Discussiones Mathematicae
General Algebra and Applications 44 (2024) 383–412
https://doi.org/10.7151/dmgaa.1458

SOME REMARKS ON THE DOMINATING SETS OF THE ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE RING

Subramanian Visweswaran

Retired Faculty, Department of Mathematics
Saurashtra University
Rajkot, India 360005

e-mail: s_visweswaran2006@yahoo.co.in

Abstract

The rings considered in this article are commutative with identity which admit at least one nonzero annihilating ideal. Let R be a ring. Let $\mathbb{A}(R)$ denote the set of all annihilating ideals of R and let us denote $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. Recall that the annihilating-ideal graph of R, denoted by $\mathbb{AG}(R)$ is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I and J are adjacent if and only if $IJ = (0)$. The aim of this article is to generalize some of the known results on the domination number of $\mathbb{AG}(R)$. We also determine the domination number of two spanning supergraphs of $\mathbb{AG}(R)$ in the case of a reduced ring R.

Keywords: annihilating-ideal graph, minimal prime ideal, maximal N-prime of (0), reduced ring, domination number of a graph.

2020 Mathematics Subject Classification: Primary: 13A15; Secondary: 05C25.

1. Introduction

The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. An ideal I of R is said to be an *annihilating ideal* of R if $Ir = (0)$ for some $r \in R\backslash\{0\}$. We denote the set of all annihilating ideals of R by $\mathbb{A}(R)$ and $\mathbb{A}(R)\backslash\{(0)\}$ by $\mathbb{A}(R)^*$. The graphs considered in this article are undirected and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Recall that the *annihilating-ideal graph* of R denoted by $\mathbb{AG}(R)$ is an undirected graph with $V(\mathbb{AG}(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent if and only if $IJ = (0)$ [5]. Several graph parameters of $\mathbb{AG}(R)$ have been determined in [5, 6]. In the literature, the graph $\mathbb{AG}(R)$ has been studied by several researchers.

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a *dominating set* of G if each $u \in V\backslash S$ has at least one $v \in S$ such that u and v are adjacent in G [3, Definition 10.2.1]. A γ*-set* of G is a minimum dominating set of G, that is, a dominating set of G whose cardinality is minimum [3, Definition 10.2.2]. The *domination number* of G is the cardinality of a minimum dominating set of G; it is denoted by $\gamma(G)$ [3, Definition 10.2.3]. A *total dominating set* of G is a set $S \subseteq V$ such that each $u \in V$ is adjacent to at least one vertex v in S. The total dominating set S of G is called a *minimum total dominating set* of G if its cardinality is minimum. The *total domination number* of G is the cardinaity of a minimum total dominating set of G; it is denoted by $\gamma_t(G)$ [14]. It is clear that $\gamma(G) \leq \gamma_t(G)$. As the graphs G considered here are simple, it follows that $\gamma_t(G)) \geq 2$.

Unless otherwise specified, the rings considered in this article are not integral domains. Let R be a ring. Several interesting results have been proved on the dominating sets and the domination number (respectively, the total domination number) of $\mathbb{AG}(R)$ by Nikandish *et al.* in [14, 15]. The aim of this article is to generalize some of the results that are proved in [14] and to determine the domination number of two spanning supergraphs of $\mathbb{AG}(R)$.

Let R be a ring. We denote the set of all prime ideals, the set of all maximal ideals, and the set of all minimal prime ideals of R by $Spec(R), Max(R)$, and $Min(R)$, respectively. We denote the set of all zero divisors of R by $Z(R)$ and $Z(R)\backslash\{0\}$ by $Z(R)^*$. We denote the group of units of R by $U(R)$ and the set of all non-units of R by $NU(R)$. We denote the nilradical of R by $nil(R)$. Recall that R is said to be *reduced* if $nil(R) = (0)$. If R is reduced, then since each prime ideal of R contains at least one member from Min(R) by [12, Theorem 10], it follows from [2, Proposition 1.8] that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. Let I be a proper ideal of R. Recall that $\mathfrak{p} \in Spec(R)$ is said to be a *maximal N-prime* of I if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z_R(\frac{R}{I}) = \{r \in R \mid rx \in I \text{ for some } x \in R\backslash I\}$[11]. Hence, $\mathfrak{p} \in Spec(R)$ is a maximal N-prime of (0) if $\mathfrak{p}$ is maximal with respect to the property of being contained in $Z(R)$. For convenience, we denote the set of all maximal N-primes of (0) in R by $MNP(R)$. Note that $S = R\backslash Z(R)$ is a multiplicatively closed subset (m.c. subset) of R. If I is an ideal of R with $I \cap S = \emptyset$, then it follows from Zorn's lemma and [12, Theorem 1] that there exists $\mathfrak{p} \in MNP(R)$ such that $I \subseteq \mathfrak{p}$. For any $x \in Z(R)$, as $Rx \cap S = \emptyset$, there exists $\mathfrak{p} \in MNP(R)$ such that $x \in \mathfrak{p}$. Therefore, if $MNP(R) = \{\mathfrak{p}_\alpha\}_{\alpha \in \Lambda}$, then it follows that $Z(R) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$. We denote the cardinalty of a set A by $|A|$. Note that $|MNP(R)| = 1$ if and only if $Z(R)$ is an ideal of R. The ring R is said to be *quasi-local* if $|Max(R)| = 1$. A Noetherian quasi-local ring is referred to as a *local* ring. Let I be an ideal of R. Recall that the *annihilator of* I, denoted by $Ann_R(I)$ or by $Ann(I)$ is defined by $Ann_R(I) = \{r \in R \mid Ir = (0)\}$. Recall that I is said to be *essential* if $I \cap J \neq (0)$ for each nonzero ideal J of R. We denote by $R[X]$ (respectively, $R[[X]]$), the polynomial (respectively, power series) ring in one variable X over R. The Krull dimension of R is simply referred to as the dimension of R and is denoted by $dimR$. We denote the set $R\backslash\{0\}$ by R^*. Let $Z(R)^* \neq \emptyset$. We denote the zero-divisor graph of R by $\Gamma(R)$. For any $n \in \mathbb{N}$ with $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.

Let us now describe briefly the results that are proved in this article. Let R be a reduced ring with $|Min(R)| < \infty$. If $\gamma(\mathbb{AG}(R)) > 1$, then it is known that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |Min(R)|$ [14, Theorem 2.4]. If $\gamma(\Gamma(R)) > 1$, then it is known that $\gamma(\Gamma(R)) = |Min(R)|$ [14, Theorem 2.5]. In Theorem 2.15 (respectively, Theorem 2.17) we generalize [14, Theorem 2.4] (respectively, [14, Theorem 2.5]) to a larger class of reduced rings. Towards that goal, in Section 2, we first introduce the concept of reduced rings satisfying $(*)$ (respectively, $(**)$). We denote $Min(R) \cap \mathbb{A}(R)$ by $\mathcal{A}(R)$. We say that R *satisfies* $(*)$ if $\mathcal{A}(R) \neq \emptyset$ and $\bigcap_{\mathfrak{p}\in\mathcal{A}(R)} \mathfrak{p} = (0)$. We say that R *satisfies* $(**)$ if R satisfies $(*)$ and $\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{q}\in\mathcal{A}(R)\backslash\{\mathfrak{p}\}} \mathfrak{q}$ for each $\mathfrak{p} \in \mathcal{A}(R)$. We first provide some examples of reduced rings satisfying $(*)$ (respectively $(**)$) (see Examples 2.1, 2.2, 2.9, and 2.10). In Example 2.11, a reduced ring R is provided such that $\mathfrak{p} \notin \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. If R satisfies $(*)$ and if $\gamma(\mathbb{AG}(R)) \geq 2$, then it is proved in Theorem 2.15 that $\gamma(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(R))$. It is noted in Remark 2.16 that Theorem 2.15 generalizes [14, Theorem 2.4]. If R satisfies $(**)$ and if $\gamma(\Gamma(R)) \geq 2$, then it is shown in Theorem 2.17 that $\gamma(\Gamma(R)) = |\mathcal{A}(R)| = \gamma_t(\Gamma(R))$. If R satisfies $(**)$ and if $\gamma(\mathbb{AG}(R)) \geq 2$, then it is proved in Corollary 2.19 that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma(\Gamma(R)) = \gamma_t(\Gamma(R))$ and is noted in Remark 2.20 that Corollary 2.19 generalizes [14, Corollary 2.6].

