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SEMIGROUPS OF PARTIAL TRANSFORMATIONS WITH INVARIANT SET: GREEN'S RELATIONS, UNIT REGULARITY AND DIRECTLY FINITENESS

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Abstract

Given a nonempty set X, and let $P(X)$ denote the partial transformation semigroup on X. For a nonempty subset Y of X, define $\overline{PT}(X,Y)$ as follows

$$\overline{PT}(X,Y) = \{\alpha \in P(X) : (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

Then $\overline{PT}(X,Y)$ is a generalization of $P(X)$, consisting of all partial transformations on X that leave Y as an invariant set. In this paper, we investigate the Green's relations and explore all unit regular elements. Additionally, we determine the necessary and sufficient conditions for $\overline{PT}(X,Y)$ to be unit regular and directly finite.

Keywords: partial transformation semigroups, Green's relations, unit regular semigroups, directly finite semigroups.

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1. Introduction and preliminaries

Let X be a nonempty set, and let $T(X)$ be the full transformation semigroup on X under the composition of mappings. In semigroup theory, the semigroup $T(X)$ holds significant importance since any semigroups can be considered as an isomorphic subsemigroup of $T(X)$. Many prominent properties have been established for $T(X)$, and several special subsemigroups of $T(X)$ have been extensively investigated.

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As far back as 1955, Doss [9] demonstrated that $T(X)$ is a regular semigroup and provided a description of Green's relations on $T(X)$. Tirasupa [20] then showed in 1979 that $T(X)$ is factorizable if and only if X is finite. Moving forward to 1980, Alarcao [1] established that $T(X)$ is directly finite if and only if X is finite.

Over time, the concept of the full transformation semigroup has undergone impressive expansion, incorporating and encompassing previous findings. One well-known generalization of $T(X)$ is a semigroup $\overline{T}(X,Y)$, where Y is a fixed nonempty subset of X, defined as follows

$$\overline{T}(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

The investigation of this semigroup was initiated by Magill [13] in 1966. Subsequently, in 2005, Nenthein *et al.* [14] obtained a characterization for the regularity of elements in $\overline{T}(X,Y)$ and illustrated that $\overline{T}(X,Y)$ is regular if and only if either $X = Y$ or Y is a singleton set. Later on, in 2007, Boonmee [2] examined the factorization properties of $\overline{T}(X,Y)$ by establishing its connection with unit regularity. In 2011, Honyam and Sangwong [10] provided a comprehensive description of Green's relations and the ideals within $\overline{T}(X,Y)$. Additional properties related to $\overline{T}(X,Y)$ have been described [3, 18, 19].

In 2013, Honyam and Sanwong [11] introduced the concept of a transformation semigroup, which demonstrates simplicity yet significant capability. This was achieved by considering a fixed subset Y of a nonempty set X and defining the set as follows

$$Fix(X,Y) = \{\alpha \in T(X) : y\alpha = y \text{ for all } y \in Y\}.$$

Notably, such a semigroup is a generalization of $T(X)$, as $Fix(X,\emptyset) = T(X)$, and it is contained within $\overline{T}(X,Y)$. The authors demonstrated that $Fix(X,Y)$ forms a regular semigroup and provided a complete description of Green's relations on $Fix(X,Y)$. Later in 2017, Chaiya et al. [5] gave necessary and sufficient conditions for $Fix(X,Y)$ to be factorizable, unit regular, and directly finite. Furthermore, several other properties concerning $Fix(X,Y)$ have been detailed in previous studies [4, 6, 15, 16].

Consider $P(X)$, the partial transformation semigroup, which comprises all functions from a subset of X to X, operating under function composition, where X is a nonempty set. It is evident that $T(X)$, $Fix(X,Y)$, and $\overline{T}(X,Y)$ are strictly contained within $P(X)$. The description of Green's relations on $P(X)$ appeared in [12]. The factorization, unit regularity, and directly finiteness of $P(X)$ were also described in [1, 20].

In 2020, Chinram and Yonthanthum [8] extended the study of the semigroup $Fix(X,Y)$ to partial transformations by generalizing it to $P(X)$. They achieved this by considering $Y \subseteq X$ and defining the set as follows

$$PFix(X,Y) = \{\alpha \in P(X) : y\alpha = y \text{ for all } y \in \operatorname{dom}\alpha \cap Y\}.$$

This construction generalizes $P(X)$, as $PFix(X,\emptyset) = P(X)$. They demonstrated that $PFix(X,Y)$ is not a regular semigroup; however, they provided a complete and accurate description of the necessary and sufficient conditions for the elements of such a semigroup to be regular. Later in 2022, Wijarejak and Chaiya [21] described Green's relations on $PFix(X,Y)$. Moreover, they showed the conditions for $PFix(X,Y)$ to be unit regular in [22].

