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ON IDEMPOTENT ELEMENTS OF DUALLY RESIDUATED LATTICE ORDERED SEMIGROUPS

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Abstract

We show that idempotent elements of a dually residuated lattice ordered semigroup (a DRl-semigroup) form a Brouwerian algebra. Further we show that for any idempotent elements x, y such that $x \leq y$ the interval $[x; y]$ is also a DRL-semigroup.

Keywords: BL-algebra, Boolean algebra, Brouwerian algebra, lattice ordered group, lattice ordered monoid, MV-algebra, dually residuated lattice ordered semigroup..

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1. Introduction

Dually residuated lattice ordered semigroups were introduced in the mid-60's by Swamy (cf. [3]) as a common generalization of commutative lattice ordered groups and Brouwerian algebras. They are closely related to the multi-valued logic. The class of dually residuated lattice ordered semigroups is a variety and it contains Boolean algebras, Brouwerian algebras, BL-algebras, MV-algebras and commutative l-groups.

Here is the original definition given in [3].

Definition. An algebra $A = (A; 0; +; -; \wedge; \vee)$ of type $\langle 0; 2; 2; 2; 2\rangle$ is a Dually Residuated Lattice Ordered Semigroup (abbreviated, a DRl-semigroup) if the following holds (cf. [3]):

1. $(A; 0; +; \wedge; \vee)$ is a commutative lattice ordered monoid i.e.,
   (i) $(A; 0; +)$ is a commutative monoid,
   (ii) $(A; \wedge; \vee)$ is a lattice (the induced order is denoted by $\leq$),

(iii) $(x \wedge y) + z = (x + z) \wedge (y + z)$ for all $x, y, z \in A$,
(iv) $(x \vee y) + z = (x + z) \vee (y + z)$ for all $x, y, z \in A$,

2. $(x - y) + y \geq x$ and if $z + y \geq x$ then $z \geq x - y$ for all $x, y, z \in A$,
3. $(x - y) \vee 0 + y \leq x \vee y$ for all $x, y \in A$,
4. $x - x \geq 0$ for each $x \in A$.

In the following theorem we summarize some basic properties of DRl-semigroups as they were shown in [3].

Theorem 1. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup and* $x, y, z \in A$. *Then the following hold:*

(i) $x - x = 0$ *and* $x - 0 = x$,
(ii) $(x + y) - y \leq x$,
(iii) $(x - y) \vee 0 + y = x \vee y$,
(iv) $(x - y) - z = x - (y + z)$,
(v) $x + y = x \wedge y + x \vee y$,
(vi) $x \leq y$ *implies* $x - z \leq y - z$ *and* $z - x \geq z - y$.

Proof. This theorem is the restatement of Lemmas 1, 2, 3, 6, 9 and 13 of [3]. ∎

Denote by $Idm(A)$ the set of all additively idempotent elements of a DRl-semigroup A, i.e., $Idm(A) = \{x \in A | x + x = x\}$. Clearly $0 \in Idm(A)$ and if there exists a greatest element in A (denoted by 1) then $1 \in Idm(A)$.

Further, for $x, y \in A$ such that $x \leq y$ denote by $[x; y]$ the interval in A with the endpoints x and y, i.e., $[x; y] = \{z \in A | x \leq z \leq y\}$.

In [2] Rachůnek showed that $Idm(A)$ in a bounded representable DRl-semigroup is a Brouwerian algebra. In this paper we will prove that Rachůnek's proposition holds in a general case, i.e., that idempotent elements in any DRl-semigroup form a Brouwerian algebra. Further, we will show that any interval between idempotent elements is also a DRl-semigroup.

Recall from [3] that a Brouwerian algebra is a system $B = (B; \leq; -)$ where $(B; \leq)$ is a lattice with a least element and for all $x, y \in B$ there exists a least element $z \in B$ such that $y \vee z \geq x$ (z is denoted by $x - y$).

2. Structure of idempotent elements

Before proceeding to the main results let us prove a few technical assertions that will be needed.

Lemma 2. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup and* $x \in Idm(A)$. *Then* $x \geq 0$.

