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ON THE VARIETIES $\mathcal{V}_n$

João Brandão[1] and Maria Borralho

Universidade do Algarve, CEOT
e-mail: jbrandao@ualg.pt
mfborralho@ualg.pt

Abstract

Here we set forth the varieties $\mathcal{V}_n$ and their connection with the varieties $\mathcal{E}_n$ of epigroups. A new congruence, *akin*, which relates similar elements in a semigroup, is introduced and used to reduce epigroups keeping their subgroup structure. We devise a recipe to study the conditions for these processes.

Keywords: semigroups, epigroups, varieties, congruences.

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1. Introduction

To introduce the varieties $\mathcal{V}_n$, we recall some standard definitions and notations. We generally follow Howie [3], although many of the results can be found in other references.

Let S be a semigroup. Here, and hereafter, unless stated otherwise S should be considered as a semigroup. An element a of S is called regular if there exists x in S such that $axa = a$. We say that $a^\dagger$ is an inverse of a regular element a if $aa^\dagger a = a$ and $a^\dagger a a^\dagger = a^\dagger$. Here we used the $\dagger$ symbol instead of the usual $'$ in order to avoid conflict with the pseudo-inverse one, see next paragraph. All regular elements have an inverse and all elements with inverse are regular. If all elements of S are regular, then S is called regular.

Whenever there is a positive integer n where a^n belongs to a subgroup of S, the element a of S is known as an epigroup element. The smallest n with this property is called the index of a and is represented by $\operatorname{ind}(a)$. If $\operatorname{ind}(a) = 1$, then a is considered as completely regular, and if all the elements of S are

[1]Corresponding author.

completely regular, then the semigroup is said to be completely regular. The Green's equivalence $\mathcal{H} - class\ H_{a^n}$ is the maximal subgroup of S containing a^n. Let e denote the identity element of H_{a^n}, then both $ae = ea$ and a^m, with $m \geq n$, are elements of H_{a^n} [4]. We define a' as pseudo-inverse of a by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of ae in the group H_{a^n} [4, 7]. If every element of a semigroup is an epigroup element, then the semigroup itself is said to be an epigroup. Every finite semigroup, and in fact every periodic semigroup, is an epigroup.

The following identities hold in all epigroups [7]:

$$x'xx' = x', \tag{1.1}$$
$$xx' = x'x, \tag{1.2}$$
$$x''' = x', \tag{1.3}$$
$$xx'x = x'', \tag{1.4}$$
$$(xy)'x = x(yx)', \tag{1.5}$$
$$(x^p)' = (x')^p. \tag{1.6}$$

Although usually quoted that p in equation (1.6) should be prime, it can be shown that it can have any natural value. Therefore, if $p = a.b$ (with a and b primes) we have:

$$(x^p)' = (x^{a.b})' = ((x^a)^b)' = ((x^a)')^b = ((x')^a)^b = (x')^{a.b} = (x')^p.$$

From equations (1.2) and (1.4) we can show that $xx'' = x''x$, as

$$xx'' = xxx'x = xx'xx = x''x,$$

and, as a consequence of this and of equation (1.3), all the multiple pseudo-inverses of the same element commute between each other.

From the above identities, other relations in epigroups important for this work can be deduced,

$$xe = x'', \tag{1.7}$$
$$x^m e \in H_{x^n}, \forall m \in \mathbb{N}, \tag{1.8}$$
$$x^m x'' \in H_{x^n}, \forall m \in \mathbb{N}, \tag{1.9}$$

where, as above, e denotes the identity element of the H_{x^n} subgroup.

We can view an epigroup $(S, \cdot)$ as a unary semigroup $(S, \cdot, ')$ where $x \mapsto x'$ is the map sending each element to its pseudo-inverse [5, 6, 7]. For each $n \in \mathbb{N}$, let $\mathcal{E}_n$ denote the variety (equational class) of all unary semigroups $(S, \cdot, ')$ satisfying equation (1.1), (1.2) and $x^{n+1}x' = x^n$. The following observation will be useful later.

Lemma 1.1 (See [2], Lemma 1). *For each $n \in \mathbb{N}$, the variety $\mathcal{E}_n$ is precisely the variety of unary semigroups satisfying* (1.1), (1.2) *and* $x^{n-1}x'' = x^n$.