The focus of this paper is on the semigroup $\overline{PT}(X,Y)$, which follows the same concept of expanding the semigroup $T(X)$ to yield $\overline{T}(X,Y)$. In 2021, Chinram [7] introduced this semigroup similarly, encompassing a fixed nonempty subset Y of X. It is defined as follows

$$\overline{PT}(X,Y) = \{\alpha \in P(X) : (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y\}.$$

In 2022, Pantarak and Chaiya [17] established that $\overline{PT}(X,Y)$ is a regular semigroup if and only if $X = Y$, and $PE = \{\alpha \in \overline{PT}(X,Y) : \operatorname{im}(\alpha) \cap Y = Y\alpha\}$ represents the set of all regular elements within $\overline{PT}(X,Y)$. In this paper, we provide a characterization of Green's relations on $\overline{PT}(X,Y)$ and discover that $\mathcal{D} = \mathcal{J}$ if and only if X is a finite set or $X = Y$. Moreover, we examine the concepts of unit regularity and direct finiteness within $\overline{PT}(X,Y)$.

In this paper, we consider the set X, which can be either finite or infinite. The cardinality of a set A is denoted by $|A|$, and the notation $X = A\dot{\cup}B$ indicates that X is a disjoint union of A and B. Additionally, we adopt the convention of writing functions on the right to the argument. Specifically, in the composition $\alpha\beta$, the transformation α is applied first. For $\alpha \in P(X)$, the domain and the image of α are denoted by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively. The inverse image of $x \in \operatorname{im}\alpha$ under α is denoted by $x\alpha^{-1}$ and it is the set $\{z \in \operatorname{dom}\alpha : z\alpha = x\}$. An equivalence relation $\ker\alpha$ on $\operatorname{dom}\alpha$ is defined to be $\ker\alpha = \{(a,b) \in \operatorname{dom}\alpha \times \operatorname{dom}\alpha : a\alpha = b\alpha\}$. Hence, the partition of $\operatorname{dom}\alpha$ induced by $\ker\alpha$ is $\{x\alpha^{-1} : x \in \operatorname{im}\alpha\}$.

We usually represent $\alpha \in P(X)$ in two-line notation as follows

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}.$$

Here, we make the assumption that the subscript i belongs to an unspecified index set I. From the notation, we can deduce that $\operatorname{dom}\alpha$ is the disjoint union $\dot{\bigcup}_{i\in I} X_i$, $a_i\alpha^{-1} = X_i$, $\{X_i : i \in I\}$ is the partition of $\operatorname{dom}\alpha$ induced by $\ker\alpha$, and $\operatorname{im}\alpha = \{a_i : i \in I\}$. To simplify notation, we represent $\{a_i : i \in I\}$ as $\{a_i\}$, and when $A \subseteq X$, we use $A\alpha$ to refer to $(\operatorname{dom}\alpha \cap A)\alpha$.

In particular, when $\alpha \in \overline{PT}(X,Y)$, we can divide the domain of α into three parts, as follows

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X\backslash Y; \{a_i\}, \{b_j\} \subseteq Y$; and $\{c_k\} \subseteq X\backslash Y$. It is important to note that the identity map on X, denoted as id_X, belongs to $\overline{PT}(X,Y)$, which implies that $\overline{PT}(X,Y)^1 = \overline{PT}(X,Y)$.

2. GREEN'S RELATIONS ON $\overline{PT}(X,Y)$

We start this section by presenting the known results from [17, 23], which serve as a crucial tool for verifying Green's relations on $\overline{PT}(X,Y)$.

Lemma 1 [17, 23]**.** *Let $\alpha, \beta \in \overline{PT}(X,Y)$.*

(1) $\alpha = \lambda\beta$ *for some* $\lambda \in \overline{PT}(X,Y)$ *if and only if* $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ *and* $Y\alpha \subseteq Y\beta$.

(2) $\alpha = \beta\mu$ *for some* $\mu \in \overline{PT}(X,Y)$ *if and only if* $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$*;* $\ker\beta \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \ker\alpha$*; and* $x \in (\operatorname{dom}\alpha \cap \operatorname{dom}\beta)\backslash Y$ *and* $x\beta \in Y$ *imply* $x\alpha \in Y$.

(3) $\alpha = \lambda\beta\mu$ *for some* $\lambda, \mu \in \overline{PT}(X,Y)$ *if and only if* $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$ *and* $|\operatorname{im}\alpha\backslash Y| \leq |\operatorname{im}\beta\backslash Y|$.

Theorem 2. *Let $\alpha, \beta \in \overline{PT}(X,Y)$. Then $\alpha\,\mathcal{L}\,\beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$ and $Y\alpha = Y\beta$.*

Proof. Assume that $\alpha\,\mathcal{L}\,\beta$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in \overline{PT}(X,Y)$. From Lemma 1(1), we get $\operatorname{im}\alpha = \operatorname{im}\beta$ and $Y\alpha = Y\beta$.

