminus G_1$ and $b \in R \setminus G_2$ such that $a \wedge b = 0$.

(2)⇒(3): Assume (2). Let G be any maximal ideal of R with $G \cap D = \emptyset$ and $R \setminus \mathcal{O}^D(P) \subseteq G$. Let G_1 be any maximal ideal of R with $G_1 \cap D = \emptyset$ and $R \setminus \mathcal{O}^D(P) \subseteq G_1$.

We prove that $G = G_1$. Suppose $G \neq G_1$. By our assumption, there exists $a \notin G_1$ and $b \notin G_2$ such that $a \wedge b = 0$. That implies $a, b \notin R \setminus \mathcal{O}^D(P)$. So that $a, b \in \mathcal{O}^D(P)$. This implies that $a \wedge b \in \mathcal{O}^D(P)$. Therefore $0 \in \mathcal{O}^D(P)$. Hence $\mathcal{O}^D(P) = R$, which is a contradiction. We conclude that $G = G_1$.

(3)⇒(1): For any maximal ideal G with $G \cap D = \emptyset$, G is the unique maximal ideal containing $R \setminus \mathcal{O}^D(P)$. Let P be a prime D-filter of R. Suppose P contains two minimal prime D-filters say Q_1 and Q_2. That is, $Q_1 \subseteq P$ and $Q_2 \subseteq P$. That implies $\mathcal{O}^D(P) \subseteq \mathcal{O}^D(Q_1)$ and $\mathcal{O}^D(P) \subseteq \mathcal{O}^D(Q_2)$. We get $P \subseteq \mathcal{O}^D(Q_1)$ and $P \subseteq \mathcal{O}^D(Q_2)$. So that $Q_2 \subseteq Q_1$ and $Q_1 \subseteq Q_2$. This concludes that $Q_1 = Q_2$. ■

Let I be an ideal of R. For any $x, y \in R$, define a binary relation ϕ_I on R as $\phi_I = \{(x, y) \in R \times R \mid x \vee a = y \vee a, \text{ for some } a \in I\}$.

Proposition 46. *For any ideal I of an associative ADL R, ϕ_I is a congruence relation on R.*

For any ADL R, it can be easily verified that the quotient R/ϕ_I is also an ADL with respect to the following operations $[a]_{\phi_I} \wedge [b]_{\phi_I} = [a \wedge b]_{\phi_I}$ and $[a]_{\phi_I} \vee [b]_{\phi_I} = [a \vee b]_{\phi_I}$ where $[a]_{\phi_I}$ is the congruence class of a modulo ϕ_I. It can be routinely verified that the mapping $\Phi : R \to R/\phi_I$ defined by $\Phi(a) = [a]_{\phi_I}$ is a homomorphism.

Theorem 47. *In an ADL R, we have the following:*

(1) *If x is a dense element of R, then $[x]_{\phi_I}$ is a dense element of R/ϕ_I,*

(2) *If G is a D-filter of R/ϕ_I, then $\Phi^{-1}(G)$ is a D-filter of R,*

(3) *If G is a prime D-filter of R/ϕ_I, then $\Phi^{-1}(G)$ is a prime D-filter of R.*

Definition. Let I be an ideal of an ADL R. For any D-filter G of R, define $\widetilde{G} = \{[a]_{\phi_I} \mid a \in G\}$.

The following result can be proved easily.

Lemma 48. *$\widetilde{G}$ is a D-filter of R/ϕ_I.*

Proposition 49. *Let G be a prime D-filter and I an ideal of an ADL R such that $G \cap I = \emptyset$. We have the following:*

(1) *$x \in G$ if and only if $[x]_{\phi_I} \in \widetilde{G}$*

(2) *$\widetilde{G} \cap \widetilde{I} = \emptyset$*

(3) *If G is a prime D-filter of R, then $\widetilde{G}$ is a prime D-filter of $R/_{\phi_I}$.*

Proof. (1) Assume that $x \in G$. Then we have $[x]_{\phi_I} \in \widetilde{G}$. Conversely assume that $[x]_{\phi_I} \in \widetilde{G}$. Then there exists $y \in G$ such that $[x]_{\phi_I} = [y]_{\phi_I}$. That implies $(x, y) \in \phi_I$. So there exists $a \in I$ such that $x \vee a = y \vee a \in G$. Since $G \cap I = \emptyset$, we get $a \notin G$. Since $x \vee a \in G$ and $a \notin G$, we get that $x \in G$.

