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COHERENT LATTICES

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Abstract

The notion of coherent lattices is introduced and established relations between a coherent lattice and that of a generalized Stone lattice, Boolean algebra, quasi-complemented lattice, and normal lattice. A set of equivalent conditions is given for every sublattice of a lattice to become a coherent lattice. Some equivalent conditions are given for every interval of a lattice to become a coherent sublattice. Coherent lattices are characterized with the help of certain properties of filters and dense elements.

Keywords: Coherent lattice, generalized Stone lattice, Boolean algebra, quasi-complemented lattice, normal lattice.

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1. Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker [11] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in

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the structures of rings as well as lattices and characterized many algebraic structures in terms of annihilators. Speed [15] and Cornish [4, 6] made an extensive study of annihilators in distributive lattices. Cornish introduced the notion of normal lattices [4] and characterized the normal lattices using minimal prime ideals and congruences. In [4], he introduced the notion of quasi-complemented lattices and characterized the class of quasi-complemented lattices using the annulets and congruences. In [4], he introduced the notion of generalized Stone lattices and studied the interconnections among generalized Stone lattices, normal lattices and quasi-complemented lattices. The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by Stone [16], Frink [8], and George Gratzer [9]. Later many authors like Speed [15], and Frink [8] etc., extended the study of pseudo-complements to characterize Stone lattices. In [3], Chajda, Halaš and Kühr extensively studied the structure of pseudo-complemented semilattices. In [12], the authors investigated extensively certain properties of D-filters of distributive lattices. In this paper, the authors given a set of equivalent conditions for a quasi-complemented lattice to become a Boolean algebra by using the D-filters. In [13], the authors investigated the properties of prime D-filters and then characterized the minimal prime D-filters of distributive lattices using certain congruences.

In this note, the concept of coherent lattices is introduced and proved that every generalized Stone lattice is a coherent lattice. Some equivalent conditions are given for every coherent lattice to become a generalized Stone lattice. Boolean algebras are characterized in terms of annulets and principal ideals of distributive lattices and then a set of equivalent conditions is given for every coherent lattice to become a Boolean algebra. A sufficient condition is given for every quasi-complemented lattice to become a coherent lattice. A sufficient condition is given for every coherent lattice to become a normal lattice. Properties of coherent lattices are generalized to the case of direct product of coherent lattices.

A set of equivalent conditions is given for every sublattice of a lattice to become a coherent sublattice. Some equivalent conditions are derived for every interval of a lattice to become a coherent lattice. Coherent lattices are characterized with the help of the properties of filters and D-filters of distributive lattices.

2. Preliminaries

The reader is referred to [1, 4, 6, 7, 10, 12, 13, 14] and [15] for the elementary notions and notations of distributive lattices. Some of the preliminary definitions and results are presented for the ready reference of the reader.

Definition [1]. An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,

(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,

(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of a lattice L is called an ideal (filter) of L if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. Define a relation $\leq$ on a lattice L by $x \leq y$ if and only if $x \vee y = y$ or equivalently $x \wedge y = x$. Then $(L, \leq)$ is a partially ordered set. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice. A proper ideal (resp. filter) P of a distributive lattice L is said to be *prime* if for any $x, y \in L$, $x \wedge y \in P$ (resp. $x \vee y \in P$) implies $x \in P$ or $y \in P$. A proper ideal (resp. filter) P of a lattice L is called *maximal* if there exists no proper ideal (resp. filter) Q such that $P \subset Q$. A proper ideal (resp. filter) P of a distributive lattice is *minimal* [4] if there exists no prime ideal (resp. filter) Q such that $Q \subset P$.

For any non-empty subset A of a distributive lattice L with 0, the annulet [6] of a is define as the set $(a)^* = \{x \in L \mid x \wedge a = 0\}$. For any $a \in L$, $(a)^*$ is an ideal of the lattice L. An element $x \in L$ is called *dense* if $(x)^* = \{0\}$ and the set of all dense elements of a lattice is denoted by D.

Proposition 1 [15]**.** *Let L be a distributive lattice with 0. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,

(2) $(a \vee b)^* = (a)^* \cap (b)^*$,

(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,

(4) $(a)^{***} = (a)^*$,

(5) $(a)^* = L$ *if and only if* $a = 0$.

Theorem 2 [10]**.** *A prime ideal P of a distributive lattice is a minimal prime ideal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$* (or equivalently, for any $x \in L$, $x \notin P$ if and only if $(x)^* \subseteq P$).