Each $\mathcal{E}_n$ is a variety of epigroups, and the inclusions $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ hold for all n. Every finite semigroup is contained in some $\mathcal{E}_n$, and $\mathcal{E}_1$ is the variety of completely regular semigroups.

2. Starting point

The variety $\mathcal{V}$ appears in [1] as a variety of unary semigroups, which also generalizes completely regular semigroups, satisfying (1.1), (1.2), $x''y = xy$ and $xy'' = xy$.

Later Kinyon and Borralho [2] introduced the family of varieties of unary semigroups. For each $n \in \mathbb{N}$, the variety $\mathcal{V}_n$ is defined by (1.1), (1.2),

$$xy^{n-1}y'' = xy^n, \text{ and} \tag{2.1}$$

$$x''x^{n-1}y = x^n y. \tag{2.2}$$

There [2], they state that completely regular semigroups can be defined conceptually (unions of groups) or as unary semigroups satisfying certain identities. The epigroup varieties $\mathcal{V}_n$ only have a definition as unary semigroups. Since they are closed under taking variants [2, Theorem 6], they are clearly interesting varieties interlacing the varieties $\mathcal{E}_n$ [See 2, 2.4]. Thus one might ask the following.

Problem 1 (See [2]). Is there a conceptual characterization of the varieties $\mathcal{V}_n$, or even just $\mathcal{V}_1$, analogous to the characterizations of $\mathcal{E}_1$?

From [2, (2.4)] we have the following chain of varieties

$$\mathcal{E}_1 \subset \mathcal{V}_1 \subset \mathcal{E}_2 \subset \mathcal{V}_2 \subset \mathcal{E}_3 \cdots .$$

3. The *akin* binary relation

To better understand the role of the $\mathcal{V}_n$ varieties, we found convenient to define the binary relation *akin*, $\mathcal{A}$, in a semigroup S as

$$\mathcal{A} = \{(a, b) \in S^2 : xa = xb \wedge ay = by, \ \forall x, y \in S\}. \tag{3.1}$$

The binary relations *leftakin* ($\mathcal{LA}$) and *rightakin* ($\mathcal{RA}$) can also be defined by using only $xa = xb$ or $ay = by$ in equation (3.1) respectively, but these relations are not important for the purpose of this work. As usual, we will quote $a\mathcal{A}b$ to express that $(a, b) \in \mathcal{A}$.

Although related to the Green's relations $\mathcal{L}$ and $\mathcal{R}$ and $\mathcal{H}$, these $\mathcal{LA}$, $\mathcal{RA}$ and $\mathcal{A}$ relations are more restrictive. They force the corresponding elements of each column, or line in the Cayley table to be equal, instead of the sets of these elements including a and b. By other words, we can state that the *akin* relation is concerned with the identity of the elements, $xa = xb$ or $ay = by$, $\forall x, y \in S$, while the Green's relations are related to the sets $S^1a = S^1b$ or $aS^1 = bS^1$.

Two extreme cases must be referred. The first one, when $a\mathcal{A}\,b \Rightarrow a = b$, which arises for example in *monoid* epigroups. In this case $\mathcal{A}$ is the *equality* relation of S, 1_S. Another extreme situation occurs in, e.g., *null* semigroups where $a\mathcal{A}\,b, \forall a, b \in S$, then $\mathcal{A} = S \times S$ is the *universal* relation in S.

Of particular importance is the case when $a\mathcal{A}\,b$ and $a \neq b$. Then the a and b columns and lines of the Cayley table of the semigroup S are, respectively, identical. The semigroup S does not need to be commutative but $a^2 = ab = ba = b^2$ and, as a consequence, all the expressions involving only a and b having the same number of terms will give the same result. Also, in this occurrence, a and b cannot belong to the same subgroup of S, which do not have identical lines or columns, neither belong to different subgroups of S as $a^2 = b^2$. In addition, if one of them, e.g., a, belongs to a subgroup of S, then ind(a)=1 and ind(b)=2, as b^2 will belong to the same group of a. Both a and b will be elements of the same K_e *unipotency class* [7] of S. In an epigroup, if none of them are elements of a subgroup of S, they will have the same index, as $a^n = b^n$. In all cases, if S is an epigroup, they will have the same pseudoinverse as $a.e_g = b.e_g$, being e_g the equipotent element of their *unipotency class*.

