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ON THE ISOMORPHISM PROBLEM FOR KNIT PRODUCTS

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Abstract

In this paper, we classify up to isomorphism the groups that can be represented as knit products of two groups. More precisely, some necessary and sufficient conditions for two knit products to be isomorphic are given. We mainly deal with isomorphisms leaving one of the two factors or even both invariant. In particular, we decide under some conditions how the knit products arise as split extensions. Furthermore, the decomposition of unfaithful knit products is investigated.

Keywords: knit product, factorization problem, lower isomorphic, upper isomorphic, diagonally isomorphic.

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1. Introduction

The classification of groups up to isomorphisms is one of the most classical problems in group theory. This problem is frequently reduced to the theory of extensions of groups and cohomology theory of groups (see [5, 8, 10–14]). This work investigate the classification of groups using a well known structure operation, namely the knit product. Knit products were introduced by Zappa in [19], and have been intensively studied starting with the classical papers by Szép [15–17]. Other terms referring to Knit products used in the literature are Zappa-Szép products, bicrossed products, general products, and factorisable groups, as stated in ([1,3,16,18] and the references therein). One of the most important examples of knit product is Hall's theorem which shows that every finite soluble group is a knit product of a Sylow p-subgroup and a Hall p-subgroup [6]. In order to fix our notation, we recall first the construction of knit products.

Let G_1 and G_2 be two groups. A group G is called the internal knit product of G_1 and G_2 if $G = G_1G_2$ and $G_1 \cap G_2 = 1$, or, equivalently, for each $g \in G$ there exists a unique $g_1 \in G_1$ and a unique $g_2 \in G_2$ such that $g = g_1g_2$. The knit product is a generalization of the semidirect product of two groups for the case when neither factor is required to be normal.

The factorization problem is one of the most famous open problems of group theory which can be divided into two distinct subproblems. The first is to describe all groups which arise as knit products of G_1 and G_2. The second is to classify up to isomorphism all the knit products of G_1 and G_2 (The isomorphism problem). This is a problem of classifying whether two knit products are isomorphic. The first problem is solved for knit products with cyclic factors. Notably, Rédei has determined the structure of the knit product of two cyclic groups which are not both finite [9]. Douglas and Huppert have studied the knit products of two finite cyclic groups (see [4, 7]). In particular, in [1, Theorem 3.1], it is proved that a knit product of two finite cyclic groups, one of them being of prime order, is isomorphic to a semidirect product of the same cyclic groups. Apart from this, the isomorphism problem is still an open question in general even for knit products with cyclic factors. In this paper, we study the isomorphism problem for knit products in some cases. More precisely, we deal with isomorphisms of certain type, namely leaving one of the two factors or both invariant. In particular, we determine how the knit product can be reduced to the semidirect product of groups. Some examples of isomorphic knit products of two finite cyclic groups are given. Furthermore, we show possibility of various decompositions of a given unfaithful knit product.

Throughout this paper, we denote by $Z(G)$, $\mathrm{Bij}(G)$, $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$, respectively, the center, the group of all bijections, the monoid of all endomorphisms, and the automorphism group of G. Let $\theta \in \mathrm{Aut}(G)$, γ_θ denotes the conjugation by θ in $\mathrm{Aut}(G)$. For an endomorphism ρ of G, we denote the fixed subgroup of ρ by $Fix_G(\rho)$. For any two groups H and K, let $\mathrm{Map}(H, K)$, $\mathrm{Hom}(H, K)$ and $\mathrm{AHom}(H, K)$ denote the set of all maps, the set of all homomorphisms and the set of all anti-homomorphism from H to K, respectively.

2. Preliminaries and properties

Let G_1 and G_2 be two groups and G an internal knit product of G_1 and G_2. For each $g_1 \in G_1$ and $g_2 \in G_2$, there exist $\alpha(g_1, g_2) \in G_1$ and $\beta(g_1, g_2) \in G_2$ such that $g_2g_1 = \alpha(g_1, g_2)\beta(g_1, g_2)$. This defines a homomorphism $\alpha : G_2 \to Bij(G_1)$ and an anti-homomorphism $\beta : G_1 \to \mathrm{Bij}(G_2)$, where $\alpha(g_2)(g_1) = \alpha(g_1, g_2)$ and $\beta(g_1)(g_2) = \beta(g_1, g_2)$, and satisfying the following conditions

$$\alpha(1)(g_1) = g_1 \text{ and } \beta(1)(g_2) = g_2, \tag{1}$$

$$\alpha(g_2)(1) = \beta(g_1)(1) = 1, \tag{2}$$

$$\alpha(g_2)(g_1 g_1') = \alpha(g_2)(g_1)\alpha(\beta(g_1)(g_2))(g_1'), \tag{3}$$

$$\beta(g_1)(g_2 g_2') = \beta(\alpha(g_2')(g_1))(g_2)\beta(g_1)(g_2') \tag{4}$$

for all g_1, $g_1' \in G_1$ and g_2, $g_2' \in G_2$. More concisely, the first condition above asserts the mapping α is a left action of G_2 on G_1 and that β is a right action of G_1 on G_2. Now, let G_1 and G_2 be two groups, and let $\alpha : G_2 \to \mathrm{Bij}(G_1)$ be a group homomorphism and $\beta : G_1 \to \mathrm{Bij}(G_2)$ an anti-homomorphism which satisfy the above conditions. Define the external bicrossed product of G_1 and G_2 induced by (α, β) as the group $G_1\ {}_\alpha\!\bowtie_\beta G_2$ with underlying set $G_1 \times G_2$ and operation given by