A distributive lattice L with 0 is called a *normal lattice* [4] if every prime ideal contains a unique minimal prime ideal. A distributive lattice L with 0 is called a *quasi-complemented lattice* [7] if to each $x \in L$, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. A distributive lattice with 0 and dense elements is quasi-complemented if and only if to each $x \in L$, there exists $x' \in L$ such that $(x)^{**} = (x')^*$. A distributive lattice L with 0 is called a *generalized Stone lattice* [7] if it satisfies the property: $(x)^* \vee (x)^{**} = L$ for all $x \in L$. The pseudo-complement b^* of an element b is the element satisfying

$$a \wedge b = 0 \;\Leftrightarrow\; a \wedge b^* = a \;\Leftrightarrow\; a \leq b^*$$

where $\leq$ is the induced order of L. Every pseudo-complemented distributive lattice is a quasi-complemented lattice. In a pseudo-complemented distributive lattice, we have $(x^*] = (x)^*$ for any $x \in L$.

Theorem 3 [4]**.** *Following are equivalent in a distributive lattice L with 0:*

(1) *L is normal;*

(2) *for any $x, y \in L$, $x \wedge y = 0$ implies $(x)^* \vee (y)^* = L$;*

(3) *for any $x, y \in L$, $(x)^* \vee (y)^* = (x \wedge y)^*$.*

Theorem 4 [4]**.** *A distributive lattice L with 0 is a generalized Stone lattice if and only if it satisfies the following conditions:*

(1) *L is quasi-complemented,*

(2) *L is normal.*

A lattice L is called relatively complemented if for any $a, b \in L$, the interval $[a, b]$ is a complemented lattice. A lattice L is relatively complemented if $[0, a]$ is complemented for any $a \in L$.

Theorem 5 [14]**.** *A distributive lattice L with 0 is relatively complemented if and only if every prime ideal of L is a minimal prime ideal.*

A filter F of a distributive lattice L is called a *D-filter* [12] if $D \subseteq F$. A prime D-filter of a distributive lattice is *minimal* if it is the minimal element in the poset of all prime D-filters. A prime D-filter of a distributive lattice is minimal [13] if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$. For any non-empty subset A of a lattice, we define $A^\circ = \{x \in L \mid x \vee y \in D \text{ for all } y \in A\}$. Clearly A° is a D-filter of L. For $A = \{a\}$, we consider $\{a\}^\circ$ by $(a)^\circ$.

Proposition 6. [13] *Let A, B be two subsets of a distributive lattice L. Then*

(1) *$A \subseteq B$ implies $B^\circ \subseteq A^\circ$,*

(2) *$A \subseteq A^{\circ\circ}$,*

(3) *$A^{\circ\circ\circ} = A^\circ$,*

(4) *$A^\circ = L$ if and only if $A \subseteq D$.*

Throughout this note, all lattices are bounded and distributive unless otherwise mentioned.

3. COHERENT LATTICES

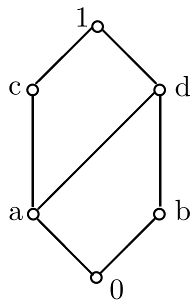
In this section, the notion of coherent lattices is introduced. Relations of the coherent lattices with the classes of generalized Stone lattices, Boolean algebras, quasi-complemented lattices, normal lattices are investigated. Coherent lattices are characterized with the help of filters and D-filters.

Definition. A lattice L is called a *coherent lattice* if, for all $x, y \in L$,

$$x \vee y \in D \text{ implies } (x)^{**} \vee (y)^{**} = L.$$

Since every non-zero element of a chain (totally ordered set) is a dense element, every chain is a coherent lattice. Obviously, every dense lattice (i.e., $(x)^* = \{0\}$ for all $0 \neq x \in L$) is coherent. In the following example, we observe a non-trivial example of a coherent lattice.

Example 7. Consider the following bounded and finite distributive lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given by:



Observed that $(a)^* = \{0, b\}$, $(b)^* = \{0, a, c\}$, $(c)^* = \{0, b\}$ and $(d)^* = \{0\}$. Clearly $d, 1$ are the only dense elements in L. Also $(a)^{**} = \{0, a, c\}$, $(b)^{**} = \{0, b\}$, $(c)^{**} = \{0, a, c\}$ and $(d)^{**} = L$. It can be routinely verified that L is coherent.