If R is an Artinian ring such that $R \ncong F_1 \times F_2$, where F_1 and F_2 are fields, then it is known that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |Min(R)|$ [14, Theorem 2.8]. Our next focus in Section 2 is to obtain a generalization of this result to any zero-dimensional ring. Towards that goal, it is first proved in Theorem 2.24 that for a ring R with $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$ and $\gamma(\mathbb{AG}(R)) > 1$, $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |MNP(R)|$. Let R be a ring such that $dimR = 0$. It is clear that $Spec(R) = Max(R) = Min(R)$. As any minimal prime ideal of a ring is contained in its set of zero divisors (see [12, Theorem 84]), it follows that $Spec(R) = Min(R) = MNP(R)$. If $\gamma(\mathbb{AG}(R)) < \infty$, then it is shown in Proposition 2.29 that $|MNP(R)| < \infty$. It is proved in Proposition 2.30(1) that $\gamma(\mathbb{AG}(R)) = 1$ if and only if either $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)$ or $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. If $|MNP(R)| < \infty$, $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$ and $R \ncong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$, then it is shown in Proposition 2.31(1) that $\gamma(\mathbb{AG}(R)) = |Min(R)|$

and if $|Min(R)| \geq 2$, then $\gamma(\mathbb{AG}(R)) = |Min(R)| = \gamma_t(\mathbb{AG}(R))$. It is noted that Proposition 2.31(1) generalizes [14, Theorem 2.8]. Let $n \in \mathbb{N}$ be such that $n \geq 3$. In Proposition 2.32, von Neumann regular rings R are characterized such that $\gamma(\mathbb{AG}(R)) = n$. Some examples are given which illustrate the results that are proved in this section (see Examples 2.25 and 2.28).

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In Section 3 of this article, we discuss some results on the domination number of two spanning supergraphs of $\mathbb{AG}(R)$. Recall that the *strongly annihilating-ideal graph* of R, denoted by $SAG(R)$ is an undirected graph with $V(SAG(R)) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent if and only if $I \cap Ann_R(J) \neq (0)$ and $J \cap Ann_R(I) \neq (0)$[16]. If distinct $I, J \in \mathbb{A}(R)^*$ are adjacent in $\mathbb{AG}(R)$, then it is known that I and J are adjacent in $SAG(R)$ [16, Lemma 2.1(2)]. As $V(\mathbb{AG}(R)) = V(SAG(R)) = \mathbb{A}(R)^*$, it follows that $SAG(R)$ is a spanning supergraph of $\mathbb{AG}(R)$.

Several connections between the graph-theoretic properties of $SAG(R)$ and the algebraic properties of R have been studied and also the conditions under which $SAG(R)$ is identical to $\mathbb{AG}(R)$ have been investigated in [16]. In Section 3, we first discuss some results on the domination number of $SAG(R)$, where R is a reduced ring such that $SAG(R) \neq \mathbb{AG}(R)$. For such a ring R, it is known that $|Min(R)| \geq 3$ by (1) $\Rightarrow$ (2) of [16, Theorem 4.1] and it is proved in Proposition 3.3 that $\gamma(SAG(R)) > 1$ and in Theorem 3.4, it is shown that $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$. If the ring R (reduced or not) admits a non-trivial idempotent element, then it is proved in Proposition 3.6 that $\gamma(SAG(R)) \leq 2$. If R is not reduced and admits an ideal I such that $Ann_R(I) = I$, then it is shown in Lemma 3.7 that $\{I\}$ is a dominating set of $SAG(R)$. Some examples are given to illustrate the results proved in this section (see Examples 3.5, 3.8, and 3.9).

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. In Section 3, we also discuss some results on the domination number of another spanning supergraph of $\mathbb{AG}(R)$. Recall that the *sum-annihilating essential ideal graph* of R, denoted by $\mathcal{AE}_R$ is an undirected graph with $V(\mathcal{AE}_R) = \mathbb{A}(R)^*$ and distinct vertices I and J are adjacent if and only if $Ann_R(I) + Ann_R(J)$ is an essential ideal of R [1]. The article [1] contains many interesting results on $\mathcal{AE}_R$. It is noted in [1, Observation 4(2)] that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathcal{AE}_R$. Among other results, the authors of [1] have established sharp bounds on the domination number of $\mathcal{AE}_R$ in [1, Section 4]. If R is a non-reduced ring, then $\gamma(\mathcal{AE}_R) = 1$ [1, Corollary 8]. If R is a Noetherian reduced ring, then $\gamma(\mathcal{AE}_R) = |Min(R)|$ [1, Proposition 2]. For any reduced ring R, we verify in Proposition 3.11 that $\mathbb{AG}(R) = \mathcal{AE}_R$ (a fact which is not noted in [1]) and results on $\gamma(\mathbb{AG}(R))$ are already contained in Section 2 of this article.

2. Some results on the domination number of $\mathbb{AG}(R)$

Unless otherwise specified, the rings considered in this article are not integral domains. Let R be a reduced ring with $|Min(R)| < \infty$. If $\gamma(\mathbb{AG}(R)) > 1$, then it is known that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |Min(R)|$ [14, Theorem 2.4]. Firstly, the aim of this section is to generalize this result. Indeed, we prove this result for reduced rings R satisfying the property $(*)$ (which we define now) and deduce [14, Theorem 2.4]. For any reduced ring R, we denote $Min(R) \cap \mathbb{A}(R) = \{\mathfrak{p} \in Min(R) \mid \mathfrak{p} \in \mathbb{A}(R)\}$ by $\mathcal{A}(R)$. We say that R *satisfies* $(*)$ if the following conditions are satisfied: (1) $\mathcal{A}(R) \neq \emptyset$ and (2) $\bigcap_{\mathfrak{p}\in\mathcal{A}(R)} \mathfrak{p} = (0)$. If a reduced ring R satisfies $(*)$ and $\gamma(\mathbb{AG}(R)) \geq 2$, then it is proved in Theorem 2.15 that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\mathcal{A}(R)|$.

Let R be a reduced ring such that $|Min(R)| < \infty$. If $\gamma(\Gamma(R)) > 1$, then it is known that $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |Min(R)|$ [14, Theorem 2.5]. Secondly, our aim is to generalize this result. We prove this result for reduced rings R satisfying the property $(**)$ (which we introduce now) and deduce [14, Theorem 2.5]. We say that a reduced ring R *satisfies* $(**)$ if the following conditions hold: (1) R satisfies $(*)$ and (2) For each $\mathfrak{p} \in \mathcal{A}(R)$, $\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{q}\in\mathcal{A}(R)\backslash\{\mathfrak{p}\}} \mathfrak{q}$. Let R be a reduced ring such that R satisfies $(**)$. If $\gamma(\Gamma(R)) > 1$, then it is shown in Theorem 2.17 that $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |\mathcal{A}(R)|$.

First, we provide some examples to illustrate the properties $(*)$ and $(**)$.

Example 2.1. Let R be a reduced ring such that $Min(R)$ is finite. Then $\mathcal{A}(R) = Min(R)$ and R satisfies $(**)$.

Proof. This is well-known. For the sake of completeness, we provide an argument. Let $|Min(R)| = n$. Since $\bigcap_{\mathfrak{p}\in Min(R)} \mathfrak{p} = (0)$ and R is not an integral domain, it follows that $n \geq 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, n\}\}$. Let $i \in \{1, 2, \ldots, n\}$. As distinct minimal prime ideals of a ring are not comparable under inclusion, it follows from [2, Proposition 1.11(ii)] that $\mathfrak{p}_i \not\supseteq \bigcap_{j\in A_i} \mathfrak{p}_j$, where $A_i = \{1, 2, \ldots, n\}\backslash\{i\}$. Let $x_i \in (\bigcap_{j\in A_i} \mathfrak{p}_j)\backslash\mathfrak{p}_i$. Then it is clear that $x_i \neq 0$ and $\mathfrak{p}_i x_i = (0)$. Hence, $\mathfrak{p}_i \in \mathbb{A}(R)$ for each $i \in \{1, 2, \ldots, n\}$. This shows that $\mathcal{A}(R) = Min(R)$ and R satisfies $(*)$. For each $i \in \{1, 2, \ldots, n\}$, it follows from [2, Proposition 1.11(i)] that $\mathfrak{p}_i \not\subseteq \bigcup_{j\in A_i} \mathfrak{p}_j$. This shows that R satisfies $(**)$. ∎

In Example 2.2, we provide a reduced ring R such that $Min(R)$ is infinite and R satisfies $(*)$ but R does not satisfy $(**)$. The reduced ring R given in Example 2.2 is due to Gilmer and Heinzer [9, Example, page 16].

Example 2.2. Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates. Let $K[[X_1, \ldots, X_n]]$ be the power series ring in $X_1, \ldots, X_n$ over a field K and let $D = \bigcup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$. Let I be the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Then R is a quasi-local reduced ring, $\mathcal{A}(R) = Min(R)$, $Min(R)$ is infinite, and R satisfies $(*)$ but it does not satisfy $(**)$.