Conversely, assume that $\operatorname{im}\alpha = \operatorname{im}\beta$ and $Y\alpha = Y\beta$. Since $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $Y\alpha \subseteq Y\beta$, as stated in Lemma 1(1), it follows that $\alpha = \lambda\beta$ for some $\lambda \in \overline{PT}(X,Y)$. Similarly, due to $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$ and $Y\beta \subseteq Y\alpha$, we can conclude that $\beta = \mu\alpha$ for some $\mu \in \overline{PT}(X,Y)$. Therefore, we have established that $\alpha\,\mathcal{L}\,\beta$, as required. ∎

Theorem 3. *Let $\alpha, \beta \in \overline{PT}(X,Y)$. Then $\alpha\,\mathcal{R}\,\beta$ if and only if all of the following conditions hold:*

(1) $\ker\alpha = \ker\beta$*;*

(2) $x\alpha \in Y$ *if and only if* $x\beta \in Y$ *for all* $x \in \operatorname{dom}\alpha\backslash Y$.

Proof. Assume that $\alpha\,\mathcal{R}\,\beta$. This means that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$ for some $\lambda, \mu \in \overline{PT}(X,Y)$. Based on the first condition of Lemma 1(2), we conclude that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. This implies $\ker\alpha = \ker\beta$ by utilizing the second condition.

Since $\operatorname{dom}\alpha \setminus Y = (\operatorname{dom}\alpha \cap \operatorname{dom}\beta) \setminus Y$, we can deduce that $x\alpha \in Y$ if and only if $x\beta \in Y$ according to the last condition.

Conversely, assume that the given conditions hold. Since $\ker\alpha = \ker\beta$, we can conclude that $\operatorname{dom}\alpha = \operatorname{dom}\beta$. As a result, α and β satisfy all the conditions in Lemma 1(2) (in the reverse direction). Consequently, $\alpha = \beta\lambda$ and $\beta = \alpha\mu$ for some $\lambda, \mu \in \overline{PT}(X, Y)$. Therefore, we have shown that $\alpha \,\mathcal{R}\, \beta$, as required. ■

Theorem 4. *Let $\alpha, \beta \in \overline{PT}(X, Y)$. Then $\alpha\mathcal{D}\beta$ if and only if $|Y\alpha| = |Y\beta|$, $|\operatorname{im}\alpha\backslash Y| = |\operatorname{im}\beta\backslash Y|$ and $|(\operatorname{im}\alpha \cap Y)\backslash Y\alpha| = |(\operatorname{im}\beta \cap Y)\backslash Y\beta|$.*

Proof. Assume that $\alpha\mathcal{D}\beta$. Then there exists some $\gamma \in \overline{PT}(X, Y)$ such that $\alpha \,\mathcal{L}\, \gamma \,\mathcal{R}\, \beta$. Since $\gamma \,\mathcal{R}\, \beta$, we can apply Theorem 3 to obtain the following expression

$$\gamma = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} A_i & B_j & C_k \\ x_i & y_j & z_k \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset; B_j, C_k \subseteq X\backslash Y; \{a_i\}, \{x_i\} \subseteq Y; \{b_j\} \subseteq Y\backslash\{a_i\}, \{y_j\} \subseteq Y\backslash\{x_i\}$. As $\alpha\mathcal{L}\gamma$, we can apply Theorem 2, which yields $\operatorname{im}\alpha = \operatorname{im}\gamma$ and $Y\alpha = Y\gamma$. Therefore, we can write

$$\alpha = \begin{pmatrix} L_i & M_j & N_k \\ a_i & b_j & c_k \end{pmatrix}$$

where $L_i \cap Y \neq \emptyset$ and $M_j, N_k \subseteq X\backslash Y$. Subsequently, we obtain

$$\begin{aligned} |Y\alpha| &= |\{a_i\}| = |\{x_i\}| = |Y\beta|, \\ |\operatorname{im}\alpha\backslash Y| &= |\{c_k\}| = |\{z_k\}| = |\operatorname{im}\beta\backslash Y|, \text{ and} \\ |(\operatorname{im}\alpha \cap Y)\backslash Y\alpha| &= |\{b_j\}| = |\{y_j\}| = |(\operatorname{im}\beta \cap Y)\backslash Y\beta|. \end{aligned}$$

Conversely, under the given conditions. We can express α and β as

$$\alpha = \begin{pmatrix} A_i & B_j & C_k \\ a_i & b_j & c_k \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} U_i & V_j & W_k \\ u_i & v_j & w_k \end{pmatrix}$$

where $A_i \cap Y \neq \emptyset \neq U_i \cap Y; B_j, C_k, V_j, W_k \subseteq X\backslash Y; \{a_i\}, \{u_i\} \subseteq Y; \{b_j\} \subseteq Y\backslash\{a_i\}, \{v_j\} \subseteq Y\backslash\{u_i\}$ and $\{c_k\}, \{w_k\} \subseteq X\backslash Y$. Since $|\{a_i\}| = |\{u_i\}|$, $|\{b_j\}| = |\{v_j\}|$ and $|\{c_k\}| = |\{w_k\}|$, we can define

$$\mu = \begin{pmatrix} U_i & V_j & W_k \\ a_i & b_j & c_k \end{pmatrix}$$

and μ belongs to $\overline{PT}(X, Y)$. Consequently, $\alpha \,\mathcal{L}\, \mu$ by Theorem 2. Additionally, we have $\ker\mu = \ker\beta$ and $x\mu \in Y$ if and only if $x\beta \in Y$ for all $x \in \operatorname{dom}\mu\backslash Y$. Therefore, $\mu \,\mathcal{R}\, \beta$ by Theorem 3. Hence, $\alpha\mathcal{D}\beta$. ■