Proposition 8. *Every generalized Stone lattice is a coherent lattice.*

Proof. Assume that L is a generalized Stone lattice. Let $x, y \in L$ be such that $x \vee y \in D$. By Theorem 4, L is quasi-complemented and normal. Then there exist $x', y' \in L$ such that $(x)^{**} = (x')^*$ and $(y)^{**} = (y')^*$. Hence, we get

$$\begin{aligned}
(x)^{**} \vee (y)^{**} &= (x')^* \vee (y')^* \\
&= (x' \wedge y')^* && \text{since } L \text{ is normal} \\
&= (x' \wedge y')^{***} && \text{by Proposition 1(4)} \\
&= \{(x')^{**} \cap (y')^{**}\}^* && \text{by Proposition 1(3)} \\
&= \{(x)^* \cap (y)^*\}^* \\
&= \{(x \vee y)^*\}^* \\
&= L && \text{since } x \vee y \in D
\end{aligned}$$

Hence $(x)^{**} \vee (y)^{**} = L$ for all $x, y \in L$ with $x \vee y \in D$. Thus L is coherent. ■

In the following theorem, a set of equivalent conditions is given for a coherent lattice to become a generalized Stone lattice.

Theorem 9. *Let L be a coherent lattice. Then the following are equivalent:*

(1) *L is a generalized Stone lattice;*

(2) *every prime D-filter is a minimal prime D-filter;*

(3) *every maximal filter is a minimal prime D-filter;*

(4) *L is quasi-complemented.*

Proof. (1)⇒(2): Assume that L is a generalized Stone lattice. Let P be a prime filter of L. Let $d \in D$ and $x \in P$. Since L is a generalized Stone lattice, we get $(x)^* \vee (x)^{**} = L$. Hence $d \in (x)^* \vee (x)^{**}$. Thus $a \vee b = d \in D$ for some $a \in (x)^*$ and $b \in (x)^{**}$. Hence $a \wedge x = 0$ and $(x)^* \subseteq (b)^*$. Suppose $a \in P$. Then $0 = a \wedge x \in P$, which is a contradiction. Thus, we must have $a \notin P$. Now,

$$\begin{aligned} a \vee b \in D &\Rightarrow (a)^* \cap (b)^* = \{0\} \\ &\Rightarrow (a)^* \cap (x)^* = \{0\} \qquad \text{since } (x)^* \subseteq (b)^* \\ &\Rightarrow (a \vee x)^* = \{0\} \end{aligned}$$

which means that $a \vee x \in D$. Therefore P is a minimal prime D-filter of L.

(2)⇒(3): Since every maximal filter is a prime D-filter, it is clear.

(3)⇒(4): Assume condition (3). Let $x \in L$. Suppose $0 \notin [x) \vee (x)^\circ$. Then there exists a maximal filter M such that $[x) \vee (x)^\circ \subseteq M$. Hence $x \in M$ and $(x)^\circ \subseteq M$. By condition (3), M will be a minimal prime D-filter. Since $(x)^\circ \subseteq M$, we get $x \notin M$, which is a contradiction. Hence $0 \in [x) \vee (x)^\circ$. Thus $x \wedge a = 0$ for some $a \in (x)^\circ$. Hence $x \vee a \in D$. Therefore L is quasi-complemented.

(4)⇒(1): Assume that L is quasi-complemented. Let $x \in L$. Since L is quasi-complemented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. Since L is coherent, we get $(x)^{**} \vee (x')^{**} = L$. Since $x \wedge x' = 0$, we get $(x')^{**} \subseteq (x)^*$. Hence $L = (x)^{**} \vee (x')^{**} \subseteq (x)^{**} \vee (x)^*$. Therefore L is a generalized Stone lattice. ■

Since every pseudo-complemented lattice is quasi-complemented, the following corollary is a direct consequence of the above theorem:

Corollary 10. *Let L be a pseudo-complemented lattice. Then L is a coherent lattice if and only if it is a generalized Stone lattice.*

Corollary 11. *A quasi-complemented and coherent lattice is normal.*

Proof. Follows from Theorem 9 and Theorem 4. ■

Corollary 12. *Any quasi-complemented and normal lattice is coherent.*

Proof. Follows from Theorem 9 and Theorem 4. ∎

Theorem 13. *A lattice L is Boolean if and only if $(x] \vee (x)^* = L$ for all $x \in L$.*

Proof. Assume that L is a Boolean algebra. Let $x \in L$. Since L is Boolean, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. Hence $x' \in (x)^*$ and $(x] \vee (x'] = (1] = L$. Thus $(x'] \subseteq (x)^*$. Therefore $(x] \vee (x)^* = L$.