As an example, consider the monogenic transformation semigroup $T = \langle\alpha\rangle = \{\alpha, \alpha^2, \alpha^3, \alpha^4\}$ with

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 4 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 4 & 5 \end{pmatrix},$$

$$\alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 4 \end{pmatrix}, \quad \alpha^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 5 & 4 & 5 \end{pmatrix}$$

and the composition operation, $\circ$. The Cayley table of this semigroup is

$$\begin{array}{c|cccc} \circ & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ \hline \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^3 \\ \alpha^2 & \alpha^3 & \alpha^4 & \alpha^3 & \alpha^4 \\ \alpha^3 & \alpha^4 & \alpha^3 & \alpha^4 & \alpha^3 \\ \alpha^4 & \alpha^3 & \alpha^4 & \alpha^3 & \alpha^4. \end{array} \tag{3.2}$$

Looking at this table we easily realise that $\alpha^2\mathcal{A}\,\alpha^4$. One of these elements, α^4, is regular and belongs to the subgroup $\{\alpha^3, \alpha^4\}$ while, as expected, ind(α^2)=2 as $\alpha^2 \circ \alpha^2 = \alpha^4$. It is interesting to see how two different maps of a set into itself can give *akin* elements on a transformation semigroup. The maps α^2 and α^4 only differ on the image of 1 which is 3 or 5, respectively, while the image of 3 is the same of the image of 5 in all the maps of this semigroup.

It is easy to find that the *akin* binary relation in an equivalence as it is reflexive, $a\mathcal{A}\,a$, symmetric, $a\mathcal{A}\,b \Rightarrow b\mathcal{A}\,a$, and transitive, $a\mathcal{A}\,b \wedge b\mathcal{A}\,c \Rightarrow a\mathcal{A}\,c$. So, the set S can be divided into equivalence classes defined as

$$A_a = \{b \in S : a\mathcal{A}\,b\}. \tag{3.3}$$

We can consider two kinds of *akin* equivalence classes. Those with one single element, which is only *akin* to itself, we call *singular akin* classes; and those with more than one element, which are *akin* between themselves, we call *pluri akin* classes.

In addition, the *akin* equivalence preserves the semigroup operation being a compatible equivalence, i.e., $a\mathcal{A}\,b \Rightarrow ax\mathcal{A}bx \wedge ya\mathcal{A}yb$, since by the definition (3.1) if $a\mathcal{A}\,b$, then $ax = bx$ and $ya = yb$ and the *akin* relation is reflexive. As a consequence, the *akin* binary relation is a congruence and defines a quotient semigroup of S, $S/\mathcal{A}$. If $\mathcal{A}$ is the *equality* relation of S, i.e., all *akin* classes are *singular*, there is no effect, and the semigroup S and $S/\mathcal{A}$ are isomorphic, but when there is at least a pair $(a, b) \in \mathcal{A} \wedge a \neq b$, i.e., at least one *akin* class is *pluri*, we call this process an *akin reduction*, or simply *reduction* if there is no confusion, of S and represent it as $S_r = S/\mathcal{A}$.

This *akin reduction* generates a new semigroup, S_r, where each *singular* classes will be represented by its own element, and each element of a *pluri* class will be replaced by a new one representing that class. As a consequence, the result of the semigroup operation in S_r will be the same as in S, if it is an element of a *singular* class, and will be the representative of the class when the result of the operation in S is an element of a *pluri* class. Accordingly, the Cayley table of the $T_r = T/\mathcal{A}$ of example 3.2 is

$$\begin{array}{c|ccc} \circ & \alpha & \alpha^{2,4} & \alpha^3 \\ \hline \alpha & \alpha^{2,4} & \alpha^3 & \alpha^{2,4} \\ \alpha^{2,4} & \alpha^3 & \alpha^{2,4} & \alpha^3 \\ \alpha^3 & \alpha^{2,4} & \alpha^3 & \alpha^{2,4}, \end{array} \tag{3.4}$$

where we used the symbol $\alpha^{2,4}$ to represent any element of the A_{α^2} *akin* class.