Due to the fact that $\operatorname{im}\alpha \cap Y = Y\alpha \dot{\cup} [(\operatorname{im}\alpha \cap Y) \setminus Y\alpha]$ for all $\alpha \in \overline{PT}(X,Y)$, when Y is a finite subset of X, the following corollary is directly obtained from Theorem 4 through straightforward set theoretical arguments.

Corollary 5. *Let $\alpha, \beta \in \overline{PT}(X,Y)$. If Y is a finite subset of X, then $\alpha\mathcal{D}\beta$ if and only if $|\operatorname{im}\alpha| = |\operatorname{im}\beta|, |Y\alpha| = |Y\beta|$ and $|\operatorname{im}\alpha \cap Y| = |\operatorname{im}\beta \cap Y|$.*

Theorem 6. *Let $\alpha, \beta \in \overline{PT}(X,Y)$. Then $\alpha\,\mathcal{J}\,\beta$ if and only if $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$, $|Y\alpha| = |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| = |\operatorname{im}\beta \backslash Y|$.*

Proof. Assume that $\alpha\,\mathcal{J}\,\beta$. Then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in \overline{PT}(X,Y)$. According to Lemma 1(3), we obtain $|\operatorname{im}\alpha| = |\operatorname{im}\beta|, |Y\alpha| = |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| = |\operatorname{im}\beta \backslash Y|$.

Conversely, assume that $|\operatorname{im}\alpha| = |\operatorname{im}\beta|, |Y\alpha| = |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| = |\operatorname{im}\beta \backslash Y|$. Since $|\operatorname{im}\alpha| \leq |\operatorname{im}\beta|, |Y\alpha| \leq |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| \leq |\operatorname{im}\beta \backslash Y|$, we can conclude that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in \overline{PT}(X,Y)$ by Lemma 1(3). In a similar manner, using the inequalities, $|\operatorname{im}\beta| \leq |\operatorname{im}\alpha|, |Y\beta| \leq |Y\alpha|$ and $|\operatorname{im}\beta \backslash Y| \leq |\operatorname{im}\alpha \backslash Y|$, we get $\beta = \lambda'\alpha\mu'$ for some $\lambda', \mu' \in \overline{PT}(X,Y)$. Therefore, $\alpha\mathcal{J}\beta$, as required. ■

Recall that $D \subseteq J$ in any semigroup. In general, $D \subsetneq J$ in $\overline{PT}(X,Y)$. Then we provide the necessary and sufficient conditions for $\mathcal{D} = \mathcal{J}$ on $\overline{PT}(X,Y)$.

Theorem 7. *$\mathcal{D} = \mathcal{J}$ on $\overline{PT}(X,Y)$ if and only if X is a finite set or $X = Y$.*

Proof. Assume that X is infinite and $Y \subsetneq X$. Two cases arise.

Case 1. Y is a finite set. Choose $y \in Y$ and $z \in X \setminus Y$. Let $X\backslash(Y \cup \{z\}) = \{x_i : i \in I\}$ and define α and β in $\overline{PT}(X,Y)$ as follow

$$\alpha = \begin{pmatrix} z & x_i \\ y & x_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} z & x_i \\ z & x_i \end{pmatrix}.$$

Then $|\operatorname{im}\alpha| = |(X\backslash(Y \cup \{z\})) \cup \{y\}| = |X \setminus Y| = |\operatorname{im}\beta|, |Y\alpha| = 0 = |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| = |X\backslash(Y \cup \{z\})| = |\operatorname{im}\beta \backslash Y|$. According to Theorem 6, we find that $\alpha\,\mathcal{J}\,\beta$. Nevertheless, since $|\operatorname{im}\alpha \cap Y| = 1 \neq 0 = |\operatorname{im}\beta \cap Y|$, we can deduce from Corollary 5 that α and β are not $\mathcal{D}$-related. Hence, $D \neq J$.