Conversely, assume the condition. Let $x \in L$. Hence $(x)^* \vee (x] = L$. Thus $1 \in (x)^* \vee (x]$. Hence $a \vee x = 1$ for some $a \in (x)^*$. Since $a \in (x)^*$, we get $a \wedge x = 0$. Thus a is the complement of x in L. Therefore L is a Boolean algebra. ∎

Proposition 14. *Every Boolean algebra is a generalized Stone lattice.*

Proof. Assume that L is a Boolean algebra. Let $x \in L$. Suppose $(x)^* \vee (x)^{**} \neq L$. Then there exists prime ideal P such that $(x)^* \vee (x)^{**} \subseteq P$. Hence $(x)^* \subseteq P$ and $x \in (x)^{**} \subseteq P$. Since L is Boolean, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. Suppose $x' \in P$. Then $1 = x \vee x' \in P$, which is a contradiction. Hence $x' \notin P$. Therefore P is minimal. Since $(x)^* \subseteq P$, we get $x \notin P$ which is a contradiction. Thus $(x)^* \vee (x)^{**} = L$. Hence L is a generalized Stone lattice. ∎

Corollary 15. *Every Boolean algebra is a coherent lattice.*

Proof. It follows from Proposition 8 and Proposition 14. ∎

In general, the converse of Proposition 14 is not true. For, consider the coherent lattice given in Example 7. Observe that L is not a Boolean algebra because the element a has no complement in L. However, a set of equivalent conditions is given for a coherent lattice to become a Boolean algebra.

Theorem 16. *A coherent lattice L is a Boolean algebra if and only if it satisfies the following conditions;*

(1) *L is quasi-complemented;*

(2) *every principal ideal is an annihilator ideal.*

Proof. Given that L is a coherent lattice. Assume that L satisfies conditions (1) and (2). Since L is quasi-complemented and coherent, by Theorem 9, we get that L is a generalized Stone lattice. Let $x \in L$. Then $(x)^* \vee (x)^{**} = L$. By (2), we get $(x] = (x)^{**}$. Hence $(x] \vee (x)^* = L$. By Theorem 13, L is a Boolean algebra.

Conversely, assume that L is Boolean. Clearly L is quasi-complemented. Let $x \in L$. By Proposition 14, L is a generalized Stone lattice. Hence $(x)^{**} \vee (x)^* = L$. By Theorem 13, we get $(x] \vee (x)^* = L$. It is clear that $(x] \cap (x)^* = \{0\}$ and $(x)^{**} \cap (x)^* = \{0\}$. Since the lattice $\mathcal{I}(L)$ of all ideals of the lattice L is distributive, by the cancellation property, we get $(x] = (x)^{**}$. Thus every principal ideal is an annihilator ideal. ∎

Corollary 17. *A generalized Stone lattice is a Boolean algebra if and only if every principal ideal of the lattice is an annihilator ideal.*

Proof. It follows from Theorem 9 and Theorem 16. ∎

Proposition 18. *A lattice in which every maximal ideal is non-dense is a coherent lattice.*

Proof. Let L be a lattice in which every maximal ideal is non-dense. Let $x, y \in L$ be such that $x \vee y \in D$. Suppose $(x)^{**} \vee (y)^{**} \neq L$. Then there exists a maximal ideal M such that $(x)^{**} \vee (y)^{**} \subseteq M$. Hence $(x)^{**} \subseteq M$ and $(y)^{**} \subseteq M$. Thus $M^* \subseteq (x)^*$ and $M^* \subseteq (y)^*$, which gives $M^* \subseteq (x)^* \cap (y)^* = (x \vee y)^* = \{0\}$ because of $x \vee y \in D$. Thus M is dense, which is a contradiction. Hence $(x)^{**} \vee (y)^{**} = L$. Therefore L is a coherent lattice. ∎

The converse of Proposition 18 is not true. For, consider the coherent lattice given in Example 7. Clearly L is possessing two maximal ideals, namely $M_1 = \{0, a, c\}$ and $M_2 = \{0, a, b, d\}$. Observe that $M_2^* = \{0\}$ but $M_1^* = \{0, b\} \neq \{0\}$. The following result about the direct products of coherent lattices is of intrinsic interest. First we need the following lemma whose proof is routine.