Proof. By Theorem 1(i) and (ii) we have $0 = x - x = (x + x) - x \leq x$, i.e., $x \geq 0$. ∎

Lemma 3. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup,* $x \in Idm(A)$, $y \in A$ *and* $y \geq 0$. *Then* $x + y = x \vee y$.

Proof. By Theorem 1(iii) and (v) we have $x + y = (x \wedge y) + (x \vee y) = (x \wedge y) + ((y - x) \vee 0) + x \leq ((y - x) \vee 0) + x + x = ((y - x) \vee 0) + x = x \vee y$. On the other hand, from $x, y \geq 0$ it follows $x + y \geq x, y$ and therefore $x + y \geq x \vee y$. ∎

Theorem 4. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup. Then* $Idm(A)$ *is a lattice ordered monoid with the least element* 0. *Moreover,*

$$x + y = x \vee y \tag{1}$$

for all $x, y \in Idm(A)$.

Proof. Clearly $0 \in Idm(A)$. Assume that $x, y \in Idm(A)$. From $(x+y)+(x+y) = (x + x) + (y + y) = x + y$ we have $(x + y) \in Idm(A)$. Further, $(x \wedge y) + (x \wedge y) \leq x + x = x$ and $(x \wedge y) + (x \wedge y) \leq y + y = y$ imply $(x \wedge y) + (x \wedge y) \leq x \wedge y$. On the other hand, $x, y \geq 0$ implies $x \wedge y \geq 0$ and therefore $(x \wedge y) + (x \wedge y) \geq x \wedge y$. Consequently $(x \wedge y) + (x \wedge y) = x \wedge y$. Finally, $x, y \geq 0$ implies $x + y \leq (x \vee y) + (x \vee y) \leq (x + y) + (x + y) = x + y$ and therefore (1) holds. ∎

Lemma 5. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup and* $x, y \in Idm(A)$. *Then* $x - y \geq 0$.

Proof. Since $(x-y)+y \geq x$ and $x \geq 0$, $(x-y)+y \geq 0$. Theorem 1(iii) and Lemma 3 imply $y \leq x \vee y = ((x-y) \vee 0)+y = ((x-y)+y) \vee y = ((x-y)+y)+y = (x-y)+y$ and therefore by Theorem 1(i) we conclude $x - y \geq y - y = 0$. ∎

Lemma 6. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup,* $x, y \in Idm(A)$, $z \in A$ *and* $0 \leq z \leq x$. *Then the following holds:*

$$(y - x) - z = y - x. \tag{2}$$

Proof. By Theorem 1(iv) we have $(y - x) - z = y - (x + z)$ and $x = x + 0 \leq x + z \leq x + x = x$ implies $x = x + z$. Hence (2) holds. ∎

Lemma 7. *Let* $A = (A; 0; +; -; \wedge; \vee)$ *be a DRl-semigroup and* $x, y \in Idm(A)$. *Then the following holds:*

$$0 \leq ((y - x) + (y - x)) - (y - x) \leq x \wedge (y - x). \tag{3}$$

Proof. By axiom 1(iii) and Lemma 5 we have $(x\wedge(y-x))+(y-x) = (x+(y-x))\wedge((y-x)+(y-x)) = (((y-x)\vee 0)+x)\wedge((y-x)+(y-x)) = (x\vee y)\wedge((y-x)+(y-x)) \geq y\wedge((y-x)+(y-x)) = (y+y)\wedge((y-x)+(y-x))$. Theorem 1(i) and (vi) imply $y = y-0 \geq y-x$ and therefore $(y+y)\wedge((y-x)+(y-x)) = (y-x)+(y-x)$. Putting it together we have $(x\wedge(y-x))+(y-x) \geq (y-x)+(y-x)$. Moreover, by Theorem 1(iii) $x\wedge(y-x) \geq ((y-x)+(y-x))-(y-x)$. Finally, by Lemma 5 we have $(((y-x)+(y-x))-(y-x))+(y-x) \geq (y-x)+(y-x) \geq y-x$ and Theorem 1(i) implies $((y-x)+(y-x))-(y-x) \geq (y-x)-(y-x) = 0$. ■

3. Results

Theorem 8. *Let $A = (A;0;+;-;\wedge;\vee)$ be a DRl-semigroup. Then $Idm(A)$ is a Brouwerian algebra.*