Case 2. Y is an infinite set. Choose distinct a and b from Y, and c from $X \setminus Y$. Let $Y \setminus \{a, b\} = \{y_i : i \in I\}$ and define α and β in $\overline{PT}(X,Y)$ as follows

$$\alpha = \begin{pmatrix} \{a, b\} & c & y_i \\ a & b & y_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a & b & y_i \\ a & b & y_i \end{pmatrix}.$$

Then, we have $|\operatorname{im}\alpha| = |Y| = |\operatorname{im}\beta|, |Y\alpha| = |Y \setminus \{b\}| = |Y| = |Y\beta|$ and $|\operatorname{im}\alpha \backslash Y| = 0 = |\operatorname{im}\beta \backslash Y|$. According to Theorem 6, we find that $\alpha\,\mathcal{J}\,\beta$. However, as $|(\operatorname{im}\alpha \cap Y) \backslash Y\alpha| = 1 \neq 0 = |(\operatorname{im}\beta \cap Y) \backslash Y\beta|$, we can conclude from Theorem 4 that α and β are not $\mathcal{D}$-related. Hence, $D \neq J$.

Conversely, assume that either X is a finite set or $X = Y$. If X is a finite set, then $\overline{PT}(X,Y)$ becomes a periodic semigroup, and consequently, $\mathcal{D} = \mathcal{J}$ (see [12, Proposition 2.1.4] for details). On the other hand, if $X = Y$, then $\overline{PT}(X,Y) = P(X)$, and again, $\mathcal{D} = \mathcal{J}$ (see [12, p.63] for details). ∎

3. UNIT REGULARITY AND DIRECTLY FINITENESS ON $\overline{PT}(X,Y)$

This section aims to present a complete description of the unit regular elements in $\overline{PT}(X,Y)$. Moreover, we establish the necessary and sufficient conditions for $\overline{PT}(X,Y)$ to be unit regular and directly finite.

Lemma 8. *Let $\alpha \in \overline{PT}(X,Y)$. Then α is a unit in $\overline{PT}(X,Y)$ if and only if α is a bijection in $\overline{T}(X,Y)$ such that $(X \setminus Y)\alpha \subseteq X \setminus Y$.*

Proof. Assume that α is a unit in $\overline{PT}(X,Y)$. Then there exists $\beta \in \overline{PT}(X,Y)$ such that $\alpha\beta = id_X = \beta\alpha$. Since $\alpha\beta = id_X$ and id_X is injective, we conclude that α is also injective, and its domain is X. Moreover, since $\beta\alpha = id_X$ and id_X is surjective, we deduce that α is also surjective. Thus, α is a bijection in $\overline{T}(X,Y)$. For each $x \in X\backslash Y$, we have $x\alpha\beta = xid_X = x \in X\backslash Y$. If $x\alpha \in Y$, then, as $\beta \in \overline{PT}(X,Y)$, we have $(x\alpha)\beta = x \in Y$, which leads to a contradiction. Therefore, $x\alpha \in X\backslash Y$, and as a result, $(X\backslash Y)\alpha \subseteq X\backslash Y$.

Conversely, assume that $\alpha \in \overline{T}(X,Y)$ is a bijection such that $(X\backslash Y)\alpha \subseteq X\backslash Y$. Let $\beta = \alpha^{-1}$. Then $\beta \in T(X)$. To demonstrate that $\beta \in \overline{T}(X,Y)$, consider $y \in Y$. There exists $y' \in Y$ such that $y'\alpha = y$, and thus $y\beta = y\alpha^{-1} = y' \in Y$. So $Y\beta \subseteq Y$, which implies $\beta \in \overline{PT}(X,Y)$. Furthermore, we can observe that $\alpha\beta = id_X = \beta\alpha$. Hence, α is a unit in $\overline{PT}(X,Y)$. ∎

For each $\alpha \in P(X)$, let $\pi_\alpha = \{x\alpha^{-1} : x \in \operatorname{im}\alpha\}$. A subset C_α of $\operatorname{dom}\alpha$ is called a *cross-section of* π_α if each class in π_α contains exactly one element of C_α, i.e., $|C_\alpha \cap x\alpha^{-1}| = 1$ for all $x\alpha^{-1} \in \pi_\alpha$. It is clear that $|C_\alpha| = |\operatorname{im}\alpha|$. Especially, for $\alpha \in \overline{PT}(X,Y)$, we use the previous notion to define $\pi_\alpha(Y) = \{x\alpha^{-1} : x \in \operatorname{im}\alpha \cap Y\}$ and $\pi_\alpha(X\backslash Y) = \{x\alpha^{-1} : x\alpha^{-1} : x \in \operatorname{im}\alpha\backslash Y\}$. Specifically, $C_\alpha(Y) \subseteq Y\alpha^{-1}$ is a *cross-section of* $\pi_\alpha(Y)$ if each class in $\pi_\alpha(Y)$ contains exactly one element of $C_\alpha(Y)$. The *cross-section of* $\pi_\alpha(X \setminus Y)$ is defined similarly. Note that $\pi_\alpha = \pi_\alpha(Y)\dot{\cup}\pi_\alpha(X \setminus Y)$. Moreover, if C_α is a cross section of π_α, then $C_\alpha = (C_\alpha \cap Y\alpha^{-1})\dot{\cup}(C_\alpha \cap (X \setminus Y)\alpha^{-1})$ such that $C_\alpha \cap Y\alpha^{-1}$ and $C_\alpha \cap (X \setminus Y)\alpha^{-1}$ are cross sections of $\pi_\alpha(Y)$ and $\pi_\alpha(X \setminus Y)$, respectively.