Proof. For each $i \in \mathbb{N}$, it is convenient to denote $X_i + I$ by x_i. Note that R is a quasi-local reduced ring with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal and $Min(R) = \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$ (see [9, Example, page 16]). Thus $Min(R)$ is infinite and indeed, $|Min(R)| = |\mathbb{N}|$. Let $i \in \mathbb{N}$, Note that $x_i \neq 0 + I$ and $\mathfrak{p}_i x_i = (0 + I)$. Therefore, $\mathfrak{p}_i \in \mathbb{A}(R)$. This shows that $Min(R)$ is infinite and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in Min(R)$. Hence, $\mathcal{A}(R) = Min(R)$. Since for any reduced ring T, $\bigcap_{\mathfrak{p} \in Min(T)} \mathfrak{p} = (0)$, it follows that R satisfies $(*)$. Let $i \in \mathbb{N}$. Let $r \in \mathfrak{p}_i$. Since $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$, it follows that there exist $j_1, \ldots, j_n \in \mathbb{N}\backslash\{i\}$ with $j_1 < \cdots < j_n$ and $r_{j_1}, \ldots, r_{j_n} \in R$ (where we can assume $j_n > i$) such that $r = \sum_{k=1}^{n} r_{j_k} x_{j_k}$. It is clear that $r \in \mathfrak{p}_j$ for any $j \in \mathbb{N}$ with $j > j_n$. This shows that $\mathfrak{p}_i \subseteq \bigcup_{j \in \mathbb{N}\backslash\{i\}} \mathfrak{p}_j$. Hence, R does not satisfy $(**)$. ■

Let R be the reduced ring mentioned in Example 2.2. We verify in Example 2.9 that $R[X]$ (respectively, $R[[X]]$) satisfies $(*)$, $R[X]$ does not satisfy $(**)$ but $R[[X]]$ satisfies $(**)$. First, we state and prove some lemmas which are useful in the verification of Example 2.9.

Lemma 2.3. *Let R be a reduced ring. Let $\mathfrak{p} \in Spec(R)$. If $\mathfrak{p} \in \mathbb{A}(R)$, then $\mathfrak{p} \in Min(R)$.*

Proof. By hypothesis, R is a reduced ring. Let $\mathfrak{p} \in Spec(R)$. Assume that $\mathfrak{p} \in \mathbb{A}(R)$. Then there exists $r \in R\backslash\{0\}$ such that $\mathfrak{p}r = (0)$. Since $\bigcap_{\mathfrak{q} \in Min(R)} \mathfrak{q} = (0)$, it follows that $r \notin \mathfrak{q}$ for some $\mathfrak{q} \in Min(R)$. From $\mathfrak{p}r = (0)$, we get that $\mathfrak{p} \subseteq \mathfrak{q}$ and so, $\mathfrak{p} = \mathfrak{q} \in Min(R)$. ■

If A is a subring of a ring B, then we assume that A contains the identity element of B. Let A be a subring of a ring B. We denote the collection $\{C \mid \text{C is a subring of } B \text{ with } A \subseteq C\}$ by $[A, B]$.

Lemma 2.4. *Let R be a reduced ring. Let $T \in [R, R[X]]$. Then $\mathcal{A}(T) = \{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$.*

Proof. By hypothesis, R is a reduced ring. Let $T \in [R, R[X]]$. Since $R[X]$ is a reduced ring, we get that T is reduced. Note that T is not an integral domain. Let $\mathfrak{p} \in \mathcal{A}(R)$. Then $\mathfrak{p}r = (0)$ for some $r \in R\backslash\{0\}$. This implies that $(\mathfrak{p}[X] \cap T)r = (0)$. Since $\mathfrak{p}[X] \in Spec(R[X])$, it follows that $\mathfrak{p}[X] \cap T \in Spec(T)$. Therefore, $\mathfrak{p}[X] \cap T \in \mathbb{A}(T) \cap Spec(T)$. Hence, $\mathfrak{p}[X] \cap T \in Min(T)$ by Lemma 2.3. Thus $\mathfrak{p}[X] \cap T \in Min(T) \cap \mathbb{A}(T) = \mathcal{A}(T)$. This shows that $\{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\} \subseteq \mathcal{A}(T)$. Let $\mathfrak{P} \in \mathcal{A}(T) = Min(T) \cap \mathbb{A}(T)$. As any element of T belongs to $R[X]$, it follows that $\mathfrak{P}f(X) = (0)$ for some $f(X) \in R[X]\backslash\{0\}$. Let $i \geq 0$ be least with the property that the coefficient of X^i in $f(X)$, say $r_i \neq 0$. Let us denote $\mathfrak{P} \cap R$ by $\mathfrak{p}$. Then $\mathfrak{p} \in Spec(R)$ and $(\mathfrak{p}[X] \cap T)r_i = (0)$. It is clear that $r_i \notin \mathfrak{p}$ and so, $r_i \notin \mathfrak{P}$. Therefore, from $(\mathfrak{p}[X] \cap T)r_i = (0)$, it follows that $\mathfrak{p}[X] \cap T \subseteq \mathfrak{P}$. Since $\mathcal{A}(T) \subseteq Min(T)$, it follows that $\mathfrak{P} \in Min(T)$ and so, $\mathfrak{P} = \mathfrak{p}[X] \cap T$. As $\mathfrak{p} \in Spec(R) \cap \mathbb{A}(R)$, we get that $\mathfrak{p} \in Min(R)$ by Lemma 2.3 and hence, $\mathfrak{p} \in \mathcal{A}(R)$. Thus $\mathfrak{P} = \mathfrak{p}[X] \cap T$ for some $\mathfrak{p} \in \mathcal{A}(R)$. This proves that $\mathcal{A}(T) \subseteq \{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$ and so, $\mathcal{A}(T) = \{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$. ∎

Lemma 2.5. *Let R be a reduced ring. Let $T \in [R, R[X]]$. Then R satisfies $(*)$ if and only if T satisfies $(*)$.*

Proof. Let $T \in [R, R[X]]$. We know from Lemma 2.4 that $\mathcal{A}(T) = \{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$. Hence, $\mathcal{A}(R) \neq \emptyset$ if and only if $\mathcal{A}(T) \neq \emptyset$. For any given non-empty family $\mathcal{F} = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\} \subseteq Spec(R)$, it is clear that $\bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha = (0)$ if and only if $\bigcap_{\alpha\in\Lambda}(\mathfrak{p}_\alpha[X] \cap T) = (0)$. From these arguments, it follows that R satisfies $(*)$ if and only if T satisfies $(*)$. ∎

If $\mathfrak{p} \in Spec(R)$ for a ring R, then it is known that $\mathfrak{p}[[X]] \in Spec(R[[X]])$.

Lemma 2.6. *Let R be a reduced ring. Then for any $T \in [R, R[[X]]]$, $\mathcal{A}(T) = \{\mathfrak{p}[[X]] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$.*

Proof. By hypothesis, R is a reduced ring. Hence, $R[[X]]$ is a reduced ring. Let $T \in [R, R[[X]]]$. So, T is a reduced ring and it is clear that T is not an integral domain. This lemma can be proved using arguments similar to those that are used in the proof of Lemma 2.4. Hence, we omit its proof. ∎

Lemma 2.7. *Let R be a reduced ring. Let $T \in [R, R[[X]]]$. Then R satisfies $(*)$ if and only if T satisfies $(*)$.*

Proof. Let $T \in [R, R[[X]]]$. For any non-empty family $\mathcal{F} = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\} \subseteq Spec(R)$, it is clear that $\bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha = (0)$ if and only if $\bigcap_{\alpha\in\Lambda}(\mathfrak{p}_\alpha[[X]] \cap T) = (0)$. With the help of Lemma 2.6, this lemma can be proved using arguments similar to those that are used in the proof of Lemma 2.5. ∎

Lemma 2.8. *Let R be a reduced ring. Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). If R satisfies $(**)$, then so does T.*

Proof. By hypothesis, R is a reduced ring. Assume that R satisfies $(**)$. Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). We know from Lemma 2.5 (respectively, Lemma 2.7) that T satisfies $(*)$. We know from Lemma 2.4 (respectively, Lemma 2.6) that $\mathcal{A}(T) = \{\mathfrak{p}[X] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$ in the case $T \in [R, R[X]]$ (respectively, $\mathcal{A}(T) = \{\mathfrak{p}[[X]] \cap T \mid \mathfrak{p} \in \mathcal{A}(R)\}$ in the case $T \in [R, R[[X]]]$). Let $\mathfrak{P} \in \mathcal{A}(T)$. Then $\mathfrak{P} = \mathfrak{p}[X] \cap T$ in the case $T \in [R, R[X]]$ (respectively, $\mathfrak{P} = \mathfrak{p}[[X]] \cap T$ in the case $T \in [R, R[[X]]]$) for some $\mathfrak{p} \in \mathcal{A}(R)$. Since R satisfies $(**)$, there exists $p \in \mathfrak{p}$ such that $p \notin \bigcup_{\mathfrak{q}\in\mathcal{A}(R)\setminus\{\mathfrak{p}\}} \mathfrak{q}$. It is clear that $p \in \mathfrak{P}$. Let $\mathfrak{P}' \in \mathcal{A}(T)$ with $\mathfrak{P}' \neq \mathfrak{P}$. Then $\mathfrak{P}' = \mathfrak{p}'[X] \cap T$ in the case $T \in [R, R[X]]$ (respectively, $\mathfrak{P}' = \mathfrak{p}'[[X]] \cap T$ in the case $T \in [R, R[[X]]]$) for some $\mathfrak{p}' \in \mathcal{A}(R)\setminus\{\mathfrak{p}\}$. By the choice of p, it is clear that $p \notin \mathfrak{p}'$ and so, $p \notin \mathfrak{P}'$. This proves that T satisfies $(**)$. ∎

Example 2.9. Let R be the reduced ring which is considered in Example 2.2. Then the following statements hold:

(1) T satisfies $(*)$ but T does not satisfy $(**)$ for each $T \in [R, R[X]]$.

