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CONGRUENCES ON A SEMI-BROUWERIAN ALMOST DISTRIBUTIVE LATTICE

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Abstract

This paper proves that the class of smart congruences on semi-Brouwerian almost distributive lattices is a permutable sublattice of the lattice of congruences. We also extract two different permutable sublattices of a semi-Brouwerian almost distributive lattice from the class of filters in a semi-Brouwerian almost distributive lattice.

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1. Introduction

The theory of lattice was developed by several mathematicians over the years, but its modern formalization can be attributed to Birkhoff [1, 4] and others in the early to mid-20th century. Birkhoff's seminal work, Lattice Theory, published in 1940, provided a comprehensive treatment of the subject and laid the foundation for further research. While lattices with join ($\vee_*$) and meet ($\wedge_*$) operations provide a general framework for studying partial orders and algebraic structures, they may not capture certain properties of logical implication. Boolean algebras [2], which include complementation (negation, $\backslash_*$) alongside $\vee_*$ and $\wedge_*$ operations, are too restrictive for some contexts. Heyting algebras [3] provide a more flexible generalization of Boolean algebras by replacing complementation with the implication operation $\rightarrow_*$. This generalization allows Heyting algebras to capture a broader range of logical structures and reasoning principles beyond the classical Boolean framework. Furthermore, a new class of algebras named semi-Heyting algebras [8] as an abstraction from Heyting algebras was introduced, and the behaviour of closed elements and congruences were studied on them. Later, semi-Brouwerian algebras were studied by considering only the maximal element $\mathfrak{u}_1$. Most authors have studied a new binary operation on distributive lattices, a stronger condition for obtaining the desired results.

Almost distributive lattices [13] are a generalization of distributive lattices that relaxes the requirement for the distributive property to hold universally. They provide a flexible framework for modelling ordered structures where the distributive property holds approximately but may have exceptions. As in lattice theory, the authors first mainly focused on only two binary operations $\vee_*$ and $\wedge_*$. Later on, an addition of another binary operation $\rightarrow_*$ on an almost distributive lattice gave us a new algebra named Heyting-almost distributive lattices [5] and almost semi-Heyting algebras [6]. The above algebras were studied considering the least element 0 and the maximal element $\mathfrak{u}_1$. Later in 2022, a few authors aimed to study algebra by considering only the maximal element $\mathfrak{u}_1$ strictly without the involvement of the least element 0, which gave rise to a new algebra named semi-Browerian almost distributive lattices [9] whose idea was taken from the concept semi-Brouwerian algebras [8] which is an abstraction of a semi-Heyting algebra and an almost distributive lattice [13]. In this algebra, the behaviour of principal ideals, some properties, and a few equivalent conditions were studied [9]. Later, the associativity and commutativity of the binary operation $\rightarrow_*$ were studied [10].

The set of all congruences of an almost distributive lattice is only a lattice with the partial order $\subseteq$ but not a distributive lattice-like in lattices. For this reason, the idea of a smart congruence was introduced to show that the set of all smart congruences is a sublattice of a set of all congruences and the set of all smart congruences of an almost distributive lattice is a distributive lattice with the partial order $\subseteq$. Here, we try to establish a lattice isomorphism between the lattice of all smart congruences of semi-Browerian almost distributive lattice and the lattice of all filters of semi-Browerian almost distributive lattices, we also show that the smart congruences are congruence permutable in a semi-Browerian almost distributive lattice. Also for any filter F on a semi-Browerian almost distributive lattice $\mathcal{S}$, we define two congruences θ_F and ϕ_F and also show that these classes are congruence permutable in a semi-Browerian almost distributive lattice.

2. Preliminaries

Let us review some important results on almost distributive lattices and semi-Brouwerian almost distributive lattices, which are often used in this paper.

Definition [13]. An almost distributive lattice (ADL) is an algebra $(\mathcal{S}, \vee_*, \wedge_*)$ of type $(2,2)$ with the following identities:

(1) $(\alpha_1 \vee_* \beta_1) \wedge_* \gamma_1 = (\alpha_1 \wedge_* \gamma_1) \vee_* (\beta_1 \wedge_* \gamma_1)$

(2) $\alpha_1 \wedge_* (\beta_1 \vee_* \gamma_1) = (\alpha_1 \wedge_* \beta_1) \vee_* (\alpha_1 \wedge_* \gamma_1)$

(3) $(\alpha_1 \vee_* \beta_1) \wedge_* \beta_1 = \beta_1$

(4) $(\alpha_1 \vee_* \beta_1) \wedge_* \alpha_1 = \alpha_1$

(5) $\alpha_1 \vee_* (\alpha_1 \wedge_* \beta_1) = \alpha_1$

for all $\alpha_1, \beta_1, \gamma_1 \in \mathcal{S}$.

Example 1 [13]. An ADL $(\mathcal{S}, \vee_*, \wedge_*)$ is named as a discrete ADL if for any $\alpha_1, \beta_1 \in \mathcal{S}$, $\alpha_1 \wedge_* \beta_1 = \beta_1, \alpha_1 \vee_* \beta_1 = \alpha_1$.

Unless otherwise specified, the term $\mathcal{S}$ in this section refers to an almost distributive lattice $(\mathcal{S}, \vee_*, \wedge_*)$. Given $\alpha_1, \beta_1 \in \mathcal{S}$, we say that α_1 is less than or equal to β_1 if and only if $\alpha_1 = \alpha_1 \wedge_* \beta_1$; or equivalently, $\alpha_1 \vee_* \beta_1 = \beta_1$, and it is denoted by $\alpha_1 \leq_* \beta_1$. Therefore, $\leq_*$ is a partial ordering on $\mathcal{S}$. An element $\mathfrak{u}_1$ is considered to be maximal if there is no element α_1 such that $\mathfrak{u}_1 < \alpha_1$.

Theorem 2 [13]. *The following are equivalent for any* $\mathfrak{u}_1 \in \mathcal{S}$*,*

(1) $\mathfrak{u}_1$ *is a maximal element*

(2) $\mathfrak{u}_1 \vee_* \alpha_1 = \mathfrak{u}_1$ *for all* $\alpha_1 \in \mathcal{S}$

(3) $\mathfrak{u}_1 \wedge_* \alpha_1 = \alpha_1$ *for all* $\alpha_1 \in \mathcal{S}$.

Theorem 3 [13]**.** *For any* $\alpha_1, \beta_1, \gamma_1 \in \mathcal{S}$*,*

(1) $\alpha_1 \vee_* \beta_1 = \alpha_1 \Leftrightarrow \alpha_1 \wedge_* \beta_1 = \alpha_1$

(2) $\alpha_1 \vee_* \beta_1 = \beta_1 \Leftrightarrow \alpha_1 \wedge_* \beta_1 = \alpha_1$

(3) $\alpha_1 \wedge_* \beta_1 = \beta_1 \wedge_* \alpha_1 = \alpha_1$ *whenever* $\alpha_1 \leq_* \beta_1$

(4) $\wedge_*$ *is associative*

(5) $\alpha_1 \wedge_* \beta_1 \wedge_* \gamma_1 = \beta_1 \wedge_* \alpha_1 \wedge_* \gamma_1$

(6) $(\alpha_1 \vee_* \beta_1) \wedge_* \gamma_1 = (\beta_1 \vee_* \alpha_1) \wedge_* \gamma_1$

(7) $\alpha_1 \wedge_* \beta_1 \leq_* \beta_1$ *and* $\alpha_1 \leq_* \alpha_1 \vee_* \beta_1$

(8) $\alpha_1 \wedge_* \alpha_1 = \alpha_1$ *and* $\alpha_1 \vee_* \alpha_1 = \alpha_1$

(9) $\alpha_1 \leq_* \gamma_1, \beta_1 \leq_* \gamma_1 \Rightarrow \alpha_1 \wedge_* \beta_1 = \beta_1 \wedge_* \alpha_1, \alpha_1 \vee_* \beta_1 = \beta_1 \vee_* \alpha_1$.