This process can be repeated if the reduced semigroup has new *pluri akin* classes. We note, however, that any *akin* class has at most one subgroup element and, as a consequence, the subgroup structure of the semigroup is conserved in these *reduction* procedures.

4. The $\mathcal{V}_n$ varieties

According to the definition of the varities $\mathcal{V}_n$ we can say that

$$S \in \mathcal{V}_n \Leftrightarrow x^{n-1}x'' \mathcal{A}\, x^n, \forall x \in S.$$

Similarly to the "index" of the elements of epigroups [7], we can define an a-index of an element x in an epigroup, S, as n such that $x^{n-1}x''\, y = x^n\, y \wedge y\, x^{n-1}x'' = y\, x^n, \forall y \in S$ or, using the *akin* relation, the smallest natural number such that $x^{n-1}x'' \mathcal{A}\, x^n$. This a-index will be denoted as a-ind(x). Also, similarly to epigroups, where ind $S = \max\{\text{ind}(x), \forall x \in S\}$, if the a-indeces of an epigroup S are bounded, we can define a v-index of this epigroup, as v-ind $S = \max\{\text{a-ind}(x), \forall x \in S\}$. The subscript $_m$ will be used to signal an element x of S with $\text{ind}(x_m) = \text{ind}\, S$ and $\text{a-ind}(x_m) = \text{v-ind}\, S$.

Although the *akin* relation could be applied to all elements of the epigroup, we are more interested in the *akin* class of x_m^n, which defines the $\mathcal{E}_n$ and $\mathcal{V}_n$ varieties. We note, however, that most of the sentences regarding the x_m can be applied to any other element of the epigroup, taking into account its own index and a-index instead of the epigroup indexes.

Regarding the relation between the $\mathcal{E}_n$ and $\mathcal{V}_n$ varieties of an epigroup, i.e., the v-index and the index of the epigroup, two different cases can occur for an epigroup S:

- In Case I, v-ind $S = n = \text{ind}\, S$, i.e., $x_m^{n-1}x_m'' \mathcal{A}\, x_m^n$ and $x_m^{n-1}x_m'' = x_m^n$. Both $x_m^{n-1}x_m''$ and x_m^n are the same element of a subgroup of S and the *akin* class of x_m^n is *singular*.
- In Case II, v-ind $S = n = \text{ind}\, S - 1$. Thus, $x_m^{n-1}x_m'' \mathcal{A}\, x_m^n$, but $x_m^{n-1}x_m'' \neq x_m^n$, being, by 1.9, $x_m^{n-1}x_m''$ an element of a subgroup of S, but not x_m^n. These semigroups can be object of *akin reduction* processes.

As stated above, all monoid epigroups will be in Case I, while the *null* epigroups will be Case II.

In addition to these general remarks, it is important to study the conditions for the relation between the v-index of an epigroup and its index.

Here and henceforth, except otherwise stated, we consider S an epigroup with index $n \geq 2$, ind $S \geq 2$. Note that if ind $S = 1$, then all the elements of S are regular and the v-index should also be one. Following Lemma 1.1, in S there will be, at least, one element $h = x_m^{n-1}x_m'' = x_m^n$. Also, in S, there are two different elements $f = x_m^{n-2}x_m''$ and $g = x_m^{n-1}$, which when operated with x_m will give $x_m f = f x_m = x_m g = g x_m = h$. f and g must be different, otherwise by 1.1 ind S should be $n-1$. In order to assess if S is a Case I or a Case II epigroup, we need to consider the conditions that must be fulfilled for these two elements to be *akin* to each other, $(f \mathcal{A}\, g)$ and $\text{a-ind}(x_m) = \text{ind}(x_m) - 1$, i.e., v-ind $S = \text{ind}\, S - 1$. For this purpose, we are going to focus our attention on the right and left products of x_m by S, x_mS and Sx_m.