(2) $R[[X]]$ satisfies $(**)$.

Proof. In the notation of the proof of Example 2.2, R is a quasi-local reduced ring with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal and $Min(R) = \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$. It is noted in the proof of Example 2.2 that $x_i \neq 0 + I$ and $\mathfrak{p}_i x_i = (0 + I)$ for each $i \in \mathbb{N}$. Hence, $\mathcal{A}(R) = Min(R)$ and it is verified in the proof of Example 2.2 that R satisfies $(*)$ but R does not satisfy $(**)$.

(1) Note that $\mathcal{A}(T) = \{\mathfrak{p}_i[X] \cap T \mid i \in \mathbb{N}\}$ by Lemma 2.4. We know from Lemma 2.5 that T satisfies $(*)$. Let $i \in \mathbb{N}$ and let $f(X) \in \mathfrak{p}_i[X] \cap T$. Then $f(X) = \sum_{t=0}^{k} r_t X^t$ with $r_t \in \mathfrak{p}_i$ for each $t \in \{0, \ldots, k\}$. We know from the proof of Example 2.2 that for each $t \in \{0, \ldots, k\}$, there exists $j_t \in \mathbb{N}$ with $j_t > i$ such that $r_t \in \mathfrak{p}_j$ for all $j > j_t$. It is now clear that $f(X) \in \mathfrak{p}_j[X] \cap T$ for all $j \in \mathbb{N}$ with $j > max(j_0, \ldots, j_k)$. This shows that $\mathfrak{p}_i[X] \cap T \subseteq \bigcup_{j \in \mathbb{N}\backslash\{i\}} (\mathfrak{p}_j[X] \cap T)$. Therefore, T does not satisfy $(**)$.

(2) Observe that $\mathcal{A}(R[[X]]) = \{\mathfrak{p}_i[[X]] \mid i \in \mathbb{N}\}$ by Lemma 2.6. We know from Lemma 2.7 that $R[[X]]$ satisfies $(*)$. Let $i \in \mathbb{N}$. Note that $x_i \notin \mathfrak{p}_i$ but $x_j \in \mathfrak{p}_i$ for all $j \in \mathbb{N}\backslash\{i\}$. Let $f_i(X) = (\sum_{j=1}^{\infty} x_j X^j) - x_i X^i$. Then it is clear that $f_i(X) \in \mathfrak{p}_i[[X]] \backslash (\bigcup_{j \in \mathbb{N}\backslash\{i\}} \mathfrak{p}_j[[X]])$. This proves that $R[[X]]$ satisfies $(**)$. ■

In Example 2.10, we mention a reduced ring R such that $\mathcal{A}(R)$ is not finite, R satisfies $(**)$, and $\mathcal{B}(R) \neq \emptyset$, where we denote $Min(R)\backslash\mathcal{A}(R)$ by $\mathcal{B}(R)$.

Let R be a ring. An element e of R is said to be *idempotent* if $e = e^2$. An idempotent element e of R is said to be *non-trivial* if $e \notin \{0, 1\}$. For idempotent elements e, f of R, it is not hard to verify that $Re = Rf$ if and only if $e = f$.

Example 2.10. Let Λ be a set such that $|\Lambda| \geq 2$. For each $\alpha \in \Lambda$, let R_α be an integral domain. Let $R = \prod_{\alpha \in \Lambda} R_\alpha$. Then R is a reduced ring such that $|\mathcal{A}(R)| = |\Lambda|$, R satisfies $(**)$, and if Λ is not finite, then $\mathcal{A}(R)$ is not finite, and $\mathcal{B}(R) \neq \emptyset$, where $\mathcal{B}(R) = Min(R)\backslash\mathcal{A}(R)$. Moreover, T satisfies $(**)$ for each $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$).

Proof. Since R_α is an integral domain for each $\alpha \in \Lambda$ by assumption, it follows that $R = \prod_{\alpha \in \Lambda} R_\alpha$ is a reduced ring. Let $\alpha \in \Lambda$. Let us denote by e_α, the element of R whose α-th coordinate equals 1 and β-th coordinate equals 0 for all $\beta \in \Lambda\backslash\{\alpha\}$ and let us denote by f_α, the element of R whose α-th coordinate equals 0 and β-th coordinate equals 1 for all $\beta \in \Lambda\backslash\{\alpha\}$. It is clear that $Re_\alpha, Rf_\alpha \in \mathbb{A}(R)^*$ and $Re_\alpha Rf_\alpha$ equals the zero ideal of R. Since R_α is an integral domain and $\frac{R}{Rf_\alpha} \cong R_\alpha$ as rings, it follows that $Rf_\alpha \in Spec(R)$. Let us denote Rf_α by $\mathfrak{p}_\alpha$.

Since $\mathfrak{p}_\alpha \in Spec(R) \cap \mathbb{A}(R)$, we obtain from Lemma 2.3 that $\mathfrak{p}_\alpha \in Min(R)$. If α and β are distinct, then f_α and f_β are distinct idempotent elements of R and so, $\mathfrak{p}_\alpha \neq \mathfrak{p}_\beta$. Observe that $\{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\} \subseteq \mathcal{A}(R)$. It is clear that $\bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ is the zero ideal of R.

We next verify that if $\mathfrak{p} \in Spec(R)\cap\mathbb{A}(R)$, then $\mathfrak{p} = \mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$. Since $\mathfrak{p} \in \mathbb{A}(R)$, there exists a nonzero element $r \in R$ such that $\mathfrak{p}r$ is the zero ideal of R. Since $\bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ is the zero ideal of R, it follows that $r \notin \mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$. As $\mathfrak{p}r$ is the zero ideal of R, we get that $\mathfrak{p} \subseteq \mathfrak{p}_\alpha$. Since $\mathfrak{p}_\alpha \in Min(R)$, it follows that $\mathfrak{p} = \mathfrak{p}_\alpha$. This shows that $\mathcal{A}(R) = Min(R)\cap\mathbb{A}(R) \subseteq Spec(R)\cap\mathbb{A}(R) \subseteq \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\} \subseteq \mathcal{A}(R)$ and so, $\mathcal{A}(R) = Spec(R) \cap \mathbb{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Therefore, $|\mathcal{A}(R)| = |\Lambda|$. It is already noted in the previous paragraph that $\bigcap_{\alpha\in\Lambda} \mathfrak{p}_\alpha$ is the zero ideal of R. Thus R satisfies $(*)$. Since $\mathfrak{p}_\alpha \in Min(R)$ is principal for each $\alpha \in \Lambda$, it follows that $\mathfrak{p}_\alpha \not\subseteq \bigcup_{\beta\in\Lambda\setminus\{\alpha\}} \mathfrak{p}_\beta$. This proves that R satisfies $(**)$.

We next verify that if Λ is not finite, then $\mathcal{A}(R)$ is not finite and $\mathcal{B}(R) = Min(R)\setminus\mathcal{A}(R) \neq \emptyset$. As $|\mathcal{A}(R)| = |\Lambda|$, it follows that $\mathcal{A}(R)$ is not finite. It is clear that $I = \sum_{\alpha\in\Lambda} Re_\alpha$ is a proper ideal of R. Hence, there exists $\mathfrak{m} \in Max(R)$ such that $I \subseteq \mathfrak{m}$ by [2, Corollary 1.4]. It follows from [12, Theorem 10] that there exists $\mathfrak{q} \in Min(R)$ such that $\mathfrak{q} \subseteq \mathfrak{m}$. Since $e_\alpha \in \mathfrak{m}$ and $\mathfrak{p}_\alpha + Re_\alpha = R$ for each $\alpha \in \Lambda$, it follows that $\mathfrak{q} \notin \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\} = \mathcal{A}(R)$. Therefore, $\mathfrak{q} \in Min(R)\setminus\mathcal{A}(R) = \mathcal{B}(R)$ and so, $\mathcal{B}(R) \neq \emptyset$. Let $\alpha \in \Lambda$. As $f_\alpha \notin \mathfrak{q}$, it follows that $e_\alpha \in \mathfrak{q}$ and so, $\mathfrak{q} \supseteq \sum_{\alpha\in\Lambda} Re_\alpha = I$.

Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). As R satisfies $(**)$, it follows from Lemma 2.8 that T satisfies $(**)$. ■

In Example 2.11, we provide a reduced ring R such that $\mathcal{A}(R) = \emptyset$.