$$(x, y) \underset{\alpha,\beta}{\cdot} (x', y') = (x\alpha(y)(x'), \beta(x')(y)y')$$

for all x, $x' \in G_1$, and y, $y' \in G_2$. The subsets $G_1 \times \{1\}$ and $\{1\} \times G_2$ are subgroups of $G_1\ {}_\alpha\!\bowtie_\beta G_2$ isomorphic to G_1 and G_2, respectively. The internal knit product and the external knit product are isomorphic and then we can identify them in the sequel (see [2, Proposition 2.4]). If α is the trivial action then β is an action by group automorphisms and the knit product $G_1\ {}_\alpha\!\bowtie_\beta G_2$ is, in fact, the right semidirect product $G_1 \rtimes_\beta G_2$. Similarly, if β is the trivial action then α is an action by group automorphisms and the knit product $G_1\ {}_\alpha\!\bowtie_\beta G_2$ is exactly the left semidirect product $G_1\ {}_\alpha\!\ltimes G_2$. In particular, we have $G_{1\alpha}\!\bowtie_\beta G_2 = G_1 \times G_2$ if and only if α and β are trivial action. If α and β are both nontrivial actions then we say that $G_{1\alpha}\!\bowtie_\beta G_2$ is a proper knit product. Further, it is easy to check that the bicrossed product $G_{1\alpha}\!\bowtie_\beta G_2$ is abelian if and only if G_1 and G_2 are abelian and the actions α and β are trivial. So, if G_1 and G_2 are both abelian, then $G_{1\alpha}\!\bowtie_\beta G_2 \cong G_1 \times G_2$ if and only if α and β are trivial actions. But, in general, it is possible for a direct product to be isomorphic to a proper knit product as shown in the following example.

Example 1. Let $U_3(\mathbb{F}_3)$ be the Heisenberg group over the finite field $\mathbb{F}_3$. This is a finite group of order 27 and a Sylow 3-subgroup of the linear group $GL_3(\mathbb{F}_3)$. The group $U_3(\mathbb{F}_3)$ has a fixed-point-free automorphism θ of order 8. Now, let $G = U_3(\mathbb{F}_3) \times U_3(\mathbb{F}_3)$ and consider the subgroups $G_1 = \{(g, g) \mid g \in U_3(\mathbb{F}_3)\}$ and $G_2 = \{(g, \theta(g)) \mid g \in U_3(\mathbb{F}_3)\}$. Clearly, we have $G_1 \cong G_2 \cong U_3(\mathbb{F}_3)$, $G_1 \cap G_2 = \{1\}$ and $G = G_1 G_2$. Thus, the group G is the proper knit product of G_1 and G_2.

Now, in view of the preceding discussion the following problem seems natural.

Problem 2 (The isomorphism problem)**.** Let G_1 and G_2 be two groups. Classify up to an isomorphism all knit products of G_1 and G_2.

3. Knit product and split extensions

Recall that a non-abelian group which has no non-trivial abelian direct factor is said to be purely non-abelian. In the next result, we give sufficient conditions for a proper knit product to be isomorphic to the direct product, for the case when one of the factors is a finite purely non-abelian group.

Proposition 3. *Let G_1 be a finite purely non-abelian group and G_2 a group. Suppose that there exist homomorphisms $\delta \in \mathrm{Hom}(G_1, G_2)$ and $\eta \in \mathrm{Hom}(G_2, Z(G_1))$ such that*

$$\alpha(y)(x) = \eta(y)x\eta(\beta(x)(y))^{-1}$$

and

$$\beta(x)(y) = \delta(\alpha(y)(x))^{-1}y\delta(x),$$

for all $x \in G_1$, and $y \in G_2$. Then the knit product $G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2$ is isomorphic to the direct product $G_1 \times G_2$.

Proof. Define a map φ between $G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1 \times G_2$ given by $\varphi(x, y) = (x\eta(y), \delta(x)y)$, for all $x \in G_1$, $y \in G_2$. By using the assumption, we check easily that φ is a group homomorphism. Now, let $\varphi(x, y) = 1$. Then $x\eta(y) = 1$ and $\delta(x)y = 1$. Thus, we get $\eta(\delta(x)) = x$. Since $\theta = \eta \circ \delta \in \mathrm{Hom}(G_1, Z(G_1))$, it follows that $\mathrm{Im}(\theta) \trianglelefteq G_1$. Therefore, using Fitting's Lemma and the fact that G_1 is purely non-abelian, we get $x = 1$ and then $y = 1$. Hence, φ is one-to-one. On the other hand, take $(g_1, g_2) \in G_1 \times G_2$ such that $\varphi(x, y) = (g_1, g_2)$. Then, $x\eta(y) = g_1$ and $\delta(x)y = g_2$, which follows that $x^{-1}\theta(x) = \eta(g_2)g_1^{-1}$. Since G_1 is purely non-abelian, it follows that the map $f_\theta : g \mapsto g^{-1}\theta(g)$ is an anti-monomorphism and therefore, it defines an anti-automorphism of G_1. Hence $x = f_\theta^{-1}(\eta(g_2)g_1^{-1})$ and $y = \delta(f_\theta^{-1}(g_1\eta(g_2^{-1})))g_2$. Thus, φ is onto and then it is a group isomorphism. As required. ■

Remark 4. The previous proposition will not be true if G_1 is not purely non-abelian. Indeed, assume that G_2 is an abelian direct factor of G_1. Let φ be the map defined in the previous proof such that $\eta(y) = \delta(y) = y^{-1}$ for all $y \in G_2$. Thus, we get $\varphi(y, y) = (1, 1)$ and therefore, φ is not an isomorphism.