(2) Suppose $\widetilde{G} \cap \widetilde{I} \neq \emptyset$. Then choose an element $x \in R$ such that $[x]_{\phi_I} \in \widetilde{G} \cap \widetilde{I}$. Then $[x]_{\phi_I} \in \widetilde{G}$ and $[x]_{\phi_I} \in \widetilde{I}$. Since $[x]_{\phi_I} \in \widetilde{G}$ and by (1), we get $x \in G$. Since $[x]_{\phi_I} \in \widetilde{I}$, there exists $y \in I$ such that $[x]_{\phi_I} = [y]_{\phi_I}$. Then $(x, y) \in \phi_I$. So there exist $a \in I$ such that $x \vee a = y \vee a$. Since $y \vee a \in I$, we get that $x \vee a \in I$. Since $x \in G$, we have that $x \vee a \in G \cap I$. That implies $G \cap I \neq \emptyset$, we get a contradiction. Hence $\widetilde{G} \cap \widetilde{I} = \emptyset$.

(3) Clearly, we have that $\widetilde{G}$ is a proper filter of $R/_{\phi_I}$. Let $[x]_{\phi_I} \in \widetilde{D}$. Then $x \in D \subseteq G$. That implies $[x]_{\phi_I} \in G$ and hence $\widetilde{G}$ is a D-filter of $R/_{\phi_I}$. Let $[x]_{\phi_I}, [y]_{\phi_I} \in R/_{\phi_I}$ such that $[x]_{\phi_I} \vee [y]_{\phi_I} \in \widetilde{G}$. Then $[x \vee y]_{\phi_I} \in \widetilde{G}$. By (1) we have that $x \vee y \in G$. Since G is prime, we get that $x \in G$ or $y \in G$. Again by (1) we get that $[x]_{\phi_I} \in \widetilde{G}$ or $[y]_{\phi_I} \in \widetilde{G}$. Hence $\widetilde{G}$ is a prime D-filter in $R/_{\phi_I}$. ∎

Proposition 50. *Let I be an ideal of an ADL R. Then there is an order isomorphism of the set of all prime D-filters of R disjoint from I onto the set of all prime D-filters of $R/_{\phi_I}$.*

Proof. Let G and H be two prime D-filters of R such that $G \cap I = \emptyset$ and $H \cap I = \emptyset$. Then by Proposition 49(1), we get that $G \subseteq H$ if and only if $\widetilde{G} \subseteq \widetilde{H}$. Let G be a prime D-filter of R with $G \cap I = \emptyset$. Then by Proposition 49(3), we get that $\widetilde{G}$ is a prime D-filter of $R/_{\phi_I}$. Let Q be a prime D-filter of $R/_{\phi_I}$. Consider $G = \{a \in R | [a]_{\phi_I} \in Q\}$. Since Q is a D-filter of $R/_{\phi_I}$, we get that G is a D-filter of R. Let $a, b \in R$ with $a \vee b \in G$. Then $[a]_{\phi_I} \vee [b]_{\phi_I} = [a \vee b]_{\phi_I} \in Q$. Since Q is prime, we get $[a]_{\phi_I} \in Q$ or $[b]_{\phi_I} \in Q$. Therefore $a \in G$ or $b \in G$. Hence G is a prime D-filter of R. Clearly $\widetilde{G} = Q$. Suppose $G \cap I \neq \emptyset$. Then choose an element $s \in G \cap I$. That implies $[s]_{\phi_I} \in Q$ and $s \in I$. Let $[b]_{\phi_I} \in R/_{\phi_I}$. Since $s \in I$ and $b \vee s = b \vee s \vee s$, we get that $(b, b \vee s) \in I$. That implies $[b]_{\phi_I} = [b \vee s]_{\phi_I} = [b]_{\phi_I} \vee [s]_{\phi_I} \in Q$. Therefore $[b]_{\phi_I} \in Q$. and hence $R/_{\phi_I} = Q$, which is a contradiction. Thus $G \cap I = \emptyset$. ■