Recall that a ring R is *von Neumann regular* if given $a \in R$, there exists $b \in R$ such that $a = a^2b$ [8, Exercise 16, page 111]. Note that a ring R is von Neumann regular if and only if $dimR = 0$ and R is reduced by $(a) \Leftrightarrow (d)$ of [8, Exercise 16, page 111]. Thus if R is von Neumann regular, then $Spec(R) = Max(R) = Min(R)$.

Let R be a von Neumann regular ring. If $a \in (R\setminus\{0\})\cap NU(R)$, then $a = ue$ for some $u \in U(R)$ and e is a non-trivial idempotent element of R by (i)⇒(iii) of [8, Exercise 29, page 113]. Let $\mathfrak{m} \in Max(R)$. If $\mathfrak{m} \in \mathbb{A}(R)$, then $\mathfrak{m}e = (0)$ for some non-trivial idempotent element e of R and so, $\mathfrak{m} \subseteq R(1-e)$. Therefore, $\mathfrak{m} = R(1-e)$. Thus if each maximal ideal of R belongs to $\mathbb{A}(R)$, then each prime ideal of R is principal and hence, every ideal of R is principal by [12, Exercise 10, page 8]. Thus if R is not Noetherian, then R admits at least one $\mathfrak{m} \in Max(R)$ such that $\mathfrak{m} \notin \mathbb{A}(R)$.

In Example 2.11, we mention a von Neumann regular R such that $\mathfrak{m} \notin \mathbb{A}(R)$ for each $\mathfrak{m} \in Max(R)$. Thus the reduced ring R is such that $\mathcal{A}(R) = \emptyset$. For an ideal I of a ring T, we denote $\{\mathfrak{p} \in Spec(T) \mid \mathfrak{p} \supseteq I\}$ by $V(I)$.

Example 2.11. Let L be the field of algebraic numbers (that is, L is the algebraic closure of $\mathbb{Q}$) and let A be the ring of all algebraic integers. Let $R = \frac{A}{\sqrt{2A}}$. Then R is a von Neumann regular ring, $Max(R)$ is uncountable, and $\mathfrak{p} \notin \mathbb{A}(R)$ for each $\mathfrak{p} \in Max(R)$.

Proof. Note that $dim A = 1$ by [8, Proposition 42.8(i)] and so, $dim R = 0$. Since $\sqrt{2A}$ is a radical ideal of A, it follows that $R = \frac{A}{\sqrt{2A}}$ is reduced. Therefore, R is von Neumann regular. Note that $Max(R) = \{\frac{\mathfrak{m}}{\sqrt{2A}} \mid \mathfrak{m} \in Max(A) \cap V(2A)\}$. Since $2 \notin U(A)$ and any non-unit of A belongs to uncountably many maximal ideals of A by [8, Proposition 42.8(i)], it follows that $Max(R)$ is uncountable. Let $\mathfrak{p} \in Max(R)$. Then $\mathfrak{p} = \frac{\mathfrak{m}}{\sqrt{2A}}$ for some $\mathfrak{m} \in Max(A)$ with $\mathfrak{m} \in V(2A)$. If $\mathfrak{p} \in \mathbb{A}(R)$, then $\mathfrak{p} = R(a + \sqrt{2A})$ for some $a \in \mathfrak{m}$, since R is von Neumann regular. This implies that $\mathfrak{m} = \sqrt{2A + aA}$. Note that A is a Bezout domain [13, see page 86]. Hence, $2A + aA = cA$ for some $c \in 2A + aA$ and therefore, $\mathfrak{m} = \sqrt{cA}$. This is impossible, since any non-unit of A belongs to uncountably many maximal ideals of A by [8, Proposition 42.8(i)]. Therefore, $\mathfrak{p} \notin \mathbb{A}(R)$ for each $\mathfrak{p} \in Max(R) = Min(R)$. Therefore, $\mathcal{A}(R) = \emptyset$. ∎

The proof of Theorems 2.15 (respectively 2.17) needs the following preliminaries. Let R be a reduced ring such that R satisfies $(*)$. Then $\bigcap_{\mathfrak{p} \in \mathcal{A}(R)} \mathfrak{p} = (0)$. It is not hard to verify that $Z(R) = \bigcup_{\mathfrak{p} \in \mathcal{A}(R)} \mathfrak{p}$. Let $\mathfrak{p} \in \mathcal{A}(R)$. Then $\mathfrak{p}r = (0)$ for some $r \in R \backslash \{0\}$. Hence, $\mathfrak{p} \subseteq ((0) :_R r)$. Since $r^2 \neq 0$, it follows that $r \notin \mathfrak{p}$. From $r((0) :_R r) = (0) \subset \mathfrak{p}$, we get that $((0) :_R r) \subseteq \mathfrak{p}$ and so, $\mathfrak{p} = ((0) :_R r)$. The proof of Proposition 2.12 (respectively, 2.13) needs the above two facts.

Proposition 2.12. *Let R be a reduced ring. If R satisfies $(*)$, then the following statements hold:*

(1) $\gamma_t(\mathbb{AG}(R)) \leq |\mathcal{A}(R)|$.

(2) $\gamma_t(\Gamma(R)) \leq |\mathcal{A}(R)|$.

Proof. By hypothesis, R is a reduced ring. Assume that R satisfies $(*)$.

(1) Let $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Since $\bigcap_{\alpha \in \Lambda} \mathfrak{p}_\alpha = (0)$ and R is not an integral domain, it follows that $|\Lambda| \geq 2$. Let $\alpha \in \Lambda$. Note that there exists $r_\alpha \in R \backslash \{0\}$ such that $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$. It is clear that $Rr_\alpha \in \mathbb{A}(R)^*$. Let $\beta \in \Lambda$ be such that $\beta \neq \alpha$. Observe that $Rr_\alpha \neq Rr_\beta$, since $((0) :_R r_\alpha) = \mathfrak{p}_\alpha \neq \mathfrak{p}_\beta = ((0) :_R r_\beta)$. Let $D = \{Rr_\alpha \mid \alpha \in \Lambda\}$. Observe that $|D| = |\Lambda|$. We claim that D is a total dominating set of $\mathbb{AG}(R)$. Let $I \in \mathbb{A}(R)^* \backslash D$. Note that there exists $r \in R \backslash \{0\}$ such that $Ir = (0)$. Observe that $r \notin \mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$. Hence, $I \subseteq \mathfrak{p}_\alpha$ and so, $IRr_\alpha = (0)$. Thus for any $I \in \mathbb{A}(R)^* \backslash D$, I and Rr_α are adjacent in $\mathbb{AG}(R)$ for some $\alpha \in \Lambda$. This proves that D is a dominating set of $\mathbb{AG}(R)$. For any distinct $\alpha, \beta \in \Lambda$, $r_\alpha r_\beta = 0$ by [4, Lemma 3.6] and so, Rr_α and Rr_β are adjacent

in $\mathbb{AG}(R)$. This shows that D is a total dominating set of $\mathbb{AG}(R)$. Therefore, $\gamma_t(\mathbb{AG}(R)) \leq |D| = |\Lambda| = |\mathcal{A}(R)|$.

(2) Let Λ, $\mathcal{A}(R)$, and r_α ($\alpha \in \Lambda$) be as in the proof of (1). Note that $r_\alpha \in Z(R)^*$ for each $\alpha \in \Lambda$ and $r_\alpha \neq r_\beta$ for all distinct $\alpha, \beta \in \Lambda$. Let $D_1 = \{r_\alpha \mid \alpha \in \Lambda\}$. Let $r \in Z(R)^* \backslash D_1$. Note that $r \in \mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$. Hence, $rr_\alpha = 0$. This shows that D_1 is a dominating set of $\Gamma(R)$. Since $r_\alpha r_\beta = 0$ for all distinct $\alpha, \beta \in \Lambda$ by [4, Lemma 3.6], it follows that D_1 is a total dominating set of $\Gamma(R)$. Hence, $\gamma_t(\Gamma(R)) \leq |D_1| = |\mathcal{A}(R)|$. ∎

Proposition 2.13. *Let R be a reduced ring such that $|\mathcal{A}(R)| \geq 3$. Then the following statements hold:*

(1) *If R satisfies $(*)$, then $|D| \geq |\mathcal{A}(R)|$ for any dominating set D of $\mathbb{AG}(R)$.*

(2) *If R satisfies $(**)$, then $|D| \geq |\mathcal{A}(R)|$ for any dominating set D of $\Gamma(R)$.*

Proof. By hypothesis, R is a reduced ring such that $|\mathcal{A}(R)| \geq 3$.

(1) Assume that R satisfies $(*)$. Let $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Note that $|\Lambda| \geq 3$ by hypothesis. Let $\alpha \in \Lambda$. Observe that there exists $r_\alpha \in R\backslash\{0\}$ such that $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$, $r_\alpha \notin \mathfrak{p}_\alpha$, and $Rr_\alpha \in \mathbb{A}(R)^*$.