Further, a proper knit product can be also isomorphic to a right or a left semidirect product. For example, [1, Theorem 3.1] states that a knit product of two cyclic groups G_1 and G_2, one of which has prime order, is isomorphic to a semidirect product of G_1 and G_2. In general, we have

Proposition 5. *Let G_1 and G_2 be two groups. Suppose that there exist a homomorphism $\delta \in \mathrm{Hom}(G_1, \mathrm{Ker}(\alpha))$ such that $\beta(x)(y) = \delta(\alpha(y)(x))^{-1}y\delta(x)$. Then the knit product $G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2$ is isomorphic to the left semidirect product $G_1 \, {}_{\alpha}\!\ltimes G_2$.*

Proof. Indeed, the bijection φ between $G_1 \;{}_\alpha\!\bowtie_\beta\; G_2$ and $G_1 \;{}_\alpha\!\ltimes\; G_2$ given by $\varphi(x, y) = (x, \delta(x)y)$ is clearly a group isomorphism. ■

Similarly, we have

Proposition 6. *Let G_1 and G_2 be two groups. Suppose that there exist a homomorphism $\eta \in Hom(G_2, \mathrm{Ker}(\beta))$ such that $\alpha(y)(x) = \eta(y)x\eta(\beta(x)(y))^{-1}$. Then the knit product $G_1 {}_\alpha\!\bowtie_\beta G_2$ is isomorphic to the right semidirect product $G_1 \rtimes_\beta G_2$.*

4. Isomorphism problem for knit products

Let $\alpha, \alpha' \in \mathrm{Hom}(G_2, \mathrm{Bij}(G_1))$ and $\beta, \beta' \in \mathrm{AHom}(G_1, \mathrm{Bij}(G_2))$. Let $pr_i : G_1 \,{}_{\alpha'}\!\bowtie_{\beta'}\, G_2 \longrightarrow G_i$ be the ith canonical projection and $t_i : G_i \longrightarrow G_1 \;{}_\alpha\!\bowtie_\beta\; G_2$ be the ith canonical injection. Let φ be a group homomorphism from $G_1 \;{}_\alpha\!\bowtie_\beta\; G_2$ to $G_1 \;{}_{\alpha'}\!\bowtie_{\beta'}\; G_2$ and set $\varphi_{ij} = pr_i \circ \varphi \circ t_j$ where $1 \le i, j \le 2$. So we can write φ in the matrix form: $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. Notice that t_j is a group homomorphism but pr_i is not. Furthermore, we have the following lemmas which we need in the sequel.

Lemma 7. *Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be a group homomorphism from $G_1 \;{}_\alpha\!\bowtie_\beta\; G_2$ to $G_1 \;{}_{\alpha'}\!\bowtie_{\beta'}\; G_2$. Then*

$$\varphi(x, y) = (\varphi_{11}(x)\alpha'(\varphi_{21}(x))(\varphi_{12}(y)),\ \beta'(\varphi_{12}(y))(\varphi_{21}(x))\varphi_{22}(y)) \tag{5}$$

for all $x \in G_1$, and $y \in G_2$.

Proof. Indeed, the required equation follows directly by applying the homomorphism φ to the formula $(x, y) = (x, 1) \underset{\alpha,\beta}{\cdot} (1, y)$ and using the equations $\varphi(x, 1) = (\varphi_{11}(x), \varphi_{21}(x))$ and $\varphi(1, y) = (\varphi_{12}(y), \varphi_{22}(y))$. ■

Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be an isomorphism between $G_1 \;{}_\alpha\!\bowtie_\beta\; G_2$ and $G_1 \;{}_{\alpha'}\!\bowtie_{\beta'}\; G_2$ and let $\varphi^{-1} = \begin{pmatrix} \varphi'_{11} & \varphi'_{12} \\ \varphi'_{21} & \varphi'_{22} \end{pmatrix}$ be its inverse. The following lemma follows directly from the matrix identities $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = \begin{pmatrix} \mathrm{Id}_{G_1} & 1 \\ 1 & \mathrm{Id}_{G_2} \end{pmatrix}$.

Lemma 8. *Keep the preceding notations. We have*

$$\varphi_{11}(\varphi'_{11}(x))\alpha'(\varphi_{21}(\varphi'_{11}(x)))(\varphi_{12}(\varphi'_{21}(x))) = x, \tag{6}$$

$$\varphi'_{11}(\varphi_{11}(x))\alpha(\varphi'_{21}(\varphi_{11}(x)))(\varphi'_{12}(\varphi_{21}(x))) = x, \tag{7}$$

$$\beta'(\varphi_{12}(\varphi'_{22}(y)))(\varphi_{21}(\varphi'_{12}(y)))\varphi_{22}(\varphi'_{22}(y)) = y, \tag{8}$$

$$\beta(\varphi'_{12}(\varphi_{22}(y)))(\varphi'_{21}(\varphi_{12}(y)))\varphi'_{22}(\varphi_{22}(y)) = y, \tag{9}$$

for all $x \in G_1$*, and* $y \in G_2$*.*

From now, if $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ is a map from $G_1 \,{}_{\alpha}\!\bowtie_{\beta} G_2$ to $G_1 \,{}_{\alpha'}\!\bowtie_{\beta'} G_2$, then φ is defined by the formula (5).