Definition [13]**.** A non-empty subset F of $\mathcal{S}$ is said to be a filter of $\mathcal{S}$ if it satisfies the following:

(1) $\alpha_1, \beta_1 \in F \Rightarrow \alpha_1 \wedge_* \beta_1 \in F$

(2) $\alpha_1 \in F, \mathfrak{p}_1 \in \mathcal{S} \Rightarrow \mathfrak{p}_1 \vee_* \alpha_1 \in F$.

Theorem 4 [4]**.** *The set* $\mathcal{F}(\mathcal{S})$ *of filters of* $\mathcal{S}$ *forms a distributive lattice in which given, for any filters* F_1 *and* F_2 *of* $\mathcal{S}$*, the g.l.d. and the l.u.b. of any filters* F_1 *and* F_2 *of* $\mathcal{S}$ *are given respectively by* $F_1 \wedge_* F_2 = F_1 \bigcap_* F_2$ *and* $F_1 \vee_* F_2 = \{\alpha_1 \wedge_* \beta_1 \mid \alpha_1 \in F_1$ *and* $\beta_1 \in F_2\}$.

Definition [4]**.** An equivalence relation θ on $\mathcal{S}$ is called a congruence relation if $(\alpha_1 \wedge_* \gamma_1, \beta_1 \wedge_* \delta_1) \in \theta$ and $(\alpha_1 \vee_* \gamma_1, \beta_1 \vee_* \delta_1) \in \theta$ for all $(\alpha_1, \beta_1), (\gamma_1, \delta_1) \in \theta$.

Theorem 5 [4]**.** *The set* $Con(\mathcal{S})$ *of congruences on* $\mathcal{S}$ *is a lattice, in which given congruences* $\theta_1, \theta_2 \in Con(\mathcal{S})$*, the g.l.d. and l.u.b. are* $\theta_1 \wedge_* \theta_2 = \theta_1 \cap_* \theta_2$ *and* $\theta_1 \vee_* \theta_2 = \{(\alpha_1, \beta_1) \mid$ *there exists a finite sequence of elements* $\alpha_1 = \gamma_{1_0}, \gamma_{1_1}, \ldots, \gamma_{1_n-1} = \beta_1 \in \mathcal{S}$ *such that* $(\gamma_{1_i}, \gamma_{1_i+1}) \in \theta_1 \cup_* \theta_2$*, for each* $0 \leq_* i \leq_* n-2\}$*, respectively.*

Definition [9]**.** $\mathcal{S}$ with a maximal element $\mathfrak{u}_1$ is said to be a semi-Brouwerian almost distributive lattice (SBADL) if there is a binary operation $\rightarrow_*$ on $\mathcal{S}$ with the following axioms:

(N_1) $(\alpha_1 \rightarrow_* \alpha_1) \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1$

(N_2) $\alpha_1 \wedge_* (\alpha_1 \rightarrow {}_*\beta_1) = \alpha_1 \wedge_* \beta_1 \wedge_* \mathfrak{u}_1$

(N_3) $\alpha_1 \wedge_* (\beta_1 \rightarrow {}_*\gamma_1) = \alpha_1 \wedge_* [(\alpha_1 \wedge_* \beta_1) \rightarrow {}_*(\alpha_1 \wedge_* \gamma_1)]$

(N_4) $(\alpha_1 \to {}_*\beta_1) \wedge_* \mathfrak{u}_1 = [(\alpha_1 \wedge_* \mathfrak{u}_1) \to {}_*(\beta_1 \wedge_* \mathfrak{u}_1)]$
for all $\alpha_1, \beta_1, \gamma_1 \in \mathcal{S}$.

Theorem 6 [9]**.** *In an SBADL $\mathcal{S}$, the following are equivalent:*

(1) $(\alpha_1 \to {}_*\beta_1) \wedge_* \mathfrak{u}_1 = (\beta_1 \to {}_*\alpha_1) \wedge_* \mathfrak{u}_1$ *for all* $\alpha_1, \beta_1 \in \mathcal{S}$.
(2) $(\alpha_1 \to {}_*\mathfrak{u}_1) \wedge_* \mathfrak{u}_1 = \alpha_1 \wedge_* \mathfrak{u}_1$ *for all* $\alpha_1 \in \mathcal{S}$.
(3) $\beta_1 \wedge_* (\alpha_1 \to {}_*\beta_1) \wedge_* \mathfrak{u}_1 = \alpha_1 \wedge_* \beta_1 \wedge_* \mathfrak{u}_1$ *for all* $\alpha_1, \beta_1 \in \mathcal{S}$.

Theorem 7 [9]**.** *If $\mathcal{S}$ is an SBADL and $\alpha_1, \beta_1, \gamma_1 \in \mathcal{S}$, then*

(1) $\alpha_1 \leq_* \beta_1$ *and* $\alpha_1 \leq_* \gamma_1 \Rightarrow \alpha_1 \wedge_* \mathfrak{u}_1 \leq_* (\alpha_1 \to {}_*\gamma_1) \wedge_* \mathfrak{u}_1$
(2) $(\alpha_1 \to_* \beta_1) \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1 \to_* \alpha_1 \wedge_* \mathfrak{u}_1 \leq_* \beta_1 \wedge_* \mathfrak{u}_1$
(3) $(\alpha_1 \to_* \beta_1) \wedge_* \gamma_1 = [(\alpha_1 \wedge_* \gamma_1 \to_* \alpha_1 \wedge_* \gamma_1)] \wedge_* \gamma_1$
(4) $\alpha_1 \wedge_* \mathfrak{u}_1 = \beta_1 \wedge_* \mathfrak{u}_1 \Leftrightarrow (\alpha_1 \to_* \beta_1) \wedge_* (\beta_1 \to_* \alpha_1) \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1$
(5) $\alpha_1 \wedge_* \mathfrak{u}_1 = \beta_1 \wedge_* \mathfrak{u}_1 \Leftrightarrow (\alpha_1 \vee_* \beta_1 \to_* \alpha_1 \wedge_* \beta_1) \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1$
(6) $\alpha_1 \leq_* \beta_1 \leq_* \gamma_1 \Rightarrow \beta_1 \wedge_* (\alpha_1 \to_* \gamma_1) = \beta_1 \wedge_* (\alpha_1 \to_* \beta_1)$
(7) $\beta_1 \leq_* (\alpha_1 \to_* \mathfrak{u}_1) \Leftrightarrow \beta_1 \leq_* (\beta_1 \wedge_* \alpha_1) \to_* (\beta_1 \wedge_* \mathfrak{u}_1)$
(8) $[\alpha_1 \to_* (\beta_1 \wedge_* \mathfrak{u}_1)] \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1 \Rightarrow \beta_1 \wedge_* \alpha_1 = \alpha_1$.

3. Congruences on an SBADL

A congruence θ on a ADL $\mathcal{S}$ is a smart congruence [7], if $(\alpha_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta$ implies $(\alpha_1, \beta_1) \in \theta$ and the authors proved that the set $Con_0(\mathcal{S})$ of smart congruences on $\mathcal{S}$ is a distributive lattice with the induced operations.

Now onwards, $\mathcal{S}$ stands for an SBADL with a maximal element $\mathfrak{u}_1$, and $\mathcal{F}(\mathcal{S})$ stands for the lattice of filters of $\mathcal{S}$. We begin by demonstrating that the smart congruences on an SBADL are obtained by filters based on the notion of smart congruences.