Corollary 51. *Let R be an ADL. Then the above map induces a one-to-one correspondence between the set of all minimal prime D-filters of R which are disjoint from I and the set of all minimal prime D-filters of $R/_{\phi_I}$.*

Theorem 52. *For any ideal I of an ADL R, the following are equivalent:*

(1) *Any two dintinct minimal prime D-filters of R are co-maximal*

(2) *any two distinct minimal prime D-filters of $R/_{\phi_I}$ are co-maximal.*

Proof. (1)⇒(2) Assume (1). Let G_1, G_2 be two distinct minimal prime D-filters of $R/_{\phi_I}$. Then by the corollary 51, there exist two minimal prime D-filters H_1 and H_2 of R such that $H_1 \cap I = \emptyset$ and $H_2 \cap I = \emptyset$. Also $\widetilde{H_1} = G_1$ and $\widetilde{H_2} = G_2$. Since G_1 and G_2 are distinct, we get that H_1 and H_2 are distinct. By the assumption, we have $H_1 \vee H_2 = R$. Let $a \in R$. There exist $a_1 \in H_1$ and $a_2 \in H_2$ such that $a = a_1 \wedge a_2$. Since $a_1 \in H_1$ and $a_2 \in H_2$ we get $[a_1]_{\phi_I} \in \widetilde{H}_1 = G_1$ and $a_2]_{\phi_I} \in \widetilde{H}_2 = G_2$. Now, $[a]_{\phi_I} = [a_1 \wedge a_2]_{\phi_I} = [a_1]_{\phi_I} \wedge [a_2]_{\phi_I} \in G_1 \vee G_2$. That implies $[a]_{\phi_I} \in G_1 \vee G_2$, for all $a \in R$. Therefore $G_1 \vee G_2 = R/_{\phi_I}$.

(2)⇒(1) Assume (2). Let P be a prime D-filter of R. Suppose P contains two distinct minimal prime D-filters, say G_1 and G_2. Consider $K = R \setminus P$. Clearly K is an ideal of R and $G_1 \cap K = \emptyset = G_2 \cap K$. By Corollary 51, we get that $\widetilde{G_1}$ and $\widetilde{G_2}$ are distinct minimal prime D-filters of $R/_{\phi_I}$ such that $\widetilde{G_1}, \widetilde{G_2} \subseteq \widetilde{P}$. That implies $\widetilde{P}$ is containing two distinct minimal prime D-filters of $R/_{\phi_I}$, which is a contradiction. Hence P contains a unique minimal prime D-filter. By Theorem 44, any two distinct minimal prime D-filters of R are co-maximal. ■

4. The space of minimal prime D-filters of an ADL

In this section, some topological properties of the space of all prime D-filters and the space of all minimal prime D-filters of an ADL are studied.

Let us denote the set of all prime D-filters of an ADL R by $Spec_F^D(R)$. For any $A \subseteq R$, define $\alpha(A) = \{P \in Spec_F^D(R) | A \nsubseteq P\}$ and for any $a \in R$,

$\alpha(a) = \{P \in Spec_F^D(R) | a \notin P\}$. Then we have the following result whose proof is straightforward.