Let D be any dominating set of $\mathbb{AG}(R)$. Let $\alpha \in \Lambda$. Let $A_\alpha = \{\mathfrak{p}_\alpha\}$ and let $B_\alpha = \{I \in \mathbb{A}(R)^* \mid I\mathfrak{p}_\alpha = (0)\}$. It is clear that $A_\alpha \neq \emptyset$ and from $Rr_\alpha \in B_\alpha$, it follows that $B_\alpha \neq \emptyset$. We claim that $D \cap (A_\alpha \cup B_\alpha) \neq \emptyset$. If $\mathfrak{p}_\alpha \in D$, then it is clear that $\mathfrak{p}_\alpha \in D \cap A_\alpha \subseteq D \cap (A_\alpha \cup B_\alpha)$. Suppose that $\mathfrak{p}_\alpha \notin D$. Since D is a dominating set of $\mathbb{AG}(R)$, there exists $I \in D$ such that $\mathfrak{p}_\alpha$ and I are adjacent in $\mathbb{AG}(R)$. Hence, $\mathfrak{p}_\alpha I = (0)$. It is clear that $I \in B_\alpha$. Hence, $D \cap (A_\alpha \cup B_\alpha) \neq \emptyset$. Let $\alpha, \beta \in \Lambda$ be distinct. We claim that $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$. It is clear that $A_\alpha \cap A_\beta = \emptyset$. As $|\Lambda| \geq 3$ and distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\mathfrak{p}_\alpha \mathfrak{p}_\beta \neq (0)$ and so, $A_\alpha \cap B_\beta = A_\beta \cap B_\alpha = \emptyset$. Let $I \in B_\alpha$ and $J \in B_\beta$. Note that $I \subset \mathfrak{p}_\beta$. As $J^2 \neq (0)$, it follows that $J \not\subseteq \mathfrak{p}_\beta$. Hence, $B_\alpha \cap B_\beta = \emptyset$. It is now clear that $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$. Therefore, for any distinct $\alpha, \beta \in \Lambda$, $(D \cap (A_\alpha \cup B_\alpha)) \cap (D \cap (A_\beta \cup B_\beta)) = \emptyset$. It is already verified that $D \cap (A_\alpha \cup B_\alpha) \neq \emptyset$ for each $\alpha \in \Lambda$. Hence, it is possible to find a subset D_1 of D such that D_1 contains exactly one element, say $I_\alpha \in A_\alpha \cup B_\alpha$ for each $\alpha \in \Lambda$. Now, $D_1 = \{I_\alpha \mid \alpha \in \Lambda\}$. Note that $|\Lambda| = |D_1| \leq |D|$. As $|\Lambda| = |\mathcal{A}(R)|$, we obtain that $|D| \geq |\mathcal{A}(R)|$.

(2) Assume that R satisfies $(**)$. Then R satisfies $(*)$ and for each $\alpha \in \Lambda$, $\mathfrak{p}_\alpha \not\subseteq \bigcup_{\beta \in \Lambda\backslash\{\alpha\}} \mathfrak{p}_\beta$. Let $\alpha \in \Lambda$. Let $x_\alpha \in \mathfrak{p}_\alpha \backslash (\bigcup_{\beta \in \Lambda\backslash\{\alpha\}} \mathfrak{p}_\beta)$. Let D be any dominating set of $\Gamma(R)$. Let $A_\alpha = \{x_\alpha\}$ and let $B_\alpha = \{y \in Z(R)^* \mid x_\alpha y = 0\}$. It is clear that $r_\alpha \in B_\alpha$ and so, $B_\alpha \neq \emptyset$. Since $|\Lambda| \geq 3$, it follows from the choice of x_α that $|V(Ry) \cap \mathcal{A}(R)| \geq 2$ for each $y \in B_\alpha$. Also, it is clear that $|V(Rx_\alpha) \cap \mathcal{A}(R)| = 1$. We claim that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$. This is clear if $x_\alpha \in D$. If $x_\alpha \notin D$, then there exists $y_\alpha \in D$ such that x_α and y_α are adjacent in $\Gamma(R)$. Hence, $x_\alpha y_\alpha = 0$ and so, $y_\alpha \in B_\alpha$. This shows that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$.

Let $\alpha, \beta \in \Lambda$ be distinct. We assert that $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$. By the choice of x_α, x_β it is clear that $x_\alpha \notin \mathfrak{p}_\beta$ and $x_\beta \in \mathfrak{p}_\beta$ and so, $x_\alpha \neq x_\beta$. Hence, $A_\alpha \cap A_\beta = \emptyset$. Since $|V(Rx_\alpha) \cap \mathcal{A}(R)| = 1 = |V(Rx_\beta) \cap \mathcal{A}(R)|$, $|V(Ry) \cap \mathcal{A}(R)| \geq 2$ (respectively, $|V(Rz) \cap \mathcal{A}(R)| \geq 2$) for each $y \in B_\alpha$ (respectively, $z \in B_\beta$), it follows that $A_\alpha \cap B_\beta = A_\beta \cap B_\alpha = \emptyset$. Let $y \in B_\alpha$. Since $\bigcap_{\mathfrak{p} \in \mathcal{A}(R)} \mathfrak{p} = (0)$ by assumption, it follows from the choice of x_α that $y \in \bigcap_{\mathfrak{p} \in \mathcal{A}(R) \setminus \{\mathfrak{p}_\alpha\}} \mathfrak{p}$. Hence, $y \in \mathfrak{p}_\beta \setminus \mathfrak{p}_\alpha$. Similarly, it follows that $z \in \mathfrak{p}_\alpha \setminus \mathfrak{p}_\beta$ for any $z \in B_\beta$. Therefore, $B_\alpha \cap B_\beta = \emptyset$. This shows that $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$ for all distinct $\alpha, \beta \in \Lambda$. Let D_1 be the subset of D obtained by choosing exactly one element, say z_α from $A_\alpha \cup B_\alpha$ for each $\alpha \in \Lambda$. Then $|D| \geq |D_1| = |\Lambda| = |\mathcal{A}(R)|$. ∎

Let R be a reduced ring such that R satisfies $(*)$. If $\gamma(\mathbb{AG}(R)) \geq 2$, then we prove in Theorem 2.15 that $\gamma(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(R))$. Moreover, we prove in Theorem 2.15 that for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T))$. We use Lemma 2.14 in the proof of Theorem 2.15.

Lemma 2.14. *Let R be a ring (R can possibly be non-reduced) such that R admits $I, J \in \mathbb{A}(R)^*$ with $I + J \notin \mathbb{A}(R)$. If $\gamma(\mathbb{AG}(R)) \geq 2$, then for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) \geq 2$.*

Proof. By hypothesis, the ring R is such that R admits $I, J \in \mathbb{A}(R)^*$ with $I + J \notin \mathbb{A}(R)$. Assume that $\gamma(\mathbb{AG}(R)) \geq 2$. Let $T \in [R, R[X]]$. We claim that $\gamma(\mathbb{AG}(T)) \geq 2$. Suppose that $\gamma(\mathbb{AG}(T)) = 1$. Let $C \in \mathbb{A}(T)^*$ be such that $\{C\}$ is a dominating set of $\mathbb{AG}(T)$. As $(I[X] \cap T) \cap R = I \neq J = (J[X] \cap T) \cap R$, it follows that $I[X] \cap T \neq J[X] \cap T$. Since $(0) \neq Ann_R(I) \subseteq Ann_T(I[X] \cap T)$, it follows that $I[X] \cap T \in \mathbb{A}(T)^*$. Similarly, we get that $J[X] \cap T \in \mathbb{A}(T)^*$. Let $f(X) \in T \setminus \{0\}$ be such that $(I[X] \cap T + J[X] \cap T) f(X) = (0)$. Let $i \geq 0$ be least with the property that the coefficient of X^i in $f(X)$, say $r_i \neq 0$. Note that $I + J \subseteq I[X] \cap T + J[X] \cap T$ and so, $(I + J) f(X) = (0)$. This implies that $(I + J) r_i = (0)$. This is impossible, since $I + J \notin \mathbb{A}(R)$ by hypothesis. Therefore, $I[X] \cap T + J[X] \cap T \notin \mathbb{A}(T)$. Hence, $C \in \{I[X] \cap T, J[X] \cap T\}$. Without loss of generality, we can assume that $C = I[X] \cap T$. Let $W \in \mathbb{A}(R)^*$ be such that $W \neq I$. Then $W[X] \cap T \in \mathbb{A}(T)^*$, $W[X] \cap T \neq I[X] \cap T$. Therefore, $(W[X] \cap T)(I[X] \cap T) = (0)$. This implies that $WI = (0)$ and so, $\{I\}$ is a dominating set of $\mathbb{AG}(R)$. This is impossible, since $\gamma(\mathbb{AG}(R)) \geq 2$ by assumption. Therefore, $\gamma(\mathbb{AG}(T)) \geq 2$.

Let $T \in [R, R[[X]]]$. By considering $I[[X]] \cap T, J[[X]] \cap T$ and using similar arguments, it can be shown that $\gamma(\mathbb{AG}(T)) \geq 2$. ∎

Theorem 2.15. *Let R be a reduced ring such that R satisfies $(*)$. If $\gamma(\mathbb{AG}(R)) \geq 2$, then $\gamma(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(R))$.*

Moreover, for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T))$.