Next, we aim to describe the properties of all unit regular elements in $\overline{PT}(X,Y)$. It is important to note that $\emptyset$ is always a unit regular element. The following theorem provides the characterization of such elements in the remaining cases.

Theorem 9. *Let $\emptyset \neq \alpha \in \overline{PT}(X, Y)$. Then α is unit regular if and only if the following conditions hold:*

(1) $\operatorname{im}\alpha \cap Y = Y\alpha$,

(2) *there exists a cross-section C_α of π_α such that $C_\alpha \cap Y$ is a cross-section of $\pi_\alpha(Y)$, $|X\backslash(Y \cup \operatorname{im}\alpha)| = |X\backslash(Y \cup C_\alpha)|$, and $|Y\backslash(\operatorname{im}\alpha \cap Y)| = |Y\backslash C_\alpha|$.*

Proof. Assume that α is unit regular. Then there exists a unit $\beta \in \overline{PT}(X, Y)$ such that $\alpha = \alpha\beta\alpha$. As α is regular, according to [17], we deduce that $\operatorname{im}\alpha \cap Y = Y\alpha$, leading to the conclusion that (1) is satisfied. Next, we consider $\operatorname{im}\alpha \cap Y = \{y_i\}$ and $\operatorname{im}\alpha\backslash Y = \{c_j\}$. Then, there exists $y'_i \in \operatorname{dom}\alpha$ such that $y'_i\alpha = y_i$ for all i, and there exists $x_j \in \operatorname{dom}\alpha$ such that $x_j\alpha = c_j$ for all j. Choose $C_\alpha = \{y_i\beta\} \cup \{c_j\beta\}$. To demonstrate that $C_\alpha \subseteq \operatorname{dom}\alpha$, we assume the contrary, i.e., $C_\alpha \nsubseteq \operatorname{dom}\alpha$. Then, either there exists $y_{i_0}\beta \in C_\alpha\backslash\operatorname{dom}\alpha$ or there exists $c_{j_0}\beta \in C_\alpha\backslash\operatorname{dom}\alpha$. In the first case, it implies that $y_i = y'_i\alpha = y'_i\alpha\beta\alpha = y_i\beta\alpha \notin \operatorname{im}\alpha$, which leads to a contradiction. Similarly, the second case also results in a contradiction. Therefore, $C_\alpha \subseteq \operatorname{dom}\alpha$. In order to show that C_α is a cross-section of $\pi_\alpha = \{y_i\alpha^{-1}\} \cup \{c_j\alpha^{-1}\}$, we first show that $C_\alpha \cap Y = \{y_i\beta\}$ is a cross-section of $\pi_\alpha(Y)$. It is clear, from $\alpha, \beta \in \overline{PT}(X, Y)$, that $(C_\alpha \cap Y)\alpha \subseteq Y$. To show $|(C_\alpha \cap Y) \cap y_i\alpha^{-1}| = 1$ for all i, we first suppose that there exists i_0 such that $(C_\alpha \cap Y) \cap\ y_{i_0}\alpha^{-1} = \emptyset$. Then, we have $y_{i_0} = y'_{i_0}\alpha = y'_{i_0}\alpha\beta\alpha = y_{i_0}\beta\alpha \neq y_{i_0}$ since $y_{i_0}\beta \notin y_{i_0}\alpha^{-1}$, leading to a contradiction. Next, we assume that $(y_{i_1}\beta)\alpha = (y_{i_2}\beta)\alpha$ for some $y_{i_1}\beta, y_{i_2}\beta \in C_\alpha \cap Y$. Then, we have $y_{i_1} = y'_{i_1}\alpha = y'_{i_1}\alpha\beta\alpha = y_{i_1}\beta\alpha = y_{i_2}\beta\alpha = y'_{i_2}\alpha\beta\alpha = y'_{i_2}\alpha = y_{i_2}$. Consequently, $|(C_\alpha \cap Y) \cap y_i\alpha^{-1}| = 1$ for all i. Similarly, we can deduce that $C_\alpha\backslash Y$ is a cross-section of $\pi_\alpha(X\backslash Y)$. Therefore, C_α serves as a cross-section of π_α. Consider, $\operatorname{dom}\beta = X = (Y\backslash\{y_i\}) \cup \{y_i\} \cup \{c_j\} \cup X\backslash(Y \cup \{c_j\})$ and $\operatorname{im}\beta = X = (Y\backslash\{y_i\beta\}) \cup \{y_i\beta\} \cup \{c_j\beta\} \cup X\backslash(Y \cup \{c_j\beta\})$. Since β is bijective, we get $\beta j_{X\backslash(Y\cup\{c_j\})} : X\backslash(Y \cup \{c_j\}) \to X\backslash(Y \cup \{c_j\beta\})$ and $\beta j_{Y\backslash\{y_i\}} : Y\backslash\{y_i\} \to Y\backslash\{y_i\beta\}$ are both bijective. Hence, $|X \setminus (Y \cup \operatorname{im}\alpha)| = |X\backslash(Y \cup \{c_j\})| = |X\backslash(Y \cup \{c_j\beta\})| = |X \setminus (Y \cup C_\alpha)|$ and $|Y \setminus (\operatorname{im}\alpha \cap Y)| = |Y\backslash\{y_i\}| = |Y\backslash\{y_i\beta\}| = |Y \setminus C_\alpha|$.