Lemma 53. *Let R be an ADL and $a, b \in R$. Then the following conditions hold:*

(1) $\bigcup_{a\in R} \alpha(a) = Spec_F^D(R)$,

(2) $\alpha(a) \cap \alpha(b) = \alpha(a \vee b)$,

(3) $\alpha(a) \cup \alpha(b) = \alpha(a \wedge b)$,

(4) $\alpha(a) = \emptyset$ *if and only if* $a \in D$,

(5) $\alpha(a) = Spec_F^D(R)$ *if and only if* $a = 0$.

From the above result, it can be easily observed that the collection $\{\alpha(a)|a \in R\}$ forms a base for a topology on $Spec_F^D(R)$. The topology generated by this base is precisely $\{\alpha(A \mid A \subseteq R\}$ and is called the hull-kernel topology on $Spec_F^D(R)$. Under this topology, we have the following result.

Theorem 54. *In an ADL R, we have the following:*

(1) *For any $a \in R, \alpha(a)$ is compact in $Spec_F^D(R)$,*

(2) *Let C be a compact open subset of $Spec_F^D(R)$. Then $C = \alpha(a)$ for some $a \in R$,*

(3) *$Spec_F^D(R)$ is a T_0-space,*

(4) *The map $a \mapsto \alpha(a)$ is an anti-homomorphism from R onto the lattice of all compact open subsets of $Spec_F^D(R)$.*

Proof. (1) Let $a \in R$. Let $X \subseteq R$ be such that $\alpha(a) \subseteq \bigcup_{x\in X} \alpha(x)$. Let F be a D-filter generated by the set X. Suppose $a \notin F$. Then there exists a prime D-filter P such that $F \subseteq P$ and $a \notin P$. Since $X \subseteq F \subseteq P$, we get $P \notin \alpha(x)$ for all $x \in X$. Since $a \notin P$, we get $P \in \alpha(a)$, which is a contradiction. Hence $a \in F$. So we can write $a = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for some $x_1, x_2, \ldots, x_n \in X$ and $n \in N$. Then, we get $\alpha(a) = \alpha(\bigwedge_{i=1}^n x_i) = \bigcup_{i=1}^n \alpha(x_i)$ which is finite subcover for $\alpha(a)$. Therefore $\alpha(a)$ is compact.

(2) Let C be a compact open subset of $Spec_F^D(R)$. Since C is open, we get $C = \bigcup_{x\in X} \alpha(x)$ for some $X \subseteq R$. Since C is compact, there exist $x_1, x_2, \ldots, x_n \in X$ such that $C = \bigcup_{i=1}^n \alpha(x_i) = \alpha(\bigwedge_{i=1}^n)$. Therefore $C = \alpha(x)$ for some $x \in R$.

(3) Let P and Q be two distinct prime D-filters of R. Without loss of generality, assume that $P \nsubseteq Q$. Choose $x \in R$ such that $x \in P$ and $x \notin Q$. Hence $P \notin \alpha(x)$ and $Q \in \alpha(x)$. Therefore $Spec_F^D(R)$ is a T_0-space.

(4) It can be obtained from (1), (2) and by the above lemma. ■

Proposition 55. *In an ADL R, the following are equivalent:*

(1) *$Spec_F^D(R)$ is a Hausdorff space.*

(2) *For each* $P \in Spec^D_F(R), P$ *is the unique member of* $Spec^D_F(R)$ *such that* $\mathcal{O}^D(P) \subseteq P$.

(3) *Every prime D-filter is minimal.*

(4) *Every prime D-filter is maximal.*

Proof. (1)⇒(2): Assume (1). Let $P \in Spec^D_F(R)$. Clearly $\mathcal{O}^D(P) \subseteq P$. Suppose $Q \in Spec^D_F(R)$ such that $Q \neq P$ and $\mathcal{O}^D(P) \subseteq Q$. Since $Spec^D_F(R)$ is Hausdorff, there exists $a, b \in R$ such that $P \in \alpha(a), Q \in \alpha(b)$ and $\alpha(a \vee b) = \alpha(a) \cap \alpha(b) = \emptyset$. Hence $a \notin P, b \notin Q$ and $a \vee b \in D$. Therefore $b \in \mathcal{O}^D(P) \subseteq Q$, which is a contradiction to that $b \notin Q$. Hence $P = Q$. Therefore P is the unique member of $Spec^F_D(R)$ such that $\mathcal{O}^D(P) \subseteq P$.