Lemma 19. *For any positive integer n, let $L_1, L_2, \ldots, L_n$ be n lattices. For any $a_1 \in L_1, a_2 \in L_2, \ldots, a_n \in L_n$, the following properties hold:*

(1) $(a_1, a_2, \ldots, a_n)^* = (a_1)^* \times (a_2)^* \times \cdots \times (a_n)^*$,

(2) $(a_1, a_2, \ldots, a_n)^* \vee (b_1, b_2, \ldots, b_n)^* = (a_1 \vee b_1, a_2 \vee b_2, \ldots, a_n \vee b_n)^*$,

(3) $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$.

Proof. The proof of (1) and (2) is routine.

(3) Let us denote $(x)_i = (x_1, x_2, \ldots, x_n)$ where $x_1 \in L_1, x_2 \in L_2, \ldots, x_n \in L_n$. For any $(a)_i \in L_1 \times L_2 \times \cdots \times L_n$, we have

$$\begin{aligned}
(x)_i \in (a)_i^{**} &\Leftrightarrow (a)_i^* \subseteq (x)_i^* \\
&\Leftrightarrow (a_1)^* \times (a_2)^* \times \cdots \times (a_n)^* \subseteq (x_1)^* \times (x_2)^* \times \cdots \times (x_n)^* \\
&\Leftrightarrow (a_i)^* \subseteq (x_i)^* \qquad \text{for } i = 1, 2, \ldots, n \\
&\Leftrightarrow (x_i)^{**} \subseteq (a_i)^{**} \qquad \text{for } i = 1, 2, \ldots, n \\
&\Leftrightarrow (x)_i \in (x)_i^{**} \subseteq (a)_i^{**}
\end{aligned}$$

Therefore $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$. ∎

Theorem 20. *Let $L_1, L_2, \ldots, L_n$ (where n is a positive integer) be a finite family of lattices. Then the product lattice $L_1 \times L_2 \times \cdots \times L_n$ (with point-wise operations) is coherent if and only if $L_1, L_2, \ldots, L_n$ are coherent.*

Proof. Assume that $L_1 \times L_2 \times \cdots \times L_n$ is a coherent lattice. Let $D_1, D_2, \ldots, D_n$ be the sets containing dense elements of $L_1, L_2, \ldots, L_n$ respectively. Let $a, b \in L_1$ be such that $a \vee b \in D_1$. Choose $d_2 \in D_2, d_3 \in D_3, \ldots, d_n \in D_n$. Then

$$(a, d_2, d_3, \ldots, d_n) \vee (b, d_2, d_3, \ldots, d_n) = (a \vee b, d_2, d_3, \ldots, d_n) \in D_1 \times D_2 \times \cdots \times D_n.$$

Since $L_1 \times L_2 \times \cdots \times L_n$ is coherent, we get

$$(a, d_2, d_3, \ldots, d_n)^{**} \vee (b, d_2, d_3, \ldots, d_n)^{**} = L_1 \times L_2 \times \cdots \times L_n.$$

Let $z \in L_1$. Then $(z, d_2, d_3, \ldots, d_n) \in L_1 \times L_2 \times \cdots \times L_n$. Hence, there exists $(s_1, s_2, \ldots, s_n) \in (a, d_2, d_3, \ldots, d_n)^{**}$ and $(t_1, t_2, \ldots, t_n) \in (b, d_2, d_3, \ldots, d_n)^{**}$ such that

$$(z, d_2, d_3, \ldots, d_n) = (s_1, s_2, \ldots, s_n) \vee (t_1, t_2, \ldots, t_n).$$

Therefore $z = s_1 \vee t_1$ where $s_1 \in (a)^{**}$ and $t_1 \in (b)^{**}$. Hence $(a)^{**} \vee (b)^{**} = L_1$. Therefore L_1 is coherent. Similarly, it can be proved that $L_2, L_3, \ldots, L_n$ are coherent lattices. Converse follows from the fact that $(a_1, a_2, \ldots, a_n)^{**} = (a_1)^{**} \times (a_2)^{**} \times \cdots \times (a_n)^{**}$ for any $(a_1, a_2, \ldots, a_n) \in L_1 \times L_2 \times \cdots \times L_n$. ■

Lemma 21. *If a lattice L is relatively complemented, then every chain has at most three elements.*

Proof. Assume that L is relatively complemented. Suppose there exist three elements $x, y, z \in L - \{0\}$ such that $0 < x < y < z$. Clearly $x \in [0, x \vee y]$. Since L is relatively complemented, there exists $t \in L$ such that $x \wedge t = 0$ and $x \vee t = x \vee y = y$. Since $x \wedge t = 0$, by the assumption, we get $y = x \vee t = 1$. This is absurd. Therefore every chain of L has at most three elements. ■

A sublattice S of a lattice L is called a *D-sublattice* if $0 \in S$ and $S \cap D \neq \emptyset$. An ideal J of a lattice is called D-ideal if $J \cap D \neq \emptyset$. Clearly every D-ideal is a D-sublattice.