Conversely, we assume that the conditions hold. According to (1), we can represent α as follows

$$\alpha = \begin{pmatrix} A_i & C_j \\ y_i & c_j \end{pmatrix},$$

where $A_i \cap Y \neq \emptyset; C_j \subseteq X\backslash Y; \{y_i\} \subseteq Y$; and $\{c_j\} \subseteq X\backslash Y$. Since $C_\alpha \cap Y$ forms a cross section of $\pi_\alpha(Y) = \{A_i\}$, it follows that $|(C_\alpha \cap Y) \cap A_i| = 1$ for all i. Let $y'_i \in (C_\alpha \cap Y) \cap A_i$. Hence, $|Y\backslash\{y_i\}| = |Y\backslash\{y'_i\}|$. Consequently, there exists a bijection $\varphi : Y\backslash\{y_i\} \to Y\backslash\{y'_i\}$. Let $Y\backslash\{y_i\} = \{w_s\}$. Moreover, since $C_\alpha\backslash Y$ forms a cross section of $\pi_\alpha(X\backslash Y) = \{C_j\}$, it follows that $|(C_\alpha\backslash Y) \cap C_j| = 1$ for all j. Let $c'_j \in (C_\alpha\backslash Y) \cap C_j$. Hence, $|X\backslash(Y \cup \{c_j\})| = |X\backslash(Y \cup \{c'_j\})|$. Consequently, there exists a bijection $\sigma : X\backslash(Y \cup \{c_j\}) \to X\backslash(Y \cup \{c'_j\})$. Let $X\backslash(Y \cup \{c_j\}) = \{z_t\}$, and define $\beta : X \to X$ as follows

$$\beta = \begin{pmatrix} y_i & w_s & c_j & z_t \\ y'_i & w_s\varphi & c'_j & z_t\sigma \end{pmatrix}.$$

Thus, β is a unit of $\overline{PT}(X,Y)$, and we have $\alpha = \alpha\beta\alpha$. Therefore, α is unit regular. ■

It is evident that if $\operatorname{im}\alpha \cap Y = Y\alpha$, then there exists a cross-section C_α of π_α such that $C_\alpha \cap Y$ is a cross-section of $\pi_\alpha(Y)$. In fact $|C_\alpha \setminus Y| + |C_\alpha \cap Y| = |C_\alpha| = |\operatorname{im}\alpha| = |\operatorname{im}\alpha \setminus Y| + |\operatorname{im}\alpha \cap Y|$ and $|C_\alpha \cap Y| = |Y\alpha|$. Particularly, in the case where X is a finite set, such a cross-section satisfies condition (2) of Theorem 9. Thus, we obtain the following corollary.

Corollary 10. *Let X be a finite set and $\alpha \in \overline{PT}(X,Y)$. Then α is unit regular if and only if α is regular.*

Lemma 8 shows that the units in $\overline{T}(X,Y)$ and $\overline{PT}(X,Y)$ coincide. As a result, an element in $\overline{T}(X,Y)$ is unit regular if and only if it possesses unit regularity in $\overline{PT}(X,Y)$. Thus, Theorem 9 offers us the characterization of unit regular elements in $\overline{T}(X,Y)$ when considering elements with their domain being X. Moreover, since $\overline{PT}(X,X) = P(X)$, we can directly deduce the unit regularity on $P(X)$ as follows.

Corollary 11. *Let $\alpha \in P(X)$. Then α is unit regular if and only if there exists a cross-section C_α such that $|X \setminus \operatorname{im}\alpha| = |X \setminus C_\alpha|$.*

Recall from [20, Theorem 3.1] that $P(X)$ is a unit regular semigroup if and only if X is finite. As for the semigroup $\overline{PT}(X,Y)$, we can observe that it is almost never unit regular, as demonstrated in the following corollary.