(2)⇒(3): Assume (2). Let P be a prime D-filter of R. Suppose P is not minimal. Let Q be a prime D-filter in R such that $Q \subseteq P$. Hence $\mathcal{O}^D(Q) \subseteq Q \subseteq P$. Therefore P is a minimal prime D-filter of R.

(3)⇒(4): It is clear.

(4)⇒(1): Assume (4). Let P and Q be two distinct elements of $Spec^D_F(R)$. Hence $\mathcal{O}^D(Q) \nsubseteq P$. Choose $a \in \mathcal{O}^D(Q)$ such that $a \notin P$. Since $a \in \mathcal{O}^D(Q)$, there exists $b \notin Q$ such that $a \in (b, D)$. Hence $a \vee b \in D$. Thus it yields, $P \in \alpha(a), Q \in \alpha(b)$. Since $a \vee b \in D$, we get that $\alpha(a) \cap \alpha(b) = \alpha(a \vee b) = \emptyset$. Therefore $Spec^D_F(R)$ is Hausdorff. ∎

Theorem 56. *For any D-filter G of an ADL R,* $(G, D) = \bigcap\{P \in Spec^D_F(R) \mid G \nsubseteq P\}$.

Proof. Let G be a D-filter of R. Consider $K = \bigcap\{P \in Spec^D_F(R) \mid G \nsubseteq P\}$. Let $P \in \alpha(G)$. Then $G \nsubseteq P$. Since $G \cap (G, D) = D \subseteq P$ and P is prime, we get $(G, D) \subseteq P$. Hence every prime D-filter P of R such that $G \nsubseteq P$ contains (G, D). Therefore $(G, D) \subseteq K$. Let $x \notin (G, D)$. Then there exists $y \in G$ such that $x \vee y \notin D$. Let $\mathcal{K} = \{G \mid G$ is a D-filter of R and $x \vee y \notin G\}$. Clearly, $D \in \mathcal{K}$ and so $P = \emptyset$. Clearly, $(\mathcal{K}, \subseteq)$ is a partially ordered set and it satisfies the hypothesis of the Zorn's lemma, $\mathcal{K}$ has a maximal element, say N. Then N is a D-filter of R and $x \vee y \notin N$. Therefore $x \notin N$ and $y \notin N$. Since $y \in G$, we get $G \nsubseteq N$. We now show that N is prime. Let $a, b \in R$ with $a \notin N$ and $b \notin N$. Then $N \subsetneq N \vee (a)^D$ and $N \subsetneq N \vee (b)^D$. By the maximality of N, we get $x \vee y \in N \vee (a)^D$ and $x \vee y \in N \vee (b)^D$. Hence, $x \vee y \in \{N \vee (a)^D\} \cap \{N \vee (b)^D\} = N \vee \{(a)^D \cap (b)^D\} = N \vee (a \vee b)^D$. If $a \vee b \in N$, then $x \vee y \in N$ which is a contradiction. Thus N is a prime D-filter of R such that $G \nsubseteq N$ and $x \notin N$. Therefore $x \notin K$. Hence $K \subseteq (G, D)$. ∎

Corollary 57. *For any ADL R and* $a \in R, (a, D) = \bigcap\{P \in Spec^D_F(R) \mid a \notin P\}$.

Let $Min^D_F(R)$ denote the set of all minimal prime D-filters of ADL R. For any $a \in R$, write $\alpha_m(x) = \alpha(x) \cap Min^D_F(R)$.

Theorem 58. *For any ADL R, the following conditions hold:*

(1) *Every prime D-filters contains a minimal prime D-filter.*

(2) $\bigcap_{P\in Min_F^D(R)} P = D$.