Definition. The groups $G_1 \,{}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1 \,{}_{\alpha'}\!\bowtie_{\beta'} G_2$ are called lower isomorphic, if there exists an isomorphism $\varphi : G_1 {}_{\alpha}\!\bowtie_{\beta} G_2 \longrightarrow G_1 {}_{\alpha'}\!\bowtie_{\beta'} G_2$ leaving G_2 invariant.

Theorem 9. *Let* G_1 *and* G_2 *be two groups. The knit products* $G_1 \,{}_{\alpha}\!\bowtie_{\beta} G_2$ *and* $G_1 \,{}_{\alpha'}\!\bowtie_{\beta'} G_2$ *are lower isomorphic if and only if there exist* $\varphi_{22} \in \mathrm{Aut}(G_2)$*,* $\varphi_{11} \in \mathrm{Bij}(G_1)$ *and a map* $\varphi_{21} \in \mathrm{Map}(G_1, G_2)$ *such that*

(i) $\varphi_{11}(xx') = \varphi_{11}(x)\alpha'(\varphi_{21}(x))(\varphi_{11}(x'))$,

(ii) $\varphi_{21}(xx') = \beta'(\varphi_{11}(x'))(\varphi_{21}(x))\varphi_{21}(x')$,

(iii) $\varphi_{22}(\beta(x)(y)) = \varphi_{21}(\alpha(y)(x))^{-1}\beta'(\varphi_{11}(x))(\varphi_{22}(y))\varphi_{21}(x)$,

(iv) $\alpha'(\varphi_{22}(y)) = \varphi_{11} \circ \alpha(y) \circ \varphi_{11}^{-1}$,

for all $x,\ x' \in G_1$ *and* $y \in G_2$*.*

Proof. Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be a group isomorphism between $G_1 \,{}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1 \,{}_{\alpha'}\!\bowtie_{\beta'} G_2$ leaving the group G_2 invariant. Evaluate the left hand side and right hand side of the formula $\varphi(x,1) \underset{\alpha',\beta'}{\cdot} \varphi(x',1) = \varphi(xx',1)$, we get the conditions (i) and (ii). Similarly, the formula $\varphi(1,y) \underset{\alpha',\beta'}{\cdot} \varphi(1,y') = \varphi(1,yy')$ implies that $\varphi_{22} \in \mathrm{End}(G_2)$. Further, the conditions (iii) and (iv) follow from the formula $\varphi(1,y) \underset{\alpha',\beta'}{\cdot} \varphi(x',1) = \varphi(\alpha(y)(x'), \beta(x')(y))$. On the other hand, by Lemma 8, the equations (6)–(9) imply that $\varphi_{11} \circ \varphi'_{11} = \varphi'_{11} \circ \varphi_{11} = \mathrm{Id}_{G_1}$ and $\varphi_{22} \circ \varphi'_{22} = \varphi'_{22} \circ \varphi_{22} = \mathrm{Id}_{G_2}$. Therefore, φ_{11} and φ_{22} are bijective. Conversely, a computation shows that the map $\varphi = \begin{pmatrix} \varphi_{11} & 1 \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ is a group homomorphism. So, it remains to prove that φ is bijective. If $\varphi(x,y) = 1$, we obtain $\varphi_{21}(x)\varphi_{22}(y) = 1$ and $\varphi_{11}(x) = 1$. So $x = 1$ and then $\varphi_{22}(y) = 1$ since φ_{21} is unitary. This implies that $y = 1$ and therefore φ is one-to-one. Now, let $(x,y) \in G_1 \,{}_{\alpha'}\!\bowtie_{\beta'} G_2$, we can quickly check that $\varphi(\varphi_{11}^{-1}(x), \varphi_{22}^{-1}(\varphi_{21}(\varphi_{11}^{-1}(x))^{-1}y)) = (x,y)$. Therefore φ is onto. Thus, the proof is completed ■

Let $G_1 =< x >$ and $G_2 =< y >$ be two cyclic groups of orders p^2 and n, where p is an odd prime dividing n. Let r and t be two numbers prime to p such that $(pr+1)^p \equiv 1 \mod n$. Consider the actions $\alpha : G_2 \to \mathrm{Bij}(G_1)$ and $\beta : G_1 \to \mathrm{Bij}(G_2)$ defined by $\alpha(y)(x) = x^t$, $\alpha(y^p)(x) = x$, $\beta(x)(y) = y^{pr+1}$ and $\beta(x)(y^p) = y^{p(pr+1)}$ such that $gcd((t-1), p^2) = p$ and $p(pr+1)^p \equiv p \mod n$. In this case, the corresponding knit product $G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2$ is denoted by $G_1\ {}_{t}\bowtie_{r}\ G_2$. Note that $G_1\ {}_{t}\bowtie_{r}\ G_2$ is the group G defined by Yacoub in [18, Theorem 5].