Lemma 8. *For $F \in \mathcal{F}(\mathcal{S}), \theta(F) = \{(\alpha_1, \beta_1) \in \mathcal{S} \times \mathcal{S} \mid \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F\}$ as a smart congruence on $\mathcal{S}$ and $\frac{\mathfrak{u}_1}{\theta(F)} = F$.*

Proof. To prove $\theta(F) \in Con_0(\mathcal{S})$, first we have to show that $\theta(F)$ is an equivalence relation, which can be obtained by considering $\alpha_1, \beta_1, \gamma_1, \in \mathcal{S}$ such that $(\alpha_1, \beta_1) \in \theta(F)$ implies $\alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1$ for some $\mathfrak{z}_1 \in F$, $(\beta_1, \gamma_1) \in \theta(F)$ implies $\beta_1 \wedge_* \mathfrak{z}_2 = \gamma_1 \wedge_* \mathfrak{z}_2$ for some $\mathfrak{z}_2 \in F$. On considering $(\alpha_1, \beta_1) \in \theta(F), (\gamma_1, \delta_1) \in \theta(F)$. Then $\alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1$ and $\gamma_1 \wedge_* \mathfrak{z}_2 = \delta_1 \wedge_* \mathfrak{z}_2$ for some $\mathfrak{z}_1, \mathfrak{z}_2 \in F$, we can show that $\theta(F)$ is compatible with $\wedge_*, \vee_*, \to {}_*$. Hence, $\theta(F) \in Con(\mathcal{S})$. Now, if $(\alpha_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta(F) \Rightarrow (\alpha_1, \beta_1) \in \theta(F)$. Hence, $\theta(F) \in Con_0(\mathcal{S})$. Finally, we prove that $\frac{\mathfrak{u}_1}{\theta(F)} = F$. Let $\alpha_1 \in \mathfrak{u}_1/\theta(F) \Rightarrow (\alpha_1, \mathfrak{u}_1) \in \theta(F)$ implies

$\alpha_1 \wedge_* \mathfrak{e}_1 = \mathfrak{u}_1 \wedge_* \mathfrak{e}_1 = \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F$. Since $\alpha_1 \wedge_* \mathfrak{e}_1 = \mathfrak{e}_1$ we get $\alpha_1 \vee_* \mathfrak{e}_1 = \alpha_1$. Therefore, $\alpha_1 \in F$. On the other side , if $\alpha_1 \in F$, then $\alpha_1 \wedge_* \alpha_1 = \mathfrak{u}_1 \wedge_* \alpha_1$ and hence, $\alpha_1 \in \frac{\mathfrak{u}_1}{\theta(F)} = F$. Thus, $\frac{\mathfrak{u}_1}{\theta(F)} = F$. ∎

By using Lemma 8 and Definition 2, we can obtain the following lemma.

Lemma 9. *For $F \in \mathcal{F}(\mathcal{S})$ and $\alpha_1, \beta_1 \in \mathcal{S}$, we have*

$$(\alpha_1, \beta_1) \in \theta(F) \Leftrightarrow (\alpha_1 \rightarrow_* \beta_1) \wedge_* (\beta_1 \rightarrow_* \alpha_1) \wedge_* \mathfrak{u}_1 \in F.$$

Theorem 10. *$Con_0(\mathcal{S}) \cong \mathcal{F}(\mathcal{S})$.*

Proof. Define $\psi : Con_0(\mathcal{S}) \to \mathcal{F}(\mathcal{S})$ by $\psi(\theta) = \frac{\mathfrak{u}_1}{\theta}$ for all $\theta \in Con_0(S)$. It is enough to prove that ψ is an order isomorphism. That is, ψ is surjection and $\theta_1 \subseteq \theta_2 \Leftrightarrow \psi(\theta_1) \subseteq \psi(\theta_2)$ for all $\theta_1, \theta_2 \in Con_0(\mathcal{S})$. First, we prove that ψ is a surjection. Let $F \in F(\mathcal{S})$. Then by Lemma 9, we have $\theta(F) = \{(\alpha_1, \beta_1) \in \mathcal{S} \times \mathcal{S} \mid \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1 \text{ for some } \mathfrak{e}_1 \in F\} \in Con_0(\mathcal{S})$ and $\psi(\theta(F)) = \frac{\mathfrak{u}_1}{\theta(F)} = F$. Thus, ψ is a surjection. Now, let $\theta_1, \theta_2 \in Con_0(\mathcal{S})$ such that $\psi(\theta_1) \subseteq \psi(\theta_2)$. That is, $\frac{\mathfrak{u}_1}{\theta_1} \subseteq \frac{\mathfrak{u}_1}{\theta_2}$. We prove that $\theta_1 \subseteq \theta_2$. Let $\alpha_1, \beta_1 \in \mathcal{S}$. Then

$$\begin{aligned}
(\alpha_1, \beta_1) \in \theta_1 &\Rightarrow (\alpha_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta_1 \\
&\Rightarrow (\alpha_1 \wedge_* \mathfrak{u}_1 \,{}_*\, \beta_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* m \,{}_*\, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta_1 \\
&\Rightarrow ((\alpha_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1, (\beta_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1) \in \theta_1 \\
&\Rightarrow ((\alpha_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1, \mathfrak{u}_1) \in \theta_1 \\
&\Rightarrow (\alpha_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1 \in \tfrac{\mathfrak{u}_1}{\theta_1} \subseteq \tfrac{\mathfrak{u}_1}{\theta_2} \\
&\Rightarrow ((\alpha_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1, \mathfrak{u}_1) \in \theta_2 \\
&\Rightarrow (\alpha_1 \wedge_* (\alpha_1 \,{}_*\, \beta_1) \wedge_* \mathfrak{u}_1, \alpha_1 \wedge_* \mathfrak{u}_1) \in \theta_2 \\
&\Rightarrow (\alpha_1 \wedge_* \beta_1 \wedge_* \mathfrak{u}_1, \alpha_1 \wedge_* \mathfrak{u}_1) \in \theta_2.
\end{aligned}$$

By symmetry, we get that $(\alpha_1 \wedge_* \beta_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta_2$. Therefore, $(\alpha_1 \wedge_* \mathfrak{u}_1, \beta_1 \wedge_* \mathfrak{u}_1) \in \theta_2$. Since $\theta_2 \in Con_0(\mathcal{S})$, we get $(\alpha_1, \beta_1) \in \theta_2$. Thus, $\theta_1 \subseteq \theta_2$. On the other hand, suppose $\theta_1 \subseteq \theta_2$ and $\alpha_1 \in \mathcal{S}$. Then

$$\begin{aligned}
\alpha_1 \in \psi(\theta_1) = \frac{\mathfrak{u}_1}{\theta_1} &\Rightarrow (\alpha_1, \mathfrak{u}_1) \in \theta_1 \subseteq \theta_2 \\
&\Rightarrow (\alpha_1, \mathfrak{u}_1) \in \theta_2 \\
&\Rightarrow \alpha_1 \in \mathfrak{u}_1/\theta_2 \\
&\Rightarrow \alpha_1 \in \psi(\theta_2).
\end{aligned}$$

Therefore, $\psi(\theta_1) \subseteq \psi(\theta_2)$. Hence, $\theta_1 \subseteq \theta_2 \Leftrightarrow \psi(\theta_1) \subseteq \psi(\theta_2)$. Thus, $Con_0(\mathcal{S}) \cong \mathcal{F}(\mathcal{S})$. ∎

Lemma 11. *For $\alpha_1, \beta_1 \in \mathcal{S}, [(\beta_1 \rightarrow_* \beta_1) \rightarrow_* \alpha_1] \wedge_* [(\alpha_1 \rightarrow_* \beta_1) \rightarrow_* \beta_1] \wedge_* (\beta_1 \vee_* \alpha_1) \wedge_* \mathfrak{u}_1 = \alpha_1 \wedge_* \mathfrak{u}_1$.*

Proof. Let $\alpha_1, \beta_1 \in \mathcal{S}$. Then

$$\begin{aligned}[(\beta_1 {}_* \beta_1) {}_* \alpha_1] \wedge_* [(\alpha_1 {}_* \beta_1) {}_* \beta_1] \wedge_* (\beta_1 \vee_* \alpha_1) \wedge_* \mathfrak{u}_1 \\ &= [\mathfrak{u}_1 {}_* \alpha_1] \wedge_* [(\alpha_1 {}_* \beta_1) {}_* \beta_1] \wedge_* (\beta_1 \vee_* \alpha_1) \wedge_* \mathfrak{u}_1 \\ &= \alpha_1 \wedge_* (\beta_1 \vee_* \alpha_1) \wedge_* \mathfrak{u}_1 \\ &= \alpha_1 \wedge_* \mathfrak{u}_1.\end{aligned}$$

■

Any two congruences θ_1 and θ_2 are said to be permutable if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. A sublattice of $Con(\mathcal{S})$ is said to be a permutable sublattice if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ for all θ_1, θ_2 in the sublattice.