Theorem 22. *The following assertions are equivalent in a lattice L:*

(1) *every D-sublattice is coherent;*

(2) *for any $x, y \in L - \{0\}$, $x \wedge y = 0$ implies $x \vee y = 1$;*

(3) *L is a dense lattice or L is relatively complemented.*

Proof. (1)⇒(2): Assume that every D-sublattice of L is coherent. Let $x, y \in L - \{0\}$ be such that $x \wedge y = 0$. Suppose that $x \vee y \neq 1$. Choose $1 \neq z \in L$ such that $x \vee y < z$. Now, consider the sublattice $L_1 = \{0, x, y, x \vee y, z\}$. Clearly $x \vee y \in D_1$ and so L_1 is a D-sublattice. Now, $(x)^{**} \vee (y)^{**} = \{0, x\} \vee \{0, y\} = L_1 - \{z\} \neq L_1$. Hence L_1 is not coherent which contradicts the assumption. Therefore $x \vee y = 1$.

(2)⇒(3): Assume condition (2). Suppose L is non-dense. Then $\{0\}$ is not a prime ideal of L. Let P be a prime ideal of L. Suppose P is not minimal. Then there exists minimal prime ideal M such that $M \subset P$. Choose $0 \neq x \in M$. Then $(x)^* \cap P = \{0\}$, otherwise $y \in (x)^* \cap P$. Then $x \wedge y = 0$ and $y \in P$. By the hypothesis, we get $x \vee y = 1$. Since $x \in P$, we get $1 = x \vee y \in P$ which is a contradiction. Hence $(x)^* \cap P = \{0\} \subseteq M$. Since M is prime and $M \subset P$, we must have $(x)^* \subseteq M$. This contradicts the fact that M is minimal. Therefore P is minimal. By Theorem 5, L is relatively complemented.

(3)⇒(1): Assume condition (3). Let L_1 be a D-sublattice of L and D_1 is the set of all dense elements of L_1. If L is dense, then we are through. Suppose L is relatively complemented. By Lemma 21, every chain in L has at most three elements. Let $x, y \in L_1$ be such that $x \vee y \in D_1$. Suppose $x \in D_1$ or $y \in D_1$. Then clearly $(x)^{**}_{L_1} \vee (y)^{**}_{L_1} = L_1$. Suppose $x \notin D_1$ and $y \notin D_1$. Suppose $0 < x \leq x \vee y$. If $x = x \vee y$, then $x \in D_1$ which is a contradiction. Hence $0 < x < x \vee y$. Since every chain has at most three elements, $x \vee y$ will be the greatest element of L_1. Hence $x \vee y \in (x)^{**}_{L_1} \vee (y)^{**}_{L_1}$. Therefore $(x)^{**}_{L_1} \vee (y)^{**}_{L_1} = L_1$. ■

Theorem 23. *The following assertions are equivalent in a lattice L:*

(1) *L is coherent;*

(2) *each proper D-ideal is a coherent sublattice;*

(3) *for each $d \in D$, $[0, d]$ is a coherent sublattice.*

Proof. (1)⇒(2): Assume that L is a coherent lattice. Let J be a D-ideal of L with $J \neq L$. Suppose $x, y \in J$ be such that $x \vee y \in D_J \subseteq D$. Since L is coherent, we get $(x)^{**} \vee (y)^{**} = L$. Write $(a)^{**}_J = J \cap (a)^{**}$ for any $a \in J$. Clearly $(a)^{**}_J$ is an ideal in J with $(a)^*_J$ is an annulet of a in J. Now, we get

$$J = J \cap L = J \cap \{(x)^{**} \vee (y)^{**}\} = \{J \cap (x)^{**}\} \vee \{J \cap (y)^{**}\} = (x)^{**}_J \vee (y)^{**}_J$$

which yields that J is a coherent sublattice of L.

(2)⇒(3): It is obvious because of $[0, d]$ is a proper D-ideal for any $d \in D$.

(3)⇒(1): By taking $d = 1$, the proof follows. ■

Definition. For any non-empty subset A of a lattice L, define

$$A^\tau = \{x \in L \mid (a)^{**} \vee (x)^{**} = L \text{ for all } a \in A\}$$

Clearly $\{0\}^\tau = D$ and $L^\tau = D$. For any $a \in L$, we denote $(\{a\})^\tau$ by $(a)^\tau$. Then it is obvious that $(0)^\tau = D$ and $(1)^\tau = L$.