Proof. By hypothesis, R is a reduced ring such that R satisfies $(*)$. Assume that $\gamma(\mathbb{AG}(R)) \geq 2$. As any total dominating set of a graph G is a dominating set of G, it follows that $\gamma(G) \leq \gamma_t(G)$. Therefore, $\gamma(\mathbb{AG}(R)) \leq \gamma_t(\mathbb{AG}(R))$. Let $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. If $|\mathcal{A}(R)| = 2$, then it follows from Proposition 2.12(1) that $\gamma_t(\mathbb{AG}(R)) \leq 2$ and so, $2 = \gamma(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(R))$. Suppose that $|\mathcal{A}(R)| \geq 3$. Then it follows from Propositions 2.12(1) and 2.13(1) that $\gamma(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(R))$.

Let $T \in [R, R[X]]$. We know from Lemma 2.5 that T satisfies $(*)$. Let $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Then $\mathcal{A}(T) = \{\mathfrak{p}_\alpha[X] \cap T \mid \alpha \in \Lambda\}$ by Lemma 2.4. Let $\alpha, \beta \in \Lambda$ be distinct. Then $\mathfrak{p}_\alpha \neq \mathfrak{p}_\beta$. As $(\mathfrak{p}_\alpha[X] \cap T) \cap R = \mathfrak{p}_\alpha$ and $(\mathfrak{p}_\beta[X] \cap T) \cap R = \mathfrak{p}_\beta$, we get that $\mathfrak{p}_\alpha[X] \cap T \neq \mathfrak{p}_\beta[X] \cap T$. Therefore, $|\mathcal{A}(R)| = |\mathcal{A}(T)|$. For distinct $\alpha, \beta \in \Lambda$, note that $\mathfrak{p}_\alpha + \mathfrak{p}_\beta \notin \mathbb{A}(R)$. Hence, we obtain from Lemma 2.14 that $\gamma(\mathbb{AG}(T)) \geq 2$. It now follows using arguments similar to those that are used in the previous paragraph that $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T))$.

Let $T \in [R, R[[X]]]$. With the help of Lemmas 2.6, 2.7, and 2.14 and using arguments similar to those that are used in the previous paragraph, it can be shown that $|\mathcal{A}(R)| = |\mathcal{A}(T)|$, $\gamma(\mathbb{AG}(T)) \geq 2$, and $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T))$. ■

Remark 2.16. Let R be a reduced ring such that $|Min(R)| < \infty$. Then $Min(R) = \mathcal{A}(R)$ and R satisfies $(*)$ by Example 2.1. Hence, if $\gamma(\mathbb{AG}(R)) \geq 2$, then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\mathcal{A}(R)| = |Min(R)|$ by Theorem 2.15. Moreover, for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) \geq 2$ and $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = |Min(R)| = |Min(T)| = \gamma_t(\mathbb{AG}(T))$. Hence, Theorem 2.15 generalizes [14, Theorem 2.4]. Let R be the reduced ring mentioned in Example 2.2. It is noted in the proof of Example 2.2 that $\mathcal{A}(R) = Min(R)$, $|Min(R)| = |\mathbb{N}|$, and R satisfies $(*)$. Hence, by Proposition 2.13(1) and Theorem 2.15, we get that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |Min(R)|$. Let R be the reduced ring mentioned in Example 2.10. In the notation of the proof of Example 2.10, $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$ and it is verified there that R satisfies $(*)$ and $\mathcal{A}(R) \subset Min(R)$ if Λ is not finite. If $|\Lambda| \geq 3$, then $\gamma(\mathbb{AG}(R)) = |\Lambda| = \gamma_t(\mathbb{AG}(R))$ by Proposition 2.13(1) and Theorem 2.15.

Let R be a reduced ring such that R satisfies $(**)$. If $\gamma(\Gamma(R)) > 1$, then we prove in Theorem 2.17 that $\gamma_t(\Gamma(R)) = \gamma(\Gamma(R)) = |\mathcal{A}(R)|$.

Theorem 2.17. *Let R be a reduced ring such that R satisfies $(**)$. If $\gamma(\Gamma(R)) \geq 2$, then $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |\mathcal{A}(R)|$.*

Moreover, for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\Gamma(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\Gamma(T))$.

Proof. By hypothesis, R is a reduced ring such that R satisfies $(**)$. Assume that $\gamma(\Gamma(R)) \geq 2$. Then $\gamma_t(\Gamma(R)) \geq 2$. If $|\mathcal{A}(R)| = 2$, then $\gamma_t(\Gamma(R)) \leq 2$ by

Proposition 2.12(2) and so, $\gamma(\Gamma(R)) = 2 = \gamma_t(\Gamma(R)) = |\mathcal{A}(R)|$. If $|\mathcal{A}(R)| \geq 3$, then we obtain from Propositions 2.12(2) and 2.13(2) that $\gamma(\Gamma(R)) = |\mathcal{A}(R)| = \gamma_t(\Gamma(R))$.

Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). We know from Lemma 2.8 that T satisfies $(**)$. It follows as in the proof of the moreover part of Theorem 2.15 that $|\mathcal{A}(R)| = |\mathcal{A}(T)|$. Suppose that $\gamma(\Gamma(T)) = 1$. Let $f(X) \in Z(T)^*$ be such that $\{f(X)\}$ is a dominating set of $\Gamma(T)$. Let $i \geq 0$ be least with the property that the coefficient of X^i in $f(X)$, say $r_i \neq 0$. Then $r_i \in Z(R)^*$. Since $\gamma(\Gamma(R)) \geq 2$ by assumption, there exists $r \in Z(R)^* \backslash \{r_i\}$ such that $rr_i \neq 0$. It is clear that $r \in Z(T)^* \backslash \{f(X)\}$. From $rr_i \neq 0$, it follows that $rf(X) \neq 0$. This contradicts the assumption $\{f(X)\}$ is a dominating set of $\Gamma(T)$. Therefore, $\gamma(\Gamma(T)) \geq 2$. It now follows using arguments similar to those that are used in the previous paragraph that $\gamma(\Gamma(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\Gamma(T))$. ■

Remark 2.18. Let R be a reduced ring such that $|Min(R)| < \infty$. Then $\mathcal{A}(R) = Min(R)$ and R satisfies $(**)$ by Example 2.1. Thus if $\gamma(\Gamma(R)) > 1$, then $\gamma(\Gamma(R)) = |\mathcal{A}(R)| = |Min(R)| = \gamma_t(\Gamma(R))$ by Theorem 2.17. Moreover, for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\Gamma(T)) > 1$ and $\gamma(\Gamma(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\Gamma(T)) = |Min(R)| = |Min(T)|$. Hence, Theorem 2.17 generalizes [14, Theorem 2.5]. Let R be the reduced ring mentioned in Example 2.10. In the notation of the proof of Example 2.10, $\mathcal{A}(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. We know from the proof of Example 2.10 that R satisfies $(**)$. If $|\Lambda| \geq 3$, then it follows from Proposition 2.13(2) and Theorem 2.17 that $\gamma(\Gamma(R)) = |\Lambda| = |\mathcal{A}(R)| = \gamma_t(\Gamma(R))$.

Let R be a ring which is not necessarily reduced such that $\gamma(\mathbb{AG}(R)) \geq 2$. Then $\gamma(\Gamma(R)) \neq 1$. For if $\gamma(\Gamma(R)) = 1$, then there exists $r \in Z(R)^*$ such that $rx = 0$ for all $x \in Z(R)^* \backslash \{r\}$. Let $I = Rr$. Then it is clear that $I \in \mathbb{A}(R)^*$. Let $J \in \mathbb{A}(R)^*$ be such that $J \neq I$. It is clear that any element of J belongs to $Z(R)$. Let $x \in J$. Then either $x \in I$ or $xr = 0$. Therefore, $J \subseteq I \cup ((0) :_R r)$. This implies that either $J \subseteq ((0) :_R r)$ or $J \subseteq I$. If $J \subseteq ((0) :_R r)$, then $Jr = JI = (0)$. If $J \subseteq I$, then $J \subset I$. Let $x \in J \backslash \{0\}$. Then as $I = Rr$, it follows that $x \neq r$ and so, $xr = 0$. Therefore, $Jr = JI = (0)$. This shows that J and I are adjacent in $\mathbb{AG}(R)$. This implies that $\gamma(\mathbb{AG}(R)) = 1$ and this contradicts the assumption $\gamma(\mathbb{AG}(R)) \geq 2$. We use this fact in the proof of Corollary 2.19 and in the proof of Proposition 2.27.

Corollary 2.19. *Let R be a reduced ring such that R satisfies $(**)$. If $\gamma(\mathbb{AG}(R)) \geq 2$, then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = \gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |\mathcal{A}(R)|$.*

Moreover, for any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T)) = \gamma(\Gamma(T)) = \gamma_t(\Gamma(T))$.