Theorem 12. *$Con_0(\mathcal{S})$ is a permutable sublattice of $Con(\mathcal{S})$.*

Proof. Let $\theta_1, \theta_2 \in Con_0(\mathcal{S})$ and $(\alpha_1, \beta_1) \in \theta_1 \circ \theta_2$. Then there exists $\gamma_1 \in \mathcal{S}$ such that $(\alpha_1, \gamma_1) \in \theta_1$ and $(\gamma_1, \beta_1) \in \theta_2$. Since $(\alpha_1, \gamma_1) \in \theta_1$, we get $(\gamma_1, \alpha_1) \in \theta_1$. From this we get that the pares $((\gamma_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1, (\alpha_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1)$, $((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \gamma_1, (\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \alpha_1)$ and $((\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1, (\alpha_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1)$ belong to θ_1. Hence, $(((\gamma_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1) \wedge_* ((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \gamma_1) \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1, ((\alpha_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1) \wedge_* ((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \alpha_1) \wedge_* (\alpha_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1) \in \theta_1$. By Lemma 11, we get that $(\beta_1 \wedge_* \mathfrak{u}_1, ((x \rightarrow_* \gamma_1) \rightarrow_* \beta_1) \wedge_* ((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \alpha_1) \wedge_* (\alpha_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1) \in \theta_1$. That is, $(\beta_1, ((\alpha_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1) \wedge_* ((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \alpha_1) \wedge_* (\alpha_1 \vee_* \beta_1)) \in \theta_1$. Similarly, $(\gamma_1, \beta_1) \in \theta_2$, we get $(((\alpha_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1) \wedge_* ((\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \alpha_1) \wedge_* (\alpha_1 \vee_* \beta_1), \alpha_1) \in \theta_2$. Therefore, $(\beta_1, \alpha_1) \in \theta_1 \circ \theta_2$ or $(\alpha_1, \beta_1) \in \theta_2 \circ \theta_1$. Hence, $\theta_1 \circ \theta_2 \subseteq \theta_2 \circ \theta_1$. Similarly, $\theta_2 \circ \theta_1 \subseteq \theta_1 \circ \theta_2$. Therefore, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Thus, $Con_0(\mathcal{S})$ is congruence permutable. ■

Lemma 11 can be obtained even with a weaker set of conditions on an ADL $\mathcal{S}$ compared to the conditions of an SBADL as shown in the following.

Lemma 13. *If $\rightarrow_*$ is a binary operation on an ADL $\mathcal{S} = (\mathcal{S}, \vee_*, \wedge_*, \mathfrak{u}_1)$ with the following properties:*

(1) $\mathfrak{u}_1 \rightarrow_* \alpha_1 = \alpha_1 \wedge_* \mathfrak{u}_1$

(2) $\alpha_1 \wedge_* \mathfrak{u}_1 \leq_* ((\alpha_1 \rightarrow_* \beta_1) \rightarrow_* \beta_1) \wedge_* \mathfrak{u}_1$

(3) $(\alpha_1 \rightarrow_* \beta_1) \wedge_* \mathfrak{u}_1 = (\alpha_1 \wedge_* \mathfrak{u}_1) \rightarrow_* (\beta_1 \wedge_* \mathfrak{u}_1)$,

then $[(\gamma_1 \rightarrow_* \gamma_1) \rightarrow_* \beta_1] \wedge_* [(\beta_1 \rightarrow_* \gamma_1) \rightarrow_* \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 = \beta_1 \wedge_* \mathfrak{u}_1$ *for* $\alpha_1, \beta_1, \gamma_1 \in \mathcal{S}$.

Proof. We can obtain $(\beta_1 \rightarrow_* \beta_1) \wedge_* \mathfrak{u}_1 = \mathfrak{u}_1$ by replacing α_1 with $\mathfrak{u}_1$ in (2). Consider,

$$\begin{aligned}
&[(\gamma_1 * \gamma_1) * \beta_1] \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \\
&= [(\gamma_1 * \gamma_1) \wedge_* \mathfrak{u}_1 * \beta_1 \wedge_* \mathfrak{u}_1] \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \\
&= [\mathfrak{u}_1 * \beta_1 \wedge_* \mathfrak{u}_1] \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \text{ (by Lemma 11)} \\
&= [\mathfrak{u}_1 * \beta_1] \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \\
&= \beta_1 \wedge_* \mathfrak{u}_1 \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \text{ (by (1))} \\
&= \beta_1 \wedge_* [(\beta_1 * \gamma_1) * \gamma_1] \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \\
&= \beta_1 \wedge_* (\gamma_1 \vee_* \beta_1) \wedge_* \mathfrak{u}_1 \text{ (by (2))} \\
&= \beta_1 \wedge_* (\beta_1 \vee_* \gamma_1) \wedge_* \mathfrak{u}_1 \\
&= \beta_1 \wedge_* \mathfrak{u}_1.
\end{aligned}$$

■

Using Lemma 13, we obtain the following theorem, the source of which is similar to Theorem 12.

Theorem 14. *If* $\rightarrow_*$ *is a binary operation on an ADL* $\mathcal{S} = (\mathcal{S}, \vee_*, \wedge_*, \mathfrak{u}_1)$ *with the following properties:*

(1) $\mathfrak{u}_1 \rightarrow_* \alpha_1 = \alpha_1 \wedge_* \mathfrak{u}_1$

(2) $\alpha_1 \wedge_* \mathfrak{u}_1 \leq_* ((x \rightarrow_* \beta_1) \rightarrow_* \beta_1) \wedge_* \mathfrak{u}_1$

(3) $(\alpha_1 \rightarrow_* \beta_1) \wedge_* \mathfrak{u}_1 = \alpha_1 \wedge_* \mathfrak{u}_1 \rightarrow_* \beta_1 \wedge_* \mathfrak{u}_1$

for all $\alpha_1, \beta_1 \in \mathcal{S}$*. Let* $Con(\mathcal{S})$ *be the set of all congruences on* $\mathcal{S}$ *with respect the operation* $\rightarrow_*$*, then* $Con_0(\mathcal{S})$ *is a permutable sub lattice of* $Con(\mathcal{S})$*.*

Given a filter F in $\mathcal{S}$, denote $\theta_F = \{(\alpha_1, \beta_1) \in \mathcal{S} \times \mathcal{S} \mid \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F\}$. Now, we have the following.