Proposition 24. *For any non-empty subset A of L, A^τ is a D-filter of L.*

Proof. Clearly $D \subseteq A^\tau$. Let $x, y \in A^\tau$. For any $a \in A$, we get $(x \wedge y)^{**} \vee (a)^{**} = \{(x)^{**} \cap (y)^{**}\} \vee (a)^{**} = \{(x)^{**} \vee (a)^{**}\} \cap \{(y)^{**} \vee (a)^{**}\} = L \cap L = L$. Hence $x \wedge y \in A^\tau$. Again, let $x \in A^\tau$ and $x \leq y$. Then $(x)^{**} \vee (a)^{**} = L$ for any $a \in A$ and $(x)^{**} \subseteq (y)^{**}$. For any $c \in A$, we get $L = (x)^{**} \vee (c)^{**} \subseteq (y)^{**} \vee (c)^{**}$. Hence $y \in A^\tau$. Therefore A^τ is a D-filter of L. ■

The following lemma is a direct consequence of the above definition.

Lemma 25. *For any two non-empty subsets A and B of a lattice L, the following properties hold:*

(1) $A^\tau = \bigcap_{a\in A} (a)^\tau$,

(2) $A \cap A^\tau \subseteq D$,

(3) $A \subseteq B$ *implies* $B^\tau \subseteq A^\tau$,

(4) $A \subseteq A^{\tau\tau}$,

(5) $A^{\tau\tau\tau} = A^\tau$,

(6) $A^\tau = L$ *if and only if* $A \subseteq D$.

In case of filters, we have the following result.

Proposition 26. *For any two filters F, G of a lattice L, $(F \vee G)^\tau = F^\tau \cap G^\tau$.*

Proof. Clearly $(F \vee G)^\tau \subseteq F^\tau \cap G^\tau$. Conversely, let $x \in F^\tau \cap G^\tau$. Let $c \in F \vee G$ be an arbitrary element. Then $c = i \wedge j$ for some $i \in F$ and $j \in G$. Now $(x)^{**} \vee (c)^{**} = (x)^{**} \vee (i \wedge j)^{**} = (x)^{**} \vee \{(i)^{**} \cap (j)^{**}\} = \{(x)^{**} \vee (i)^{**}\} \cap \{(x)^{**} \vee (j)^{**}\} = L \cap L = L$. Thus $x \in (F \vee G)^\tau$ and therefore $(F \vee G)^\tau = F^\tau \cap G^\tau$. ∎

The following corollary is a direct consequence of the above results.

Corollary 27. *Let L be a lattice and $a, b \in L$. Then the following hold:*

(1) $a \leq b$ *implies* $(a)^\tau \subseteq (b)^\tau$,

(2) $(a \wedge b)^\tau = (a)^\tau \cap (b)^\tau$,

(3) $(a)^\tau = L$ *if and only if a is dense,*

(4) $a \in (b)^\tau$ *implies* $a \vee b \in D$,

(5) $(a)^* = (b)^*$ *implies* $(a)^\tau = (b)^\tau$.

For any filter F of a lattice L, it can be easily observed that $F^\tau \subseteq F^\circ$. However, we derive a set of equivalent conditions for every filter to satisfy the reverse inclusion which leads to a characterization of coherent lattices.

Theorem 28. *The following assertions are equivalent in a lattice L:*

(1) *L is a coherent lattice;*

(2) *for any two filters F, G of L, $F \cap G \subseteq D$ if and only if $F \subseteq G^\tau$;*

(3) *for any filter F of L, $F^\tau = F^\circ$;*

(4) *for any $a \in L$, $(a)^\tau = (a)^\circ$.*

Proof. (1)$\Rightarrow$(2): Assume that L is coherent. Let F and G be two filters of L. Suppose $F \cap G \subseteq D$. Let $x \in F$. For any $a \in G$, we get $x \vee a \in F \cap G \subseteq D$. Hence $x \vee a \in D$. Since L is coherent, we get $(x)^{**} \vee (a)^{**} = L$ for all $a \in G$. Thus $x \in G^{\tau}$. Therefore $F \subseteq G^{\tau}$. Conversely, suppose that $F \subseteq G^{\tau}$. Let $x \in F \cap G$. Then $x \in F \subseteq G^{\tau}$. Hence $x \in G \cap G^{\tau} \subseteq D$. Therefore $F \cap G \subseteq D$.