Theorem 4.1 (Necessary condition)**.** *For v-ind$S$ = ind$S - 1$, it is necessary that $x_m \notin x_mS \wedge x_m \notin Sx_m$.*

Proof. Supposing that there exists an element $u \in S$ such that $x_mu = x_m$, then

$$\begin{aligned} fu &= x_m^{n-2}\underbrace{x''_m}u = x_m^{n-2}\underbrace{x_mx'_mx_m}u = x_m^{n-2}x_mx'_m\underbrace{(x_mu)} = x_m^{n-2}\underbrace{x_mx'_mx_m} \\ &= x_m^{n-2}x''_m = f \\ gu &= \underbrace{x_m^{n-1}}u = x_m^{n-2}x_mu = x_m^{n-2}\underbrace{(x_mu)} = \underbrace{x_m^{n-2}x_m} = x_m^{n-1} = g. \end{aligned}$$

As a consequence, the right multiplications of these two elements by u should give different results and they wouldn't be *akin* to each other. We should attain the same conclusion with the left multiplication of x_m. ■

We can express this necessary condition as

$$\text{a-ind}(x_m) = \text{ind}(x_m) - 1 \Rightarrow x_mS \subseteq S \setminus \{x_m\} \wedge Sx_m \subseteq S \setminus \{x_m\}. \tag{4.1}$$

Also by using these products, we can find a sufficient condition for $f\mathcal{A}g$.

Theorem 4.2 (Sufficient condition)**.** *For an epigroup $S \in \mathcal{E}_n$, the condition $x_mS = Sx_m = S \setminus \{x_m\}$ is a sufficient condition for $S \in \mathcal{V}_{n-1}$.*

Proof. As $x_mS = Sx_m = S \setminus \{x_m\}$, all the products $x_mu, u \in S$ (and those of $ux_m, u \in S$) will be different except for $u \in \{f, g\}$. We can say it because $\#(Sx_m) = \#(S \setminus \{x_m\}) = \#S - 1$. Then only two elements of Sx_m can be equal and these are $fx_m = gx_m = h$. This result can be expressed by

$$x_mu = x_mv \Rightarrow u = v \vee \{u, v\} = \{f, g\}, \forall u, v \in S. \tag{4.2}$$

As a consequence, we can also say that

$$\forall y \in S \setminus \{x_m, h\}\, \exists! u \in S : y = x_mu, \tag{4.3}$$

and conclude that when the two elements, f and g, are right (or left) multiplied by any other element of S, say y, the result will be the same. This can be seen as:

- If $y = x_m$ then $fx_m = gx_m = h$.
- if $y = f$ then $ff = x_m^{n-2}x''_mx_m^{n-2}x''_m$. Considering that $x''_m = x_me_g$, where e_g is the idempotent of the group of $x_m^n = h$, then

$$x_m^{n-2}x''_mx_m^{n-2}x''_m = x_m^{2n-2}e_g^2 = x_m^{2n-2}$$

and, by the same rationality, $gf = x_m^{n-1}x_m^{n-2}x''_m = x_m^{2n-2}$. So $ff = gf$.

- Similarly, if $y = g$ then $gf = gg$.
- Otherwise, using $y = x_m u$,

$$fy = x_m^{n-2} x_m'' y = x_m^{n-2} \underbrace{x_m'' x_m} u = x_m^{n-2} x_m x_m'' u = x_m^{n-1} x_m'' u = hu$$
$$gy = x_m^{n-1} y = \underbrace{x_m^{n-1} x_m} u = x_m^n u = hu,$$

  and $fy = gy$.

A similar result should be obtained by left multiplication. Then

$$Sx_m = x_m S = S \setminus \{x_m\} \Rightarrow x_m^{n-2} x_m'' \mathcal{A}\, x_m^{n-1}, \tag{4.4}$$

and a-ind$(x_m) = n - 1$, i.e., $S \in \mathcal{V}_{n-1}$. ■

As a consequence, when an epigroup S satisfies the condition $Sx_m = x_m S = S \setminus \{x_m\}$, we can apply the *reduction* process to define a new epigroup $S_r = S/\mathcal{A}$. As described above, in this process the two distinct f and g elements of S, $f = x_m^{n-2} x_m'' \mathcal{A}\, g = x_m^{n-1}$, will be replaced by a representative of their *akin* class, $w = x_m^{n-2} x_m'' = x_m^{n-1}$, which, by 1.9, is an subgroup element of S_r. Thus, in the S_r epigroup ind$(x_m) = n - 1$.

If the index of S is greater or equal to 3, then ind $S_r \geq 2$ and we can focus our attention on this S_r epigroup, again.