Theorem 15. *For any filter* F *on* $\mathcal{S}$*,* θ_F *is a congruence on* $\mathcal{S}$ *and* θ_F *is the smallest congruence in* $\mathcal{S}$ *such that* $F \times F \subseteq \theta_F$*.*

Proof. It is easy to verify that θ_F is an equivalence relation on $\mathcal{S}$. Let (α_1, β_1), $(\gamma_1, \delta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1$ and $\gamma_1 \wedge_* \mathfrak{z}_2 = \delta_1 \wedge_* \mathfrak{z}_2$ for some $\mathfrak{z}_1, \mathfrak{z}_2 \in F$. Since F is a filter of $\mathcal{S}$, we have $\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \in F$. Now,

$$\begin{aligned}
\alpha_1 \wedge_* \gamma_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 &= \gamma_1 \wedge_* \alpha_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \\
&= \gamma_1 \wedge_* \beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \text{ (since } \alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1) \\
&= \beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \gamma_1 \wedge_* \mathfrak{z}_2 \\
&= \beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \delta_1 \wedge_* \mathfrak{z}_2 \text{ (since } \gamma_1 \wedge_* \mathfrak{z}_2 = \delta_1 \wedge_* \mathfrak{z}_2) \\
&= \beta_1 \wedge_* \delta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2.
\end{aligned}$$

Also,

$$\begin{aligned}
&(\alpha_1 \vee_* \gamma_1) \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \\
&= (\alpha_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \vee_* (\gamma_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \\
&= (\beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \vee_* (\mathfrak{z}_1 \wedge_* \gamma_1 \wedge_* \mathfrak{z}_2) \text{ (since } \alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1) \\
&= (\beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \vee_* (\mathfrak{z}_1 \wedge_* \delta_1 \wedge_* \mathfrak{z}_2) \text{ (since } \gamma_1 \wedge_* \mathfrak{z}_2 = \delta_1 \wedge_* \mathfrak{z}_2) \\
&= (\beta_1 \wedge_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2)) \vee_* (\delta_1 \wedge_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2)) \\
&= (\beta_1 \vee_* \delta_1) \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha_1 \ast \gamma_1) \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \\
&= [(x \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \ast (\gamma_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2)] \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \text{ (by (3) of Theorem 7)} \\
&= [(\beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \ast (\mathfrak{z}_1 \wedge_* \gamma_1 \wedge_* \mathfrak{z}_2)] \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \text{ (since } \alpha_1 \wedge_* \mathfrak{z}_1 = \beta_1 \wedge_* \mathfrak{z}_1) \\
&= [(\beta_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \ast (\mathfrak{z}_1 \wedge_* \delta_1 \wedge_* \mathfrak{z}_2] \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \text{ (since } \gamma_1 \wedge_* \mathfrak{z}_2 = \delta_1 \wedge_* \mathfrak{z}_2) \\
&= [(\beta_1 \wedge_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2)) \ast (\delta_1 \wedge_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2))] \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \\
&= (\beta_1 \ast \delta_1) \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2.
\end{aligned}$$

Therefore, $(\alpha_1 \wedge_* \gamma_1, \beta_1 \wedge_* \delta_1), (\alpha_1 \vee_* \gamma_1, \beta_1 \vee_* \delta_1)$ and $(\alpha_1 \rightarrow_* \gamma_1, \beta_1 \rightarrow_* \delta_1) \in \theta_F$. Hence, θ_F is a congruence relation on $\mathcal{S}$. Let $\mathfrak{z}_1, \mathfrak{z}_2 \in F$. Since F is a filter of $\mathcal{S}, \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \in F$. Since $\mathfrak{z}_1 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 = \mathfrak{z}_2 \wedge_* \mathfrak{z}_1 \wedge_* \mathfrak{z}_2$ and $\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \in F, (\mathfrak{z}_1, \mathfrak{z}_2) \in \theta_F$. Therefore, $F \times F \subseteq \theta_F$. On the other hand, let θ be a congruence on $\mathcal{S}$ such that $F \times F \subseteq \theta$. Let $(\alpha_1, \beta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F$. Since $\alpha_1 \vee_* (\beta_1 \vee_* \alpha_1)$ and $\beta_1 \vee_* (\alpha_1 \vee_* \mathfrak{e}_1) \in F$, we get that $(\alpha_1 \vee_* (\beta_1 \vee_* \mathfrak{e}_1), \mathfrak{e}_1); (\beta_1 \vee_* (\alpha_1 \vee_* \mathfrak{e}_1), \mathfrak{e}_1) \in F \times F \subseteq \theta$. Since θ is a congruence on $\mathcal{S}, (\alpha_1 \wedge_* (\alpha_1 \vee_* (\beta_1 \vee_* \mathfrak{e}_1)), \alpha_1 \wedge_* \mathfrak{e}_1), (\beta_1 \wedge_* (\beta_1 \vee_* (\alpha_1 \vee_* \mathfrak{e}_1)), \beta_1 \wedge_* \mathfrak{e}_1) \in \theta$. Therefore, $(\alpha_1, \alpha_1 \wedge_* \mathfrak{e}_1), (\beta_1, \beta_1 \wedge_* \mathfrak{e}_1) \in \theta$. Since $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$, we get that $(\alpha_1, \beta_1) \in \theta$. Hence, $\theta_F \subseteq \theta$. Thus, θ_F is the smallest congruence on $\mathcal{S}$ containing $F \times F$. ■

Theorem 16. *For any filter F of $\mathcal{S}$, $\mathcal{S}/\theta_F$ is a lattice.*

Proof. Let F be a filter of $\mathcal{S}$. Let $\alpha_1, \beta_1 \in \mathcal{S}$. Since $F \neq \emptyset$, we can choose $\mathfrak{e}_1 \in F$. Then $\alpha_1 \wedge_* \beta_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \alpha_1 \wedge_* \mathfrak{e}_1$. Therefore, $(\alpha_1 \wedge_* \beta_1, \beta_1 \wedge_* \alpha_1) \in \theta_F$. Hence, $\alpha_1/\theta_F \wedge_* \beta_1/\theta_F = (\alpha_1 \wedge_* \beta_1)/\theta_F = (\beta_1 \wedge_* \alpha_1)/\theta_F = \beta_1/\theta_F \wedge_* \alpha_1/\theta_F$. Thus, $\mathcal{S}/\theta_F$ is a lattice. ■

Given a filter F on $\mathcal{S}$, denote $\phi_F = \{(\alpha_1, \beta_1) \in \mathcal{S} \times \mathcal{S} \mid \mathfrak{e}_1 \wedge_* \alpha_1 = \mathfrak{e}_1 \wedge_* \beta_1$ for some $\mathfrak{e}_1 \in F\}$. Now, we have the following.

Theorem 17. *For any filter F on $\mathcal{S}$, ϕ_F is a congruence on $\mathcal{S}$.*

Proof. It is easy to prove that ϕ_F is an equivalence relation on $\mathcal{S}$. Let (α_1, β_1), $(\gamma_1, \delta_1) \in \phi_F$. Then $\mathfrak{z}_1 \wedge_* \alpha_1 = \mathfrak{z}_1 \wedge_* \beta_1$ and $\mathfrak{z}_2 \wedge_* \gamma_1 = \mathfrak{z}_2 \wedge_* \delta_1$ for some $\mathfrak{z}_1, \mathfrak{z}_2 \in F$. Then $\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \in F$ and

$$\begin{aligned}
\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \alpha_1 \wedge_* \gamma_1 &= \mathfrak{z}_1 \wedge_* \alpha_1 \wedge_* \mathfrak{z}_2 \wedge_* \gamma_1 \\
&= \mathfrak{z}_1 \wedge_* \beta_1 \wedge_* \mathfrak{z}_2 \wedge_* \delta_1 \text{ (since } \mathfrak{z}_1 \wedge_* \alpha_1 = \mathfrak{z}_1 \wedge_* \beta_1) \\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \beta_1 \wedge_* \delta_1. \text{ (since } \mathfrak{z}_2 \wedge_* \gamma_1 = \mathfrak{z}_2 \wedge_* \delta_1).
\end{aligned}$$