Example 10. Keep the above notation. For two different numbers pairs (r,t) and (r',t'), suppose that $jt'^s \equiv jt \mod p^2$ and $s(pr'+1)^j \equiv s(pr+1) \mod n$ for some numbers s and j such that $gcd(j,p^2)=1$ and $gcd(s,n)=1$. Then, the knit products $G_1\ {}_{t}\bowtie_{r}\ G_2$ and $G_1\ {}_{t'}\bowtie_{r'}\ G_2$ are lower isomorphic.

Proof. Indeed, consider the automorphisms $\varphi_{11} \in \mathrm{Aut}(G_1)$ and $\varphi_{22} \in \mathrm{Aut}(G_2)$ defined by $\varphi_{22}(y) = y^s$ and $\varphi_{11}(x) = x^j$. Define the map $\varphi_{21} : G_1 \to G_2$ by $\varphi_{21}(x^k) = y^{p\sum_{v=0}^{k-1}(pr+1)^{jv}}$. Inductively, using (3), we have $\alpha'(y^p)(x^u) = x^u$ and then $\alpha'(y^v)(x^u) = x^{ut'^v}$ for all u and v. So $\alpha'(\varphi_{21}(x))\circ\varphi_{11} = \varphi_{11}$ and then we get the condition (i). Similarly, by using (4), we get $\beta'(x^u)(y^{\lambda p}) = y^{\lambda p(pr'+1)^u}$ for all u and λ, and then we obtain (ii). Furthermore, the equation (iv) follows directly from the condition $jt'^s \equiv jt \mod p^2$. Now, the condition $p(pr+1)^p \equiv p \mod n$ implies that $\varphi_{21}(\alpha(y^v)(x^u)) = \varphi_{21}(x^u)$ for all u and v. Since $(pr+1)^{t-1} \equiv 1 \mod n$ and $(pr'+1)^{t-1} \equiv 1 \mod n$, it follow from (4) that $\beta(x^u)(y^v) = y^{v(pr+1)^u}$ and $\beta'(x^u)(y^v) = y^{v(pr'+1)^u}$ for all u and v. Hence, the condition $s(pr'+1)^j \equiv s(pr+1) \mod n$ gives us $\varphi_{22}(\beta(x^u)(y^v)) = \beta'(\varphi_{11}(x^u))(\varphi_{22}(y^v))$ for all u and v. Thus, we obtain (iii). Therefore, by the previous theorem, the knit products $G_1\ {}_{t}\bowtie_{r}\ G_2$ and $G_1\ {}_{t'}\bowtie_{r'}\ G_2$ are lower isomorphic. ■

As direct consequences of Theorem 9, we have

Corollary 11. *Let G_1 and G_2 be two groups. The groups $G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2$ and $G_1\ {}_{\alpha'}\ltimes\ G_2$ are lower isomorphic if and only if there exist $\rho \in \mathrm{Aut}(G_2)$, $\delta \in \mathrm{Hom}(G_1, G_2)$ and a bijective 1-cocycle $\sigma \in Z^1(G_1, G_1, \alpha'\circ\delta)$ such that*

$$\begin{aligned} \rho(\beta(x)(y)) &= \delta(\alpha(y)(x))^{-1}\rho(y)\delta(x), \\ \alpha'(\rho(y)) &= \sigma\circ\alpha(y)\circ\sigma^{-1}, \end{aligned}$$

for all $x \in G_1$ and $y \in G_2$.

Corollary 12. *Let G_1 and G_2 be two groups. The groups $G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2$ and $G_1 \rtimes_{\beta'} G_2$ are lower isomorphic if and only if the action α is trivial and there exist $\sigma \in \mathrm{Aut}(G_1)$, $\rho \in \mathrm{Aut}(G_2)$ and a 1-cocycle $\delta \in Z^1(G_1, G_2, \beta'\circ\sigma)$ such that $\rho(\beta(x)(y)) = \delta(x)^{-1}\beta'(\sigma(x))(\rho(y))\delta(x)$ for all $x \in G_1$ and $y \in G_2$.*

Definition. The knit products $G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2$ and $G_1\ {}_{\alpha'}\bowtie_{\beta'}\ G_2$ are called upper isomorphic, if there exists an isomorphism $\varphi : G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2 \longrightarrow G_1\ {}_{\alpha'}\bowtie_{\beta'}\ G_2$ leaving G_1 invariant. If in addition the isomorphism φ leaves G_2 invariant, then $G_1\ {}_{\alpha}\bowtie_{\beta}\ G_2$ and $G_1\ {}_{\alpha'}\bowtie_{\beta'}\ G_2$ are said to be diagonally isomorphic.

Theorem 13. *Let G_1 and G_2 be two groups. The knit products $G_1\ {}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1\ {}_{\alpha'}\!\bowtie_{\beta'} G_2$ are upper isomorphic if and only if there exist $\varphi_{11} \in \mathrm{Aut}(G_1)$, $\varphi_{22} \in \mathrm{Bij}(G_2)$ and $\varphi_{12} \in \mathrm{Map}(G_2, G_1)$ such that*

(i) $\varphi_{22}(yy') = \beta'(\varphi_{12}(y'))(\varphi_{22}(y))\varphi_{22}(y')$,

(ii) $\varphi_{12}(yy') = \varphi_{12}(y)\alpha'(\varphi_{22}(y))(\varphi_{12}(y'))$,

(iii) $\varphi_{11}(\alpha(y)(x')) = \varphi_{12}(y)\alpha'(\varphi_{22}(y))(\varphi_{11}(x'))\varphi_{12}(\beta(x')(y))^{-1}$,

(iv) $\beta'(\varphi_{11}(x')) = \varphi_{22} \circ \beta(x') \circ \varphi_{22}^{-1}$,

for all x, $x' \in G_1$ and y, $y' \in G_2$.