(3) *For any subset A with $D \subseteq A, (A, D) = \bigcap_{P\in\alpha_m(A)}(P)$.*

Proof. (1) Let P be a prime D-filter of R. Consider $X = \{N \in Spec_F^D(R) \mid N \subseteq P\}$. Clearly X is a partially ordered set under set inclusion and hence it satisfies the hypothesis of the Zorn's lemma, X has a minimal element say M. Clearly M will be the required minimal prime D-filter of R.

(2) Since D is contained in every minimal prime D-filter of R and so contained in the intersection of all minimal prime D-filters. Let $x \notin D$. Then there exists a prime D-filter P of R such that $x \notin P$. By (1), there exists a minimal prime D-filter of R such that $M \subseteq P$. Since $x \notin P$, we get $x \notin M$. That implies M is a minimal prime D-filter of R such that $x \notin D$-filters of R. Hence x is not in the intersection of all minimal prime. Thus intersection of all minimal prime D-filters of R is equal to D.

(3) Let $P \in Min_F^D(R)$ such that $A \nsubseteq P$. Choose $x \in A$ such that $x \notin P$. Then $(A, D) \subseteq (x, D) \subseteq P$. That implies (A, D) is contained in every minimal prime D-filter of R such that $A \nsubseteq P$. Hence $(A, D) \subseteq \bigcap_{P\in\alpha_m(A)}(P)$. Let $x \notin (A, D)$. Then $x \vee y \notin D$, for some $y \in A$. By the condition (2), there exists a minimal prime D-filter P of R such that $x \vee y \notin P$. That implies $x \notin P$ and $y \notin P$. Therefore $x \notin \bigcap_{P\in\alpha_m(A)} P$ and hence $(A, D) = \bigcap_{P\in\alpha_m(A)} P$. ∎

Lemma 59. *For any $a, b \in R$, we have following:*

(1) $(a, D) \subseteq (b, D)$ *if and only if* $\alpha_m(b) \subseteq \alpha_m(a)$

(2) $\alpha_m(a) = \emptyset$ *if and only if* $a \in D$

(3) $\alpha_m(a) = Min_F^D(R)$ *if and only if* $(a, D) = D$.

Proof. (1) Let $a, b \in R$. Assume that $(a, D) \subseteq (b, D)$. Let $P \in \alpha_m(b)$ Then $b \notin P$. That implies $(a, D) \subseteq (b, D) \subseteq P$. Therefore $a \notin P$ and hence $P \in \alpha_m(a)$. Thus $\alpha_m(b) \subseteq \alpha_m(a)$. Conversely, assume that $\alpha_m(b) \subseteq \alpha_m(a)$. Now, $(a, D) = \bigcap_{P\in\alpha_m(a)} P \subseteq \bigcap_{P\in\alpha_m(b)} P = (b, D)$. Hence $(a, D) \subseteq (b, D)$.

(2) Suppose $Min_F^D(R) = \emptyset$. Then $a \in P$ for all $P \in Min_F^D(R)$. That implies $a \in \bigcap_{P\in Min_F^D(R)} P$. Since $a \in \bigcap_{P\in Min_F^D(R)} P = D$, we get $a \in D$. The converse is clear.

(3) Assume $\alpha_m(a) = Min_F^D(R)$. Then $(a, D) = \bigcap_{P\in\alpha_m(a)} P = \bigcap_{P\in Min_F^D(R)} P = D$. Therefore $(a, D) = D$. Conversely, assume $(a, D) = D$. Then $(a, D) = D \subseteq P$. That implies $a \notin P$, for all $P \in Min_F^D(R)$. Therefore $\alpha_m(a) = Min_F^D(R)$. ∎

For any D-filter G of an ADL R, define $\beta_m(G) = \{P \in Min_F^D(R) \mid G \subseteq P\}$.