Also,

$$\begin{aligned}
&\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* (\alpha_1 \vee_* \gamma_1) \\
&= (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \alpha_1) \vee_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \gamma_1) \\
&= (\mathfrak{z}_2 \wedge_* \mathfrak{z}_1 \wedge_* \alpha_1) \vee_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \delta_1) \text{ (since } \mathfrak{z}_2 \wedge_* \gamma_1 = \mathfrak{z}_2 \wedge_* \delta_1 \text{)} \\
&= (\mathfrak{z}_2 \wedge_* \mathfrak{z}_1 \wedge_* \gamma_1) \vee_* (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \delta_1) \text{ (since } \mathfrak{z}_1 \wedge_* \alpha_1 = \mathfrak{z}_1 \wedge_* \beta_1) \\
&= ((\mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \wedge_* \beta_1) \vee_* ((\mathfrak{z}_1 \wedge_* \mathfrak{z}_2) \wedge_* \delta_1) \\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* (\beta_1 \vee_* \delta_1)
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* (\alpha_1 \,_* \, \gamma_1)\\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* [(\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \alpha_1) \,_* \, (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \gamma_1)]\\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* [(\mathfrak{z}_2 \wedge_* \mathfrak{z}_1 \wedge_* \alpha_1) \,_* \, (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \gamma_1)]\\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* [(\mathfrak{z}_2 \wedge_* \mathfrak{z}_1 \wedge_* \beta_1) \,_* \, (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \delta_1)]\\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* [(\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \beta_1) \,_* \, (\mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* \delta_1)]\\
&= \mathfrak{z}_1 \wedge_* \mathfrak{z}_2 \wedge_* (\beta_1 \,_* \, \delta_1).
\end{aligned}$$

Therefore, $(\alpha_1 \wedge_* \gamma_1, \beta_1 \wedge_* \delta_1), (\alpha_1 \vee_* \gamma_1, \beta_1 \vee_* \delta_1)$ and $(\alpha_1 \rightarrow_* \gamma_1, \beta_1 \rightarrow_* \delta_1) \in \phi_F$. Thus, ϕ_F is a congruence relation on $\mathcal{S}$. ■

Theorem 18. *$\phi_F \subseteq \theta_F$ for any filter F of $\mathcal{S}$.*

Proof. Suppose F is a filter of $\mathcal{S}$. Let $(\alpha_1, \beta_1) \in \phi_F$. Then $\mathfrak{e}_1 \wedge_* \alpha_1 = \mathfrak{e}_1 \wedge_* \beta_1$ for some $\mathfrak{e}_1 \in F$. Consider,

$$\begin{aligned}
\alpha_1 \wedge_* \mathfrak{e}_1 &= \alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_1\\
&= \mathfrak{e}_1 \wedge_* \alpha_1 \wedge_* \mathfrak{e}_1\\
&= \mathfrak{e}_1 \wedge_* \beta_1 \wedge_* \mathfrak{e}_1 \text{ (since } \mathfrak{e}_1 \wedge_* \alpha_1 = \mathfrak{e}_1 \wedge_* \beta_1)\\
&= \beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_1\\
&= \beta_1 \wedge_* \mathfrak{e}_1.
\end{aligned}$$

Therefore, $(\alpha_1, \beta_1) \in \theta_F$. Thus, $\phi_F \subseteq \theta_F$. ■

For any filter F, θ_F may not be contained in ϕ_F. But in a lattice, $\theta_F = \phi_F$. Now, we prove the following.

Theorem 19. *$\mathcal{S}$ is a lattice if and only if $\phi_F = \theta_F$ for all filter F of $\mathcal{S}$.*

Proof. If $\mathcal{S}$ is a lattice, then it is clear that $\phi_F = \theta_F$ for all filter F of $\mathcal{S}$. On the other hand, suppose $\phi_F = \theta_F$ for all filter F of $\mathcal{S}$. Let $\alpha_1, \beta_1 \in \mathcal{S}$. Put $F = [\alpha_1)$, the filter generated by α_1, where $[\alpha_1) = \{\mathfrak{t}_1 \vee_* \alpha_1 \mid \mathfrak{t}_1 \in \mathcal{S}\}$. Since $(\beta_1, \beta_1 \wedge_* \alpha_1) \in \theta_F$, $(\beta_1, \beta_1 \wedge_* \alpha_1) \in \phi_F$. Then $(\mathfrak{t}_1 \vee_* \alpha_1) \wedge_* \beta_1 = (\mathfrak{t}_1 \vee_* \alpha_1) \wedge_* \beta_1 \wedge_* \alpha_1$ for some $\mathfrak{t}_1 \in \mathcal{S}$. Therefore, $\alpha_1 \wedge_* (\mathfrak{t}_1 \vee_* \alpha_1) \wedge_* \beta_1 = \alpha_1 \wedge_* (\mathfrak{t}_1 \vee_* \alpha_1) \wedge_* \beta_1 \wedge_* \alpha_1$ and hence, $\alpha_1 \wedge_* \beta_1 = \beta_1 \wedge_* \alpha_1$. Thus, $\mathcal{S}$ is a lattice. ■

Corollary 20. *Let $\mathcal{S}$ be an SBADL. Then $\mathcal{S}$ is a lattice if and only if for any $\alpha_1, \beta_1, \mathfrak{e}_1 \in \mathcal{S}$, $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ implies $\mathfrak{e}_1 \wedge_* \alpha_1 = \mathfrak{e}_1 \wedge_* \beta_1$.*

Theorem 21. *The following are equivalent for any filter F of $\mathcal{S}$.*

(1) $F \times F \subseteq \phi_F$

(2) $\theta_F \subseteq \phi_F$

(3) $\theta_F = \phi_F$.

Proof. Let F be a filter of $\mathcal{S}$.

(1)⇒(2) By Theorem 15, θ_F is the smallest congruence on $\mathcal{S}$ such that $F \times F \subseteq \theta_F$. Since $F \times F \subseteq \phi_F$, we get that $\theta_F \subseteq \phi_F$.

(2)⇒(3) Suppose $\theta_F \subseteq \phi_F$. By Theorem 18, we have $\phi_F \subseteq \theta_F$. Therefore, $\theta_F = \phi_F$.

(3)⇒(1) It follows from the fact that $F \times F \subseteq \theta_F$. ■

Theorem 22. *$\theta_F = \phi_G$ implies $F = G$ for all filters F and G of $\mathcal{S}$.*

Proof. Let F and G be two filters of $\mathcal{S}$. Choose $\gamma_1 \in F \cap G$. Now, for any $\mathfrak{e}_1 \in \mathcal{S}$,

$$\begin{aligned} \mathfrak{e}_1 \in F &\Rightarrow (\mathfrak{e}_1, \gamma_1) \in F \times F \subseteq \theta_F = \phi_G \\ &\Rightarrow \alpha_1 \wedge_* \mathfrak{e}_1 = \alpha_1 \wedge_* \gamma_1 \text{ for some } \alpha_1 \in G \\ &\Rightarrow \alpha_1 \wedge_* \mathfrak{e}_1 = \alpha_1 \wedge_* \gamma_1 \in G \text{ (since } \alpha_1, \gamma_1 \in G) \\ &\Rightarrow \mathfrak{e}_1 = (\alpha_1 \wedge_* \mathfrak{e}_1) \vee_* \mathfrak{e}_1 \in G. \end{aligned}$$

Therefore, $F \subseteq G$. Also, for any $\mathfrak{e}_2 \in \mathcal{S}$,

$$\begin{aligned} \mathfrak{e}_2 \in G &\Rightarrow (\mathfrak{e}_2 \wedge_* \gamma_1, \gamma_1) \in \phi_G = \theta_F \\ &\Rightarrow \mathfrak{e}_2 \wedge_* \gamma_1 \wedge_* \alpha_1 = \gamma_1 \wedge_* \alpha_1 \text{ for some } \alpha_1 \in F \\ &\Rightarrow \mathfrak{e}_2 \wedge_* \gamma_1 \wedge_* \alpha_1 = \gamma_1 \wedge_* \alpha_1 \in F \text{ (since } \alpha_1, \gamma_1 \in F) \\ &\Rightarrow \mathfrak{e}_2 = \mathfrak{e}_2 \vee_* (\mathfrak{e}_2 \wedge_* \gamma_1 \wedge_* \alpha_1) \in F. \end{aligned}$$

Therefore, $G \subseteq F$. Hence, $F = G$. ■

Let us recall that $\chi = \{(\alpha_1, \beta_1) \in \mathcal{S} \times \mathcal{S} \mid \alpha_1 \wedge_* \beta_1 = \beta_1 \text{ and } \beta_1 \wedge_* \alpha_1 = \alpha_1\}$ is a congruence on an ADL $\mathcal{S}$. It is easily observed that χ is also a congruence on an SBADL $\mathcal{S}$. Moreover, χ is the smallest congruence on $\mathcal{S}$ such that $\mathcal{S}/\chi$ is a lattice. In this context, we have the following.