Proof. Let φ be a map between $G_1\ {}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1\ {}_{\alpha'}\!\bowtie_{\beta'} G_2$. By applying the same arguments as those used in the proof of Theorem 9, we claim that the map φ is a group homomorphism leaving the group G_1 invariant if and only if $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ 1 & \varphi_{22} \end{pmatrix}$ such that $\varphi_{11} \in \mathrm{End}(G_1)$, $\varphi_{22} \in \mathrm{Map}(G_2, G_2)$ and $\varphi_{12} \in \mathrm{Map}(G_2, G_1)$ satisfying the conditions (i)–(iv). It remains to prove that φ is bijective if and only if φ_{11} and φ_{22} are bijective. If φ is bijective, by Lemma 8, the maps φ_{11} and φ_{22} are clearly bijective. Conversely, suppose that φ_{11} and φ_{22} are bijective and let $(x, y) \in G_1\ {}_{\alpha'}\!\bowtie_{\beta'} G_2$. We see that $\varphi(\varphi_{11}^{-1}(x\varphi_{12}(\varphi_{22}^{-1}(y))^{-1}), \varphi_{22}^{-1}(y)) = (x, y)$ which implies that φ is surjective. The injectivity is clear and then φ is bijective. As required. ∎

Example 14. Let (r, t) and (r', t') be the pairs given in Example 10. The knit products $G_1\ {}_{t}\!\bowtie_{r} G_2$ and $G_1\ {}_{t'}\!\bowtie_{r'} G_2$ are also upper isomorphic. Indeed, consider the automorphisms $\varphi_{11} \in Aut(G_1)$ and $\varphi_{22} \in Aut(G_2)$ defined in Example 10 and define the map $\varphi_{12} : G_2 \to G_1$ by $\varphi_{12}(y^k) = x^{kp}$ for all k. Using $t \equiv 1 \mod p$, we get (ii). Furthermore, the condition (i) follows by using $(pr' + 1)^p \equiv 1 \mod n$. Similarly, the relation $jt'^s \equiv jt \mod p^2$ gives us the condition (iii). Finally, the condition (iv) follows immediately from the relation $s(pr' + 1)^j \equiv s(pr + 1) \mod n$. Thus, by the previous result, the knit products $G_1\,{}_{t}\!\bowtie_{r} G_2$ and $G_1\,{}_{t'}\!\bowtie_{r'} G_2$ are upper isomorphic.

Now, as consequences of Theorem 13, we give the following results.

Corollary 15. *Let G_1 and G_2 be two groups. The groups $G_1\,{}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1\,{}_{\alpha'}\!\ltimes G_2$ are upper isomorphic if and only if the action β is trivial and there exist $\sigma \in \mathrm{Aut}(G_1)$, $\rho \in \mathrm{Aut}(G_2)$ and a 1-cocycle $\eta \in Z^1(G_2, G_1, \alpha' \circ \rho)$ such that $\sigma(\alpha(y)(x)) = \eta(y)\alpha'(\rho(y))(\sigma(x))\eta(y)^{-1}$ for all $x \in G_1$ and $y \in G_2$.*

Corollary 16. *Let G_1 and G_2 be two groups. The groups $G_1\,{}_{\alpha}\!\bowtie_{\beta} G_2$ and $G_1 \rtimes_{\beta'} G_2$ are upper isomorphic if and only if there exist $\sigma \in \mathrm{Aut}(G_1)$, $\eta \in \mathrm{Hom}(G_2, G_1)$*

and a bijective 1*-cocycle* $\rho \in Z^1(G_2, G_2, \beta' \circ \eta)$ *such that*

$$\begin{aligned}\sigma(\alpha(y)(x)) &= \eta(y)\sigma(x)\eta(\beta(x)(y))^{-1},\\ \beta'(\sigma(x)) &= \rho \circ \beta(x) \circ \rho^{-1},\end{aligned}$$

for all $x \in G_1$ *and* $y \in G_2$*.*

Corollary 17. *Let* G_1 *and* G_2 *be two groups. The knit products* $G_1 {}_{\alpha}\bowtie_{\beta} G_2$ *and* $G_1 {}_{\alpha'}\bowtie_{\beta'} G_2$ *are diagonally isomorphic if and only if there exist* $\sigma \in \mathrm{Aut}(G_1)$ *and* $\rho \in \mathrm{Aut}(G_2)$ *such that* $\alpha' \circ \rho = \gamma_\sigma \circ \alpha$ *and* $\beta' \circ \sigma = \gamma_\rho \circ \beta$*.*

Example 18. Let $G_1 =< x >$ and $G_2 =< y >$ be two cyclic groups of orders 12 and 3, respectively. Consider the actions α, $\alpha' : G_2 \to \mathrm{Bij}(G_1)$ and $\beta : G_1 \to \mathrm{Aut}(G_2)$ defined by

$$\beta(x)(y) = y^{-1},$$

$$\alpha(y)(x^k) = \begin{cases} x^k, & k \text{ even} \\ x^{k+4}, & k \text{ odd} \end{cases}$$

and

$$\alpha'(y)(x^k) = \begin{cases} x^k, & k \text{ even} \\ x^{k+8}, & k \text{ odd}\,. \end{cases}$$

Now, consider the automorphisms $\sigma \in \mathrm{Aut}(G_1)$ and $\rho \in \mathrm{Aut}(G_2)$ defined by $\rho(y) = y^2$ and $\sigma(x) = x^7$. By a simple computation, we get $\alpha' \circ \rho = \gamma_\sigma \circ \alpha$ and $\beta \circ \sigma = \gamma_\rho \circ \beta$. Hence, by the previous corollary, the knit products $G_1 {}_{\alpha}\bowtie_{\beta} G_2$ and $G_1 {}_{\alpha'}\bowtie_{\beta} G_2$ are diagonally isomorphic.