Taking into account that $x_m S = Sx_m = S \setminus \{x_m\}$ and that $S_r = S \setminus A_f \cup \{w\}$ we can conclude that $x_m S_r = S_r x_m = S_r \setminus \{x_m\}$.

As stated above when proving Theorem 4.2, all the products $x_m u, u \in S$ are different except for $u \in \{f, g\}$, which when operated with x_m give h and none produces x_m. So, there are two different elements in S, $u = x_m^{n-3} x_m''$ and $v = x_m^{n-2}$, which when operated with x_m give f and g. In S_r, the elements f and g have been replaced by w. As a consequence, in this epigroup S_r, u, and v when operated with x_m give the same result, w, and all the others will give different results but none produce x_m. We conclude that $\#(S_r x_m) = \#(S_r \setminus \{x_m\}) = \#S_r - 1$, and $S_r x_m = S_r \setminus \{x_m\}$.

Then, by Theorem 4.2 $u = x_m^{n-3} x_m'' \mathcal{A}\, v = x_m^{n-2}$ and a-ind$(x_m) = n - 2$.

The new epigroup S_r can be an object of another *reduction* process and so on. In general, we can say that, when an epigroup S, with ind$(x_m) \geq 2$, satisfies the condition $Sx_m = x_m S = S \setminus \{x_m\}$, we can apply the *akin reduction* process successively until ind$(x_m) = 1$.

The above referred monogenic transformation semigroup $(T, \circ)$, with $T = \langle \alpha \rangle = \{\alpha, \alpha^2, \alpha^3, \alpha^4\}$ and $\circ$ defined by the Cayley table 3.2, can be seen as an example of the application of Theorems 4.1 and 4.2.

This semigroup T is an epigroup with a subgroup $G = \{\alpha^3, \alpha^4\}$. As $\alpha \circ \alpha \circ \alpha = \alpha^3$, we conclude that ind $T = $ ind$(\alpha) = 3$ with $x_m = \alpha$, $x_m'' = \alpha'' = \alpha^3$. $T \in \mathcal{E}_3$

and verifies the condition $\alpha \circ \alpha \circ \alpha^3 = \alpha \circ \alpha \circ \alpha$. From the Cayley table 3.2, we conclude that $\alpha \circ T = T \circ \alpha = \{\alpha^2, \alpha^3, \alpha^4\} = T \setminus \{\alpha\}$, which satisfies both the necessary and sufficient conditions for v-ind $T = \operatorname{ind} T - 1$.

The two above referred *akin* elements α^2 and α^4 are respectively $\alpha \circ \alpha^3$ and $\alpha \circ \alpha$. So v-ind T = a-ind$(\alpha) = 2$ and $T \in \mathcal{V}_2$, being v-ind $T = \operatorname{ind} T - 1$, as expected from Theorem 4.2.

This result supports that the semigroup T can be reduced until $\operatorname{ind}(\alpha) = 1$. A further reduction of T_r, see example 3.4, will give the semigroup T_{rr},

(4.5)

$\circ$	$\alpha^{1,3}$	$\alpha^{2,4}$
$\alpha^{1,3}$	$\alpha^{2,4}$	$\alpha^{1,3}$
$\alpha^{2,4}$	$\alpha^{1,3}$	$\alpha^{2,4}$,

where we used the symbols $\alpha^{1,3}$ and $\alpha^{2,4}$ to represent any element of the A_α and A_{α^2} *akin* classes respectively. T_{rr} is now a completely regular semigroup and, as a consequence, $T_{rr} \in \mathcal{E}_1$ and $T_{rr} \in \mathcal{V}_1$. We can see that the group structure of the semigroup T has been conserved in T_{rr}, as stated before. The information of this *reduction* process can be complemented by the computation of $T^3 = \{\alpha^3, \alpha^4\}$. We can see that in these reduction processes the group elements are conserved.

If we add the identity element α^0,

$$\alpha^0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix},$$

to T, we obtain the monoid semigroup T^1 whose Cayley table is

(4.6)

$\circ$	α^0	α	α^2	α^3	α^4
α^0	α^0	α	α^2	α^3	α^4
α	α	α^2	α^3	α^4	α^3
α^2	α^2	α^3	α^4	α^3	α^4
α^3	α^3	α^4	α^3	α^4	α^3
α^4	α^4	α^3	α^4	α^3	α^4.