Theorem 23. *For any filter F of $\mathcal{S}$, $\chi \subseteq \theta_F$.*

Proof. This follows from the fact that $\mathcal{S}/\theta_F$ is a lattice (refer to Theorem 16). ■

Theorem 24. *For any filter F of $\mathcal{S}$, $\chi \subseteq \phi_F$ if and only if $\phi_F = \theta_F$.*

Proof. Let F be a filter of $\mathcal{S}$. Suppose $\chi \subseteq \theta_F$. Clearly $\phi_F \subseteq \theta_F$. Let $(\alpha_1, \beta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F$. Since $(\alpha_1 \wedge_* \beta_1, \beta_1 \wedge_* \alpha_1) \in \chi$, $(\alpha_1 \wedge_* \beta_1, \beta_1 \wedge_* \alpha_1) \in \phi_F$. Therefore, there exists $\mathfrak{e}_2 \in F$ such that $\mathfrak{e}_2 \wedge_* \alpha_1 \wedge_* \beta_1 = \mathfrak{e}_2 \wedge_* \beta_1 \wedge_* \alpha_1$. Since F is a filter of $\mathcal{S}$, $\mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \in F$. Now,

$$\begin{aligned}
&\mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \alpha_1\\
&= \mathfrak{e}_2 \wedge_* \mathfrak{e}_1 \wedge_* \alpha_1\\
&= \mathfrak{e}_2 \wedge_* \alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \alpha_1\\
&= \mathfrak{e}_2 \wedge_* \beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \alpha_1 \text{ (since } \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1)\\
&= \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \beta_1 \wedge_* \alpha_1\\
&= \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \alpha_1 \wedge_* \beta_1 \text{ (since } \mathfrak{e}_2 \wedge_* \beta_1 \wedge_* \alpha_1 = \mathfrak{e}_2 \wedge_* \alpha_1 \wedge_* \beta_1)\\
&= \mathfrak{e}_2 \wedge_* \alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* y\\
&= \mathfrak{e}_2 \wedge_* \beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \beta_1 \text{ (since } \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1)\\
&= \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \beta_1.
\end{aligned}$$

Therefore, $(\alpha_1, \beta_1) \in \phi_F$. Hence, $\phi_F = \theta_F$.

The converse follows from Theorem 23. ■

Earlier we have observed that $\phi_F \subseteq \theta_F$ and $\chi \subseteq \theta_F$ for any filter F. In the fact, θ_F is the supremum of ϕ_F, χ in the lattice of congruences on $\mathcal{S}$. This is proved in the following.

Theorem 25. *Let F be any filter in $\mathcal{S}$. Then $\theta_F = \phi_F \vee_* \chi$, the supremum of ϕ_F and χ in $Con(\mathcal{S})$, where $Con(\mathcal{S})$ is the lattice of all congruences on $\mathcal{S}$.*

Proof. Clearly, $\phi_F \vee_* \chi \subseteq \theta_F$. On the other hand, suppose $(\alpha_1, \beta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* f$ for some $\mathfrak{e}_1 \in F$. Now consider the sequence $\alpha_1, \mathfrak{e}_1 \wedge_* \alpha_1, \alpha_1 \wedge_* \mathfrak{e}_1, \beta_1 \wedge_* \mathfrak{e}_1, \mathfrak{e}_1 \wedge_* \beta_1, \beta_1$. In this, any consequence pair belongs to ϕ_F or to χ. Therefore, $(\alpha_1, \beta_1) \in \phi_F \vee_* \chi$. Thus, $\theta_F = \phi_F \vee_* \chi$. ■

Theorem 26. *Let $\mathcal{M}$ be the set of maximal elements in $\mathcal{S}$. Then $\mathcal{M}$ is a filter in $\mathcal{S}$, $\phi_{\mathcal{M}} = \Delta$ and $\theta_{\mathcal{M}} = \chi$.*

Proof. Note that if $\mathfrak{u}_1$ is a maximal element and α_1 is any element, then $\mathfrak{u}_1 \vee_* \alpha_1 = \mathfrak{u}_1$ and $\alpha_1 \vee_* \mathfrak{u}_1$ is maximal. Also, if $\mathfrak{u}_{1_1}$ and $\mathfrak{u}_{1_2}$ are maximal elements, then so is $\mathfrak{u}_{1_1} \wedge_* \mathfrak{u}_{1_2}$. From there, it follows that $\mathcal{M}$ is a filter of $\mathcal{S}$. Let $(\alpha_1, \beta_1) \in \phi_{\mathcal{M}}$. Then $\mathfrak{u}_1 \wedge_* \alpha_1 = \mathfrak{u}_1 \wedge_* \beta_1$ for some $\mathfrak{u}_1 \in \mathcal{M}$. Therefore, $\alpha_1 = \beta_1$. Hence, $\phi_{\mathcal{M}} = \Delta$. By Theorem 23, we get that $\chi \subseteq \theta_{\mathcal{M}}$. Let $(\alpha_1, \beta_1) \in \theta_{\mathcal{M}}$. Then $\alpha_1 \wedge_* \mathfrak{u}_1 = \beta_1 \wedge_* \mathfrak{u}_1$ for some $\mathfrak{u}_1 \in \mathcal{M}$. Now,

$$\begin{aligned}
\alpha_1 \wedge_* \beta_1 &= \alpha_1 \wedge_* \mathfrak{u}_1 \wedge_* \beta_1\\
&= \beta_1 \wedge_* \mathfrak{u}_1 \wedge_* \beta_1 \text{ (since } \alpha_1 \wedge_* \mathfrak{u}_1 = \beta_1 \wedge_* \mathfrak{u}_1)\\
&= \mathfrak{u}_1 \wedge_* \beta_1 \wedge_* \beta_1\\
&= \beta_1
\end{aligned}$$

and

$$\begin{aligned}
\beta_1 \wedge_* \alpha_1 &= \beta_1 \wedge_* \mathfrak{u}_1 \wedge_* \alpha_1\\
&= \alpha_1 \wedge_* \mathfrak{u}_1 \wedge_* \alpha_1 \text{ (since } \beta_1 \wedge_* \mathfrak{u}_1 = \alpha_1 \wedge_* \mathfrak{u}_1)\\
&= \mathfrak{u}_1 \wedge_* \alpha_1\\
&= \alpha_1.
\end{aligned}$$

Therefore, $(\alpha_1, \beta_1) \in \chi$. Hence, $\theta_{\mathcal{M}} = \chi$. ■

Theorem 26 is strengthened in the following.

Theorem 27. *For any filter F of $\mathcal{S}$, the following are equivalent:*

(1) $\theta_F = \chi$

(2) $F = \mathcal{M}$

(3) $\phi_F = \Delta$.

Proof. Let F be a filter of $\mathcal{S}$.

(1)⇒(2) Suppose $\theta_F = \chi$. Let $\mathfrak{e}_1 \in F$. Then for any $\mathfrak{e}_2 \in \mathcal{S}$, $\mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_1 = \mathfrak{e}_2 \wedge_* \mathfrak{e}_1$ and hence, $(\mathfrak{e}_1 \wedge_* \mathfrak{e}_2, \mathfrak{e}_2) \in \theta_F = \chi$. So that $\mathfrak{e}_1 \wedge_* \mathfrak{e}_2 = \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_2 = \mathfrak{e}_2$. Hence, $\mathfrak{e}_1$ is a maximal element in $\mathcal{S}$. Since all maximal elements necessarily belong to every filter, it follows that (2) holds.

(2)⇒(3) It follows from Theorem 26.