Remark 19. Under some conditions, it is possible for two isomorphic knit products to be upper, lower or diagonally isomorphic. Indeed, suppose that G_1 and G_2 have coprime order. Let $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ be an isomorphism between $G_1 {}_{\alpha}\bowtie_{\beta} G_2$ and $G_1 {}_{\alpha'}\bowtie_{\beta'} G_2$. By evaluating the left hand side and the right hand side of the formulas $\varphi(x, 1) \underset{\alpha',\beta'}{\cdot} \varphi(x', 1) = \varphi(xx', 1)$ and $\varphi(1, y) \underset{\alpha',\beta'}{\cdot} \varphi(1, y') = \varphi(1, yy')$, we get the condition (ii) of Theorem 9 and the condition (ii) of Theorem 13. If $\mathrm{Im}(\varphi_{11}) \leq \mathrm{Ker}(\beta')$, then φ_{21} is group homomorphism and therefore it must be trivial. That is $G_1 {}_{\alpha}\bowtie_{\beta} G_2$ and $G_1 {}_{\alpha'}\bowtie_{\beta'} G_2$ are lower isomorphic. Similarly, if $\mathrm{Im}(\varphi_{22}) \leq \mathrm{Ker}(\alpha')$ then they must be upper isomorphic. Hence, if we have the both conditions, the isomorphic knit products are in fact diagonally isomorphic.

Remark 20. Let G_1 and G_2 be two groups. Suppose that the knit products $G_1 {}_{\alpha}\bowtie_{\beta} G_2$ and $G_1 {}_{\alpha'}\bowtie_{\beta'} G_2$ are diagonally isomorphic. In view of the preceding corollary, one can find automorphisms $\sigma \in Aut(G_1)$ and $\rho \in Aut(G_2)$ so that

$\alpha'(\rho(G_2)) = \sigma\circ\alpha(G_2)\circ\sigma^{-1}$ and $\beta'(\sigma(G_1)) = \rho\circ\beta(G_1)\circ\rho^{-1}$. Since $\rho(G_2) = G_2$ and $\sigma(G_1) = G_1$, it follows that the images $\alpha'(G_2)$ and $\alpha(G_2)$ are conjugate subgroups of $Aut(G_1)$, and $\beta'(G_1)$ and $\beta(G_1)$ are conjugate subgroups of $Aut(G_2)$.

Conversely, the conjugacy of the images of the corresponding actions does not necessarily give us isomorphic knit products. For example, let $G_1 = \langle g\rangle$ be the cyclic group of order 7 and $G_2 = \langle a, b \mid a^3 = b^7 = 1, a^{-1}ba = b^2\rangle$. Let β and β' be trivial actions and define α such that $\alpha(a)(g) = g^2$ and $\alpha(b) = \mathrm{Id}_{G_1}$. Similarly, we define α' such that $\alpha'(a)(g) = g^4$ and $\alpha'(b) = \mathrm{Id}_{G_1}$. We have $\alpha'(G_2) = \alpha(G_2)$ and $\beta'(G_1) = \beta(G_1) = \{\mathrm{Id}_{G_2}\}$, but the corresponding knit products

$$\left\langle a, b, g \mid a^3 = b^7 = g^7 = 1, bg = gb, a^{-1}ba = b^2, a^{-1}ga = g^2\right\rangle$$

and

$$\left\langle a, b, g \mid a^3 = b^7 = g^7 = 1, bg = gb, a^{-1}ba = b^2, a^{-1}ga = g^4\right\rangle$$

are not isomorphic.

5. Unfaithful knit product decompositions

Definition. Let $G = G_1\ {}_{\alpha}\!\bowtie_{\beta} G_2$ be a knit product of G_1 and G_2. We call G a faithful knit product if the actions α and β are faithful, that is α is a monomorphism and β is an anti-monomorphism.

Let $G_1\ {}_{\alpha}\!\bowtie_{\beta} G_2$ be an unfaithful knit product. Take $H_1 = \mathrm{Ker}(\beta)$ and $H_2 = \mathrm{Ker}(\alpha)$. Let π_i be the canonical projection of G_i onto G_i/H_i and let $s_i : G_i/H_i \to G_i$ be a group homomorphism such that $\pi_i\circ s_i = \mathrm{Id}_{G_i/H_i}$ and $\mathrm{Im}(s_i\circ\pi_i) \leq Z(G_i)$. Define the maps $f_x : G_2 \to G_2$ and $f_y : G_1 \to G_1$ by $f_x(y) = y\beta(x)(y)^{-1}$ and $f_y(x) = \alpha(y)(x)^{-1}x$. The following result shows that the characterization of isomorphism classes of the unfaithful knit product $G_1\ {}_{\alpha}\!\bowtie_{\beta} G_2$ is reduced to that of the faithful knit product $G_1/H_1\ {}_{\overline{\alpha}}\!\bowtie_{\overline{\beta}} G_2/H_2$ with $\overline{\alpha}\circ\pi_2(y)\circ\pi_1 = \pi_1\circ\alpha(y)$ and $\overline{\beta}\circ\pi_1(x)\circ\pi_2 = \pi_2\circ\beta(x)$ for all $x \in G_1$ and $y \in G_2$.