This T^1 semigroup does not satisfy the necessary condition 4.1 as $\alpha \circ T^1 = T^1 \circ \alpha = \{\alpha, \alpha^2, \alpha^3, \alpha^4\} \nsubseteq T^1 \setminus \{\alpha\}$, i.e., $\alpha \in \alpha \circ T^1$. Now, α^2 is not *akin* to α^4, the *akin* classes of all elements of T^1 are *singular* and v-ind $T = \operatorname{ind} T$.

5. GENERALISING

In this work we started studying the varieties $\mathcal{E}_n$ and $\mathcal{V}_n$. As a result, we took particular attention on x_m, which determines the varieties of the epigroup. Despite that, most of the above considerations can be applied to any element of the epigroup, a, considering its own index and a-index, independently of the index and v-index of the epigroup, S.

Adapting the above statements about x_m we can say.

Proposition 5.1. *The expressions, a-ind$(a) = n =$ ind$(a) - 1$ and $a^{n-1}a''\mathcal{A}\,a^n$, but $a^{n-1}a'' \neq a^n$, are equivalent.*

In this case, the epigroup S can be object of an *akin reduction* process.

And the necessary and sufficient conditions will be:

Proposition 5.2 (necessary)**.** *It is necessary that $a \notin aS \wedge a \notin Sa$ to a-ind$(a) = n =$ ind$(a) - 1$.*

Proposition 5.3 (sufficient)**.** *It is sufficient that $Sa = aS = S \setminus \{a\}$ for the expression $a^{n-1}a''\mathcal{A}\,a^n \wedge a^{n-1}a'' \neq a^n$ to be accomplished.*

We can add that, when an epigroup S, with ind$(a) \geq 2$, satisfies this sufficient condition, we can apply the *akin reduction* process successively until ind$(a) = 1$.

The semigroup $(U, \circ)$, where $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\circ$ is defined by the Cayley table

(5.1)

$\circ$	1	2	3	4	5	6	7	8
1	3	2	4	2	2	2	2	2
2	2	2	2	2	2	2	2	2
3	4	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	2
5	2	2	2	2	6	7	8	7
6	2	2	2	2	7	8	7	8
7	2	2	2	2	8	7	8	7
8	2	2	2	2	7	8	7	8 ,

illustrates this generalization.

This semigroup has two subgroups, namely, $\{2\}$ and $\{7, 8\}$. As ind$(1) = 4$ and a-ind$(1) = 3$, we conclude that $x_m = 1$, v-ind U = ind $U - 1$, and $2\mathcal{A}\,4$ (as $1^2\,1'' = 2$ and $1^3 = 4$). In addition to this x_m other element of U satisfy similar relations, ind$(5) = 3$ and a-ind$(5) = 2 =$ ind$(5) - 1$ and $6\mathcal{A}\,8$ (as $5\,5'' = 8$ and $5^2 = 6$). Both 1 and 5 satisfy the Proposition 5.1 $a^{n-1}a''\mathcal{A}\,a^n$, but $a^{n-1}a'' \neq a^n$, and the Proposition 5.2, $a \notin aU \wedge a \notin Ua$. As a consequence, both can be used for *akin reduction* processes

After some *akin reduction* processes, we obtain the semigroup $(U^{red}, \circ)$ whose Cayley table is

(5.2)

$\circ$	$\bar{2}$	$\bar{7}$	$\bar{8}$
$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$
$\bar{7}$	$\bar{2}$	$\bar{8}$	$\bar{7}$
$\bar{8}$	$\bar{2}$	$\bar{7}$	$\bar{8}$,

where $\bar{2}$ stands for an element of $\{1, 2, 3, 4\}$, $\bar{7}$ for an element of $\{5, 7\}$, and $\bar{8}$ for an element of $\{6, 8\}$.

6. Conclusion

We have shown that the *akin* congruence relation can be used to define the varieties $\mathcal{V}_n$ and study their connection with the varieties $\mathcal{E}_n$ of epigroups. This new congruence, *akin*, which relates similar elements in a semigroup, can be used to reduce the epigroups keeping their subgroup structure. We have demonstrated that the products aS and Sa can be used to define a necessary and a sufficient condition for these processes.

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