(3)⇒(1) Suppose $\phi_F = \Delta$. Clearly, $\chi \subseteq \theta_F$. Let $(\alpha_1, \beta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F$. Now,

$$\begin{aligned} \mathfrak{e}_1 \wedge_* \alpha_1 \wedge_* \beta_1 &= \alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \beta_1 \\ &= \beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \beta_1 \text{ (since } \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1) \\ &= \mathfrak{e}_1 \wedge_* \beta_1 \end{aligned}$$

and

$$\begin{aligned} \mathfrak{e}_1 \wedge_* \beta_1 \wedge_* \alpha_1 &= \beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \alpha_1 \\ &= \alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \alpha_1 \text{ (since } \beta_1 \wedge_* \mathfrak{e}_1 = \alpha_1 \wedge_* \mathfrak{e}_1) \\ &= \mathfrak{e}_1 \wedge_* \alpha_1. \end{aligned}$$

Therefore, $(\alpha_1 \wedge_* \beta_1, \beta_1), (\beta_1 \wedge_* \alpha_1, \alpha_1) \in \phi_F$. Since $\phi_F = \Delta$, we get that $\alpha_1 \wedge_* \beta_1 = \beta_1$ and $\beta_1 \wedge_* \alpha_1 = \alpha_1$. Hence, $(\alpha_1, \beta_1) \in \chi$. Thus, $\theta_F = \chi$. ■

Lemma 28. *Given filters F and G of $\mathcal{S}$, $F \subseteq G$ implies $\theta_F \subseteq \theta_G$ and $\phi_F \subseteq \phi_G$.*

Proof. Let F and G be two filters of $\mathcal{S}$ such that $F \subseteq G$. Let $(\alpha_1, \beta_1) \in \theta_F$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ for some $\mathfrak{e}_1 \in F$. Since $F \subseteq G$, $(\alpha_1, \beta_1) \in \theta_G$. Therefore, $\theta_F \subseteq \theta_G$. Similarly, $\phi_F \subseteq \phi_G$. ■

Theorem 29. *Given filters F and G of $\mathcal{S}$, we have*

(1) $\theta_F \cap_* \theta_G = \theta_{F \cap_* G}$ *and* $\theta_F \circ \theta_G = \theta_{F \vee_* G}$

(2) $\phi_F \cap_* \phi_G = \phi_{F \cap_* G}$ *and* $\phi_F \circ \phi_G = \phi_{F \vee_* G}$.

Proof. (1) By Lemma 28, we have $\theta_{F \cap_* G} \subseteq \theta_F \cap_* \theta_G$. Let $(\alpha_1, \beta_1) \in \theta_F \cap_* \theta_G$. Then $\alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1$ and $\alpha_1 \wedge_* \mathfrak{e}_2 = \beta_1 \wedge_* \mathfrak{e}_2$ for some $\mathfrak{e}_1 \in F$ and $\mathfrak{e}_2 \in G$. Then $\mathfrak{e}_1 \vee_* \mathfrak{e}_2 \in F \cap_* G$. Now,

$$\begin{aligned} \alpha_1 \wedge_* (\mathfrak{e}_1 \vee_* \mathfrak{e}_2) &= (\alpha_1 \wedge_* \mathfrak{e}_1) \vee_* (\alpha_1 \wedge_* \mathfrak{e}_2) \\ &= (\beta_1 \wedge_* \mathfrak{e}_1) \vee_* (\beta_1 \wedge_* \mathfrak{e}_2) \text{ (since } \alpha_1 \wedge_* \mathfrak{e}_1 = \beta_1 \wedge_* \mathfrak{e}_1 \text{)} \\ &= \beta_1 \wedge_* (\mathfrak{e}_1 \vee_* \mathfrak{e}_2). \text{ (since } \alpha_1 \wedge_* \mathfrak{e}_2 = \beta_1 \wedge_* \mathfrak{e}_2 \text{)} \end{aligned}$$

Therefore, $(\alpha_1, \beta_1) \in \theta_{F\cap G}$. Hence, $\theta_F \cap \theta_G = \theta_{F\cap_* G}$. Since $F, G \subseteq F \vee_* G$ (by Lemma 28), we get that $\theta_F, \theta_G \subseteq \theta_{F\vee_* G}$. Therefore, $\theta_F \circ \theta_G \subseteq \theta_{F\vee_* G}$. On the other hand, let $(\alpha_1, \beta_1) \in \theta_{F\vee_* G}$. Then $\alpha_1 \wedge_* \mathfrak{h}_1 = \beta_1 \wedge_* \mathfrak{h}_1$ for some $\mathfrak{h}_1 \in F \vee_* G$. Then $\mathfrak{h}_1 = \mathfrak{e}_1 \wedge_* \mathfrak{e}_2$ for some $\mathfrak{e}_1 \in F$ and $\mathfrak{e}_2 \in G$. Put $\gamma_1 = (\alpha_1 \wedge_* \mathfrak{e}_2) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1)$. Then

$$\begin{aligned}
\gamma_1 \wedge_* \mathfrak{e}_1 &= [(\alpha_1 \wedge_* \mathfrak{e}_2) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1)] \wedge_* \mathfrak{e}_1 \\
&= (\alpha_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_1) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_1) \\
&= (\alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_1) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1) \\
&= (\beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_1) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1) \text{ (by (5) of Theorem 3)} \\
&= ((\mathfrak{e}_1 \wedge_* \mathfrak{e}_2) \wedge_* (\mathfrak{e}_2 \wedge_* \mathfrak{e}_1)) \vee_* (\mathfrak{e}_2 \wedge_* \mathfrak{e}_1) \\
&= \beta_1 \wedge_* \mathfrak{e}_1
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1 \wedge_* \mathfrak{e}_2 &= [(\alpha_1 \wedge_* \mathfrak{e}_2) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1)] \wedge_* \mathfrak{e}_2 \\
&= (\alpha_1 \wedge_* \mathfrak{e}_2 \wedge_* \mathfrak{e}_2) \vee_* (\beta_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_2) \\
&= (\alpha_1 \wedge_* \mathfrak{e}_2) \vee_* (\alpha_1 \wedge_* \mathfrak{e}_1 \wedge_* \mathfrak{e}_2) \text{ (by (5) of Theorem 3)} \\
&= (\alpha_1 \wedge_* \mathfrak{e}_2) \vee_* [(\alpha_1 \wedge_* \mathfrak{e}_2) \wedge_* (\mathfrak{e}_1 \wedge_* \mathfrak{e}_2)] \\
&= \alpha_1 \wedge_* \mathfrak{e}_2.
\end{aligned}$$

Therefore, $(\gamma_1, \beta_1) \in \theta_F$ and $(\alpha_1, \gamma_1) \in \theta_G$ (since $\mathfrak{e}_1 \in F$, $\mathfrak{e}_2 \in G$). Hence, $(\alpha_1, \beta_1) \in \theta_F \circ \theta_G$. Thus, $\theta_F \circ \theta_G = \theta_{F\vee_* G}$. Similarly, $\phi_F \cap_* \phi_G = \phi_{F\cap_* G}$ and $\phi_F \circ \phi_G = \phi_{F\vee_* G}$. ■

Theorem 30. *In $\mathcal{S}$, we have the following:*

(1) $\{\theta_F \mid F \in \mathcal{F}(\mathcal{S})\}$ *forms a permutable sublattice of* $Con(\mathcal{S})$

(2) $\{\phi_F \mid F \in \mathcal{F}(\mathcal{S})\}$ *forms a permutable sublattice of* $Con(\mathcal{S})$.

4. Conclusions

This paper extensively studied the classification of smart congruences on semi-Brouwerian almost distributive lattices as a permutable sublattice of the lattice of congruences and also extracted two different permutable sublattices of a semi-Brouwerian almost distributive lattices from the class of filters in a semi-Brouwerian almost distributive lattice. In future work, we will try to study the behaviour of an SBADL in terms of implicative filters.

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