Proposition 21. *Keep the above notations and assumptions and let G_1 be a group and G_2 an abelian group. Suppose that $\mathrm{Im}(f_x) \leq \mathrm{Fix}_{G_2}(s_2\circ\pi_2)$ and $\mathrm{Im}(f_y) \leq \mathrm{Fix}_{G_1}(s_1\circ\pi_1)$ for all $x \in G_1$ and $y \in G_2$. Then the knit product $G_1/H_1\ {}_{\overline{\alpha}}\!\bowtie_{\overline{\beta}} G_2/H_2$ is a direct factor of G.*

Proof. Indeed, it is directly checked that $\overline{\alpha}(\pi_2(y)) \in \mathrm{Epi}(G_1/H_1)$. Now, if $\overline{\alpha}(\pi_2(y))(\pi_1(x)) = H_1$ then $\alpha(y)(x) \in H_1$. But, it follows from the equation (4) that $\beta\circ\alpha(y) = \beta$ for all $y \in G_2$, so $\beta(x) = \mathrm{Id}_{G_2}$ and then $x \in H_1$. Hence $\overline{\alpha}(\pi_2(y)) \in \mathrm{Aut}(G_1/H_1)$. Similarly, we get $\overline{\beta}(\pi_1(x)) \in \mathrm{Aut}(G_2/H_2)$. Furthermore, it is obvious to see that $\overline{\alpha} : G_2/H_2 \to \mathrm{Aut}(G_1/H_1)$ is a group homomorphism and the map $\overline{\beta} : G_1/H_1 \to \mathrm{Aut}(G_2/H_2)$ is an anti-homomorphism. Now, define the bijection $\varphi : G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2 \longrightarrow H_1 \times (G_1/H_1 \, {}_{\overline{\alpha}}\!\bowtie_{\overline{\beta}} G_2/H_2) \times H_2$ by

$$\varphi(x,y) = (xs_1(\pi_1(x^{-1})), (\pi_1(x), \pi_2(y)), ys_2(\pi_2(y^{-1})))$$

for all $x \in G_1$, $y \in G_2$. Let x, $x' \in G_1$ and y, $y' \in G_2$, we have

$$\begin{aligned}
\varphi((x,y) \underset{\alpha,\beta}{\cdot} (x',y')) &= \varphi(x\alpha(y)(x'), \beta(x')(y)y') \\
&= (x\alpha(y)(x')s_1(\pi_1(\alpha(y)(x')^{-1}x^{-1})), \\
&\quad (\pi_1(x)\pi_1(\alpha(y)(x')), \pi_2(\beta(x')(y))\pi_2(y')), \\
&\quad \beta(x')(y)y's_2(\pi_2(y'^{-1}\beta(x')(y)^{-1}))) \\
\text{using the assumption} &= (xs_1(\pi_1(x^{-1}))x's_1(\pi_1(x'^{-1})), \\
&\quad (\pi_1(x)\overline{\alpha}(\pi_2(y))(\pi_1(x')), \overline{\beta}(\pi_1(x'))(\pi_2(y))\pi_2(y')), \\
&\quad ys_2(\pi_2(y^{-1})y's_2(\pi_2(y'^{-1}))) \\
&= (xs_1(\pi_1(x^{-1}))x's_1(\pi_1(x'^{-1})), \\
&\quad (\pi_1(x), \pi_2(y)) \underset{\overline{\alpha},\overline{\beta}}{\cdot} (\pi_1(x'), \pi_2(y')), \\
&\quad ys_2(\pi_2(y^{-1})y's_2(\pi_2(y'^{-1}))) \\
&= \varphi(x,y)\varphi(x',y').
\end{aligned}$$

Thus φ is a group homomorphism and then it is a group isomorphism, as required. ∎

Using a similar computation as in the previous proof, the following proposition provides another factorisation of $G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2$.

Proposition 22. *Let G_1 and G_2 be two groups. Suppose that $\mathrm{Im}(f_x) \leq \mathrm{Fix}_{G_2}(s_2 \circ \pi_2)$ and $\mathrm{Im}(f_y) \leq \mathrm{Fix}_{G_1}(s_1 \circ \pi_1)$ for all $x \in G_1$ and $y \in G_2$. Then*

$$G_1 \, {}_{\alpha}\!\bowtie_{\beta} G_2 \cong (H_2 \times G_1/H_1) \, {}_{\widetilde{\alpha}}\!\bowtie_{\widetilde{\beta}} (G_2/H_2 \times H_1)$$

where $\widetilde{\alpha}(\pi_2(y), h_1)(h_2, \pi_1(x)) = (h_2, \pi_1(\alpha(y)(x)))$ and $\widetilde{\beta}(h_2, \pi_1(x))(\pi_2(y), h_1) = (\pi_2(\beta(x)(y)), h_1)$.

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