Corollary 12. *$\overline{PT}(X,Y)$ is unit regular if and only if X is finite and $X = Y$.*

Proof. If X is finite and $X = Y$, then $\overline{PT}(X,Y) = P(X)$ is unit regular by the previous note. In contrast, if $\overline{PT}(X,Y)$ is unit regular, it is also regular, which leads to the conclusion that $X = Y$, as mentioned in [17, Corollary 3.1.5]. Consequently, it follows that $PT(X,Y) = P(X)$, implying the finiteness of X, based on the earlier note. ■

In 1980, Alarcao [1] provided a characterization for the conditions under which a semigroup S with identity is unit regular and directly finite, respectively.

Proposition 13 [1]**.** *Let S be a semigroup with identity 1.*

(1) *S is unit regular if and only if it is factorizable.*

(2) *S is directly finite if and only if $H_1 = D_1$.*

By combining Proposition 13 and Theorem 12, we derive the following corollary.

Corollary 14. *The following statements are equivalent.*

(1) *$\overline{PT}(X,Y)$ is unit regular;*

(2) *$\overline{PT}(X,Y)$ is factorizable;*

(3) *X is finite and $X = Y$.*

In [1, Proposition 5], the author demonstrated that $P(X)$ is unit regular if and only if it is directly finite. However, we will show that this property does not hold for $\overline{PT}(X,Y)$. Nonetheless, the finiteness of X remains a sufficient condition, as illustrated in the following example.

Theorem 15. *$\overline{PT}(X,Y)$ is directly finite if and only if X is a finite set.*

Proof. Assume that X is a finite set. Let $\alpha, \beta \in \overline{PT}(X,Y)$ be such that $\alpha\beta = id_X$. Then α and β are bijective, and therefore they are group elements. Consequently, $\alpha = \beta^{-1}$, and thus $\beta\alpha = id_X$.

Conversely, we assume that X is an infinite set. In order to demonstrate that $\overline{PT}(X,Y)$ is not directly finite, we will consider two cases.

Case 1. Y is finite. Let $Y = \{y_1, y_2, \ldots, y_k\}$. Thus, $X\backslash Y$ is infinite. Choose $b \in X\backslash Y$. Then $|X\backslash(Y \cup \{b\})| = |X\backslash Y|$, and there exists a bijection $\sigma : X\backslash(Y \cup \{b\}) \to X\backslash Y$. Let $X\backslash(Y \cup \{b\}) = \{x_j : j \in J\}$. Now, fix $j_0 \in J$ and let $J' = J\backslash\{j_0\}$. Define β as follows

$$\beta = \begin{pmatrix} y_1 & \cdots & y_k & \{x_{j_0}, b\} & x_{j'} \\ y_1 & \cdots & y_k & x_{j_0}\sigma & x_{j'}\sigma \end{pmatrix}.$$

Then, $\beta \in \overline{PT}(X,Y)$ and $\operatorname{im}\beta = X$. So $|Y\beta| = |Y| = |Y id_X|, |\operatorname{im}\beta\backslash Y| = |X\backslash Y| = |X id_X\backslash Y| = |\operatorname{im} id_X\backslash Y|$, and $|(\operatorname{im}\beta \cap Y)\backslash Y\beta| = |Y\backslash Y| = |(X id_X \cap Y)\backslash Y id_X| = |(\operatorname{im} id_X \cap Y)\backslash Y id_X|$. By Theorem 4, we obtain $\beta \in D_{id_X}$. However, $\beta \notin H_{id_X}$ since $\ker\beta \neq \ker id_X$. Hence, $D_{id_X} \neq H_{id_X}$. By Proposition 13(2), we get $\overline{PT}(X,Y)$ is not directly finite.

Case 2. Y is infinite. We choose $a \in Y$. Then $|Y\backslash\{a\}| = |Y|$, so there exists a bijection $\varphi : Y\backslash\{a\} \to Y$. Let $Y\backslash\{a\} = \{y_i : i \in I\}$. Now, fix $i_0 \in I$, and let $I' = I\backslash\{i_0\}$. Let $X \setminus Y = \{x_j : j \in J\}$ and define $\alpha \in \overline{PT}(X,Y)$ as follows

$$\alpha = \begin{pmatrix} \{y_{i_0}, a\} & y_{i'} & x_j \\ y_{i_0}\varphi & y_{i'}\varphi & x_j \end{pmatrix}.$$

Thus, α is surjective, and hence $|Y\alpha| = |Y| = |Yid_X|, |\mathrm{im}\,\alpha\backslash Y| = |X\backslash Y| = |Xid_X\backslash Y| = |\mathrm{im}\,id_X\backslash Y|$, and $|(\mathrm{im}\,\alpha \cap Y\backslash Y\alpha)| = |Y\backslash Y| = |(Xid_X \cap Y)\backslash Yid_X| = |(\mathrm{im}\,id_X \cap Y)\backslash Yid_X|$. By Theorem 4, deduce that $\alpha \in D_{id_X}$. However, we see that $\ker\alpha \neq \ker id_X$, so $\alpha \notin H_{id_X}$. Thus $D_{id_X} \neq H_{id_X}$. According to Proposition 13(2), we conclude that $\overline{PT}(X,Y)$ is not directly finite. ■

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