Lemma 60. *Let G be a D-filter of an ADL R. If $\beta_m(G) = \emptyset$, then $(G, D) = D$.*

Proof. Let $\beta_m(G) = \emptyset$. Then $\beta_m(G) = Min_F^D(R)$. That implies $(G, D) = \bigcap_{P \in \alpha_m(F)} P \subseteq \bigcap_{P \in Min_F^D(R)} P = D$. Therefore $(G, D) = D$. ■

For any ADL R, define $E = \{x \in R \mid (x, D) = D\}$.

Lemma 61. *For any ADL R, E is an ideal.*

Proof. Clearly $0 \in E$. Let $x, y \in E$. Then $((x \vee y, D), D) = ((x, D), D) \cap ((y, D), D) = (D, D) \cap (D, D) = R \cap R = R$. That implies $((x \vee y), D) = (R, D) = D$. Therefore $x \vee y \in E$. Let $x \in E$. Then $(x, D) = D$. Let $y \in R$. Now, $(x \wedge y, D) = (x, D) \cap (y, D) = D \cap (y, D) = D$. Therefore $x \wedge y \in E$. Hence E is an ideal of R. ■

Theorem 62. *Let G be a D-filter of an ADL R. Then $Min_F^D(R)$ is compact if and only if $\beta_m(G) = \emptyset$ implies $G \cap E \neq \emptyset$.*

Proof. Assume that $Min_F^D(R)$ is compact. Let G be a D-filter R such that $\beta_m(G) = \emptyset$. Then $\alpha_m(G) = Min_F^D(R)$. Since $Min_F^D(R)$ is compact, there exists $a \in G$ such that $\alpha_m(a) = Min_F^D(R)$. That implies $(a, D) = D$. Therefore $a \in E$ and hence $G \cap E \neq \emptyset$. Conversely, assume that for any D-filter G of $R, \beta_m(G) = \emptyset$ implies $G \cap E \neq \emptyset$. Let $A \subseteq R$ be such that $Min_F^D(R) = \bigcup_{a \in A} \alpha_m(A) = \alpha_m(A) = \alpha_m(G)$ where $G = (A)^D$. Since $Min_F^D(R) = \alpha_m(G)$, we get $\beta_m(G) = \emptyset$. By the assumption, we get $G \cap E \neq \emptyset$. Choose $d \in G \cap E$. Since $d \in G$ and $G = (A)^E$, there exists $a_1, a_2, \ldots, a_n \in A$ such that $d \vee (a_1 \wedge a_2 \wedge \cdots \wedge a_n) = d$. Since $d \in E$, $Min_F^D(R) = \alpha_m(d) \subseteq \alpha_m(\bigwedge_{i=1}^n a_i) = \bigcup_{i=1}^n \alpha_m(a_i)$. Hence $Min_F^D(R)$ is compact. ■

Theorem 63. *Let R be an ADL. For any $Y \subseteq Min_F^D(R)$, the closure of Y in $Min_F^D(R)$ is $\beta_m(\bigcap_{P \in Y} P)$ and, in particular, $\overline{\alpha_m(F)} = \beta_m((G, D))$, for any $D \subseteq G \subseteq R$.*

Proof. Let $Y \subseteq Min_F^D(R)$. Then $\overline{Y}$ in $Min_F^D(R) = \{\overline{Y}$ in $Spec_F^D(R)\} \cap Min_F^D(R) = H(\bigcap_{P \in Y} P) \cap Min_F^D(R) = \beta_m(\bigcap_{P \in Y} P)$. In particular, for any $D \subseteq G \subseteq R$, we have $\overline{\alpha_m(G)} = \beta_m(\bigcap_{P \in \alpha_m(G)} P) = \beta_m(\bigcap_{I \nsubseteq P,\ P \in Min_F^D(R)} P) = \beta_m((F, D))$. ■

Proposition 64. *Let F, G be two D-filters of an ADL R. Then the following are equivalent:*

(1) $G \subseteq (F, D)$

(2) $G \cap F = D$

(3) $\alpha_m(G) \cap \alpha_m(F) = \emptyset$.