Proof. By Theorem 4 it follows that $Idm(A)$ is closed under operations $+,\wedge$ and $\vee$ and that $0 \in Idm(A)$. Now we will show that $Idm(A)$ is also closed under $-$. Assume that $x,y \in Idm(A)$ and denote $\alpha = ((y-x)+(y-x))-(y-x)$. By Lemma 7 we have $0 \leq \alpha \leq x\wedge(y-x) \leq x$ and therefore $x = x+0 \leq x+\alpha \leq x+x = x$, i.e., $x = x+\alpha$. Further, by Theorem 1(iii) and Lemmas 5, 6 and 7 we have $y-x = (y-x)\vee\alpha = (((y-x)-\alpha)\vee 0)+\alpha = ((y-x)\vee 0)+\alpha = (y-x)+\alpha = (y-x)+((y-x)+(y-x))-(y-x) \geq (y-x)+(y-x)$, i.e., $(y-x) \geq (y-x)+(y-x)$. The identity $(y-x) \leq (y-x)+(y-x)$ follows by Lemma 5. ■

Lemma 9. *Let $A = (A;0;+;-;\wedge;\vee)$ be a DRl-semigroup, $x,y \in Idm(A)$ and $x \leq y$. Then the interval $[x;y]$ equipped with the operations $+,\wedge$ and $\vee$ is a commutative lattice ordered monoid with the least element x and the greatest element y.*

Proof. Assume that $u,v \in [x;y]$. From $x \leq u \leq y$ and $x \leq v \leq y$ it follows $x \leq u\wedge v \leq u\vee v \leq u+v \leq y+y = y$ and therefore $[x;y]$ is closed under $+,\wedge$ and $\vee$. Obviously, x and y is the least and the greatest element of $[x;y]$, respectively. Further, by Lemma 3 we have $u+x = u\vee x = u$, i.e., x is the neutral element of $[x;y]$. ■

Remark 10. The interval $[x;y]$ from Lemma 9 may not be closed under the operation $-$ i.e., $[x;y]$ may not be a DRl-semigroup. Indeed, if $x > 0$ and $z \in [x;y]$ then $z-z = 0 \notin [x;y]$.

However, the following theorem shows that $[x;y]$ equipped with a naturally modified operation $-$ (denoted by $-^*$) is a DRl-semigroup.

Theorem 11. *Let $A = (A;0;+;-;\wedge;\vee)$ be a DRl-semigroup, $x,y \in Idm(A)$ and $x \leq y$. Then the structure $([x;y];x;+;-^*;\wedge;\vee)$ where $u -^* v = (u-v)\vee x$*

for all $u, v \in [x; y]$ *is a DRl-semigroup with the least element* x *and the greatest element* y.

Proof. By Lemma 9 we know that $([x; y]; x; +; \wedge; \vee)$ is a commutative lattice ordered monoid with the least element x and the greatest element y. Hence the axiom (1) is satisfied. Assume that $u, v \in [x; y]$ and denote $u -^* v = (u - v) \vee x$. By Theorem 1(vi) we have $y = y \vee x \geq (y - v) \vee x \geq (u - v) \vee x \geq x$, i.e., $(u -^* v) \in [x; y]$. Further, Theorem 1(iii), Lemma 3 and Lemma 5 imply $(u -^* v) + v = ((u - v) \vee x) + v = ((u - v) + v) \vee (x + v) = (((u - v) \vee 0) + v) \vee (x \vee v) = (u \vee v) \vee v = u \vee v \geq u$. If $z \in [x; y]$ and $z + v \geq u$ then obviously $z \geq u - v$ and $z \geq x$, i.e., $z \geq (u - v) \vee x = u -^* v$. Hence the axiom (2) is satisfied. By Lemmas 2, 3 and 5 we have $((u -^* v) \vee 0) + v = (((u - v) \vee x) \vee 0) + v = ((u - v) \vee x) + v = ((u - v) + x) + v = (u - v) + (x + v) = (u - v) + (x \vee v) = (u - v) + v = ((u - v) \vee 0) + v \leq u \vee v$. Hence the axiom (3) is satisfied. The axiom (4) is redundant and is implicitly satisfied (cf. [1]). ■

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