(2)$\Rightarrow$(3): Assume condition (2). Let F be a filter of L. Clearly $F^{\tau} \subseteq F^{\circ}$. Conversely, let $x \in F^{\circ}$. Hence, for any $a \in F$, we have

$$\begin{aligned} x \vee a \in D &\Rightarrow [x) \cap [a) \subseteq D \\ &\Rightarrow [x) \subseteq [a)^{\tau} \subseteq (a)^{\tau} \qquad \text{by (2)} \\ &\Rightarrow (x] \subseteq \bigcap_{a \in F} (a)^{\tau} = F^{\tau} \\ &\Rightarrow x \in F^{\tau} \end{aligned}$$

which concludes that $F^{\circ} \subseteq F^{\tau}$. Therefore $F^{\circ} = F^{\tau}$.

(3)$\Rightarrow$(4): Assume condition (3). Let $a \in L$. Clearly $(a)^{\tau} \subseteq (a)^{\circ}$. Conversely, let $x \in (a)^{\circ}$. Since $([a))^{\circ} = (a)^{\circ}$, by (3), we get $x \in ([a))^{\circ} = ([a))^{\tau}$. Since $\{a\} \subseteq [a)$, we get $x \in ([a))^{\tau} \subseteq (\{a\})^{\tau} = (a)^{\tau}$.

(4)$\Rightarrow$(1): Assume condition (4). Let $a, b \in L$ and suppose $a \vee b \in D$. Then $a \in (b)^{\circ} = (b)^{\tau}$. Hence $(a)^{**} \vee (b)^{**} = L$. Therefore L is a coherent lattice. ∎

References

[1] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloq. XXV (Providence, USA, 1976).

[2] S. Burris and H.P. Sankappanavar, A Course in Univerasal Algebra (Springer Verlag, 1981).

[3] I. Chajda, R. Halaš and J. Kühr, Semilattice structures (Heldermann Verlag, Germany, 2007).
ISBN 978-3-88538-230-0

[4] W.H. Cornish, *Normal lattices*, J. Austral. Math. Soc. **14** (1973) 167–179.
https://doi.org/10.1017/S1446788700010041

[5] W.H. Cornish, *Congruences on distributive pseudo-complemented lattices*, Bull. Austral. Math. Soc. **8** (1973) 161–179.
https://doi.org/10.1017/S0004972700042404

[6] W.H. Cornish, *Annulets and α-ideals in distributive lattices*, J. Austral. Math. Soc. **15** (1973) 70–77.
https://doi.org/10.1017/S1446788700012775

[7] W.H. Cornish, *Quasi-complemented lattices*, Comm. Math. Univ. Carolinae **15(3)** (1974) 501–511.
http://dml.cz/dmlcz/105573

[8] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. **29** (1962) 505–514. https://doi.org/10.1215/S0012-7094-62-02951-4

[9] G. Gratzer, General lattice theory (Academic Press, New York, San Francisco, USA, 1978).

[10] J. Kist, *Minimal prime ideals in commutative semigroups*, Proc. London Math. Soc., Sec. B **13** (1963) 31–50.
https://doi.org/10.1112/plms/s3-13.1.31

[11] M. Mandelker, *Relative annihilators in lattices*, Duke Math. J. **37** (1970) 377-386. https://doi.org/10.1215/S0012-7094-70-03748-8

[12] A.P. Paneendra Kumar, M. Sambasiva Rao and K. Sobhan Babu, *Filters of distributive lattices generated by dense elements*, Palestine J. Math. **11(2)** (2022) 45–54.

[13] A.P. Paneendra Kumar, M. Sambasiva Rao and K. Sobhan Babu, *Generalized prime D-filters of distributive lattices*, Archivum Mathematicum **57(3)** (2021) 157–174. https://doi.org/10.5817/AM2021-3-157

[14] M. Sambasiva Rao, *A note on σ-ideals of distributive lattices*, Alg. Struct. and their Appl. **9(2)** (2022) 163–179.
https://doi.org/10.22034/AS.2022.2720

[15] T.P. Speed, *Some remarks on a class of distributive lattices*, Jour. Aust. Math. Soc. **9** (1969) 289–296.
https://doi.org/10.1017/S1446788700007205

[16] M.H. Stone, *A theory of representations for Boolean algebras*, Tran. Amer. Math. Soc. **40** (1936) 37–111.
https://doi.org/10.2307/1989664

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