Proof. (1)⇒(2): Assume that $G \subseteq (F, D)$. Then $G \cap F \subseteq (F, D) \cap F = D$. Therefore $G \cap F = D$.

(2)⇒(3): Assume that $G \cap F = D$. Let $P \in \alpha_m(G) \cap \alpha_m(F) = \alpha_m(G \cap F)$. Then $D = G \cap F \nsubseteq P$, which is a contradiction. Therefore $\alpha_m(G) \cap \alpha_m(F) = \emptyset$.

(3)⇒(1): Assume that $\alpha_m(G) \cap \alpha_m(F) = \emptyset$. Let $x \in G$. Suppose $x \notin (F, D)$. Then there exists $y \in F$ such that $x \vee y \notin D$. Then there exists $P \in Min_F^D(R)$ such that $x \vee y \notin P$. That implies $x \notin P$ and $y \notin P$. Hence $G \nsubseteq P$ and $F \nsubseteq P$. Therefore $P \in \alpha_m(G)$ and $P \in \alpha_m(F)$. Therefore $P \in \alpha_m(G) \cap \alpha_m(F)$, which is a contradiction. So $x \in (F, D)$. Therefore $G \subseteq (F, D)$. ∎

Corollary 65. *Let G be a D-filter of an ADL R and $x \in R$. Then $x \in (G, D)$ if and only if $\alpha_m(x) \cap \alpha_m(G) = \emptyset$.*

Proof. By taking $G = \{x\}$, in the above proposition. ∎

Theorem 66. *Every open subset of $Min_F^D(R)$ is closed if and only if for any D-filter of R, $(G, D) = D$ implies $\beta_m(G) = \emptyset$.*

Proof. Assume that every open set of $Min_F^D(R)$ is closed. Let G be a D-filter of R. Then $\beta_m(G)$ is an open set in $Min_F^D(R)$. Now, $\beta_m(G) \neq \emptyset$. Then there exists $x \in R \setminus D$ such that $\alpha_m(x) \subseteq \beta_m(G)$. That implies $\alpha_m(x) \cap \alpha_m(G) = \emptyset$. Therefore $x \in (G, D)$ and $x \notin D$. Hence $(G, D) \neq D$. Thus $(G, D) = D$, which gives $\beta_m(G) = \emptyset$. Conversely, assume that the condition holds. Let H be an open subset of $Min_F^D(R)$. Then $H = \alpha_m(G)$, for some D-filter G of L. By Theorem-63, we have $\overline{\alpha_m(G)} = \beta_m((G, D))$. It is enough to show that $\beta_m((G, D)) = \alpha_m(G)$. Since $((G \vee (G, D)), D) = D$, by the assumption, we get $\beta_m(G \vee (G, D)) = \emptyset$. Now, for any $P \in Min_F^D(R)$, we have, $P \in \alpha_m(G) \Leftrightarrow G \nsubseteq P \Leftrightarrow (G, D) \subseteq P \Leftrightarrow P \in \beta_m(G)$. Hence $\alpha_m(G) = \beta_m(G)$. Therefore H is closed in $Min_F^D(R)$. ∎

Theorem 67. *In an ADL R, $Min_F^D(R)$ is a Hausdorff space.*

Proof. Let P and Q be distinct elements of $Min_F^D(R)$. Then there exists $a \in P$ such that $a \notin Q$. Since P is minimal, we get $(a, D) \nsubseteq P$. Then there exists $b \in (a, D)$ such that $b \notin P$. That implies $a \vee b \in D$ and hence $\alpha_m(a) \cap \alpha_m(b) = \emptyset$. Since $a \notin Q$ and $b \notin P$, we get $Q \in \alpha_m(a)$ and $P \in \alpha_m(b)$. Therefore $Min_F^D(R)$ is a Hausdorff space. ∎

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Received 9 January 2023
Revised 27 May 2023
Accepted 29 May 2023