Proof. By hypothesis, R is a reduced ring such that R satisfies $(**)$. Assume that $\gamma(\mathbb{AG}(R)) \geq 2$. Then $\gamma(\Gamma(R)) \geq 2$. It follows from Theorems 2.15 and 2.17 that $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = |\mathcal{A}(R)| = \gamma(\Gamma(R)) = \gamma_t(\Gamma(R))$.

Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). From the moreover part of Theorems 2.15 and 2.17, we obtain that $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T)) = \gamma(\Gamma(T)) = \gamma_t(\Gamma(T))$. ■

Remark 2.20. Let R be a reduced ring such that $|Min(R)| < \infty$. Then R satisfies $(**)$ and $\mathcal{A}(R) = Min(R)$ by Example 2.1. Thus if $\gamma(\mathbb{AG}(R)) \geq 2$, then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = \gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = |Min(R)|$ by Corollary 2.19. For any $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$), we obtain from the moreover part of Corollary 2.19 that $\gamma(\mathbb{AG}(T)) = |\mathcal{A}(T)| = |\mathcal{A}(R)| = \gamma_t(\mathbb{AG}(T)) = \gamma(\Gamma(T)) = \gamma_t(\Gamma(T)) = |Min(R)| = |Min(T)|$. Hence, Corollary 2.19 generalizes [14, Corollary 2.6].

Let R be a ring such that $dim R = 0$. Then $Spec(R) = Max(R) = Min(R)$. It is known that any minimal prime ideal of a ring is contained in its set of zero-divisors [12, Theorem 84]. Therefore, $Spec(R) = MNP(R)$. Let R be an Artinian ring. If R is not isomorphic to direct product of two fields, then it is known that $\gamma(\mathbb{AG}(R)) = |Min(R)|$ [14, Theorem 2.8]. It is known that a ring R is Artinian if and only if R is Noetherian and $dim R = 0$ [2, Theorem 8.5]. Thus if R is Artinian which is not isomorphic to direct product of two fields, then [14, Theorem 2.8] can be stated as $\gamma(\mathbb{AG}(R)) = |MNP(R)|$. Our aim is to generalize this result to any ring R such that the zero ideal of R admits a strong primary decomposition. First, we prove some lemmas which are needed to prove a generalization of [14, Theorem 2.8].

For idempotent elements e, f of a ring R, it is not hard to verify that $Re \cap Rf = Ref$. Thus if $ef = 0$, then $Re \cap Rf = (0)$. This fact will be used in the proof of Lemma 2.21.

Lemma 2.21. *Let Λ be a set such that $|\Lambda| \geq 3$. Let R_α be a ring for each $\alpha \in \Lambda$ and let $R = \prod_{\alpha \in \Lambda} R_\alpha$. Then the following statements hold:*

(1) *If D is any dominating set of $\mathbb{AG}(R)$, then $|D| \geq |\Lambda|$.*

(2) *If D is any dominating set of $\Gamma(R)$, then $|D| \geq |\Lambda|$.*

Proof. For each $\alpha \in \Lambda$, let e_α denote the element of R whose α-th coordinate equals 1 and β-th coordinate equals 0 for all $\beta \in \Lambda \backslash \{\alpha\}$. Let f_α denote the element of R whose α-th coordinate equals 0 and β-th coordinate equals 1 for all $\beta \in \Lambda \backslash \{\alpha\}$. It is clear that $e_\alpha f_\alpha$ equals the zero element of R and so, $Re_\alpha, Rf_\alpha \in \mathbb{A}(R)^*$ (respectively, $f_\alpha, e_\alpha \in Z(R)^*$).

(1) Let D be any dominating set of $\mathbb{AG}(R)$. Let $\alpha \in \Lambda$. Let $A_\alpha = \{Rf_\alpha\}$ and let $B_\alpha = \{I \in \mathbb{A}(R)^* \mid I \subseteq Re_\alpha\}$. It is clear that $Re_\alpha \in B_\alpha$ and so, $B_\alpha \neq \emptyset$.

We claim that $D \cap (A_\alpha \cup B_\alpha) \neq \emptyset$. This is clear if $Rf_\alpha \in D$. If $Rf_\alpha \notin D$, then there exists $I \in D$ such that I and Rf_α are adjacent in $\mathbb{AG}(R)$. It is then clear that $I \subseteq Re_\alpha$ and so, $I \in B_\alpha$. This shows that $D \cap (A_\alpha \cup B_\alpha) \neq \emptyset$. Let $\alpha, \beta \in \Lambda$ be distinct. We claim that $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$. As $Rf_\alpha \neq Rf_\beta$, it follows that $A_\alpha \cap A_\beta = \emptyset$. Since $|\Lambda| \geq 3$ by hypothesis, it follows that f_α has at least two nonzero coordinates. Hence, $A_\alpha \cap B_\beta = \emptyset$. Similarly, it follows that $A_\beta \cap B_\alpha = \emptyset$. From $Re_\alpha \cap Re_\beta$ equals the zero ideal of R, we get that $B_\alpha \cap B_\beta = \emptyset$. This shows that for any distinct $\alpha, \beta \in \Lambda$, $(A_\alpha \cup B_\alpha) \cap (A_\beta \cup B_\beta) = \emptyset$ and so, $(D \cap (A_\alpha \cup B_\alpha)) \cap (D \cap (A_\beta \cup B_\beta)) = \emptyset$. It is possible to form a subset D_1 of D consisting of exactly one element, say I_α from $A_\alpha \cup B_\alpha$ for each $\alpha \in \Lambda$. Note that $D_1 = \{I_\alpha \mid \alpha \in \Lambda\}$. Hence, $|D| \geq |D_1| = |\Lambda|$.

(2) Let D be any dominating set of $\Gamma(R)$. Let $\alpha \in \Lambda$. Let $A_\alpha = \{f_\alpha\}$ and let $B_\alpha = \{y \in Z(R)^* \mid Ry \subseteq Re_\alpha\}$. It can be shown as in the proof of (1) that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$ for each $\alpha \in \Lambda$ and $(D \cap (A_\alpha \cup B_\alpha)) \cap (D \cap (A_\beta \cup B_\beta)) = \emptyset$ for all distinct $\alpha, \beta \in \Lambda$. Let D_1 be a subset of D which contains exactly one element, say $z_\alpha \in D \cap (A_\alpha \cup B_\alpha)$ for each $\alpha \in \Lambda$. Then $|D| \geq |D_1| = |\Lambda|$. ∎

Let I be an ideal of a ring R such that $I \subseteq Z(R)$. Let $S = R \backslash Z(R)$. Then S is a multiplicatively closed subset (m.c. subset) of R and $I \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [12, Theorem 1] that there exists $\mathfrak{p} \in MNP(R)$ such that $I \subseteq \mathfrak{p}$.

Let $\mathfrak{p} \in MNP(R)$. If $\mathfrak{p} \in \mathbb{A}(R)$, then there exists $r \in R\backslash\{0\}$ such that $\mathfrak{p}r = (0)$. Hence, $\mathfrak{p} \subseteq ((0) :_R r)$. As $((0) :_R r) \subseteq Z(R)$ and $\mathfrak{p} \in MNP(R)$, it follows that $\mathfrak{p} = ((0) :_R r)$. We use this fact and the fact mentioned in the previous paragraph whenever we need in our discussion.

Lemma 2.22. *Let R be a ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$. Then the following statements hold:*

(1) $\gamma(\mathbb{AG}(R)) = 1$ *if* $|MNP(R)| = 1$ *and* $\gamma_t(\mathbb{AG}(R)) \leq |MNP(R)|$ *if* $|MNP(R)| \geq 2$.

(2) $\gamma(\Gamma(R)) = 1$ *if* $|MNP(R)| = 1$ *and* $\gamma_t(\Gamma(R)) \leq |MNP(R)|$ *if* $|MNP(R)| \geq 2$.

Proof. Let $MNP(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Observe that $|MNP(R)| = |\Lambda|$. Let $\alpha \in \Lambda$. As $\mathfrak{p}_\alpha \in \mathbb{A}(R)$ by hypothesis, there exists $r_\alpha \in R\backslash\{0\}$ such that $\mathfrak{p}_\alpha = ((0) :_R r_\alpha)$. It is clear that $Rr_\alpha \in \mathbb{A}(R)^*$ (respectively, $r_\alpha \in Z(R)^*$) and for all distinct $\alpha, \beta \in \Lambda$, $Rr_\alpha \neq Rr_\beta$ (respectively, $r_\alpha \neq r_\beta$).

(1) Let $D = \{Rr_\alpha \mid \alpha \in \Lambda\}$. Note that $|D| = |\Lambda|$. Since given any $I \in \mathbb{A}(R)^*$ is contained in $\mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$ and $r_\alpha r_\beta = 0$ for all distinct $\alpha, \beta \in \Lambda$ by [4, Lemma 3.6], it can be shown as in the proof of Proposition 2.12(1) that D is a dominating set of $\mathbb{AG}(R)$ if $|\Lambda| = 1$ and D is a total dominating set of $\mathbb{AG}(R)$ if

$|\Lambda| \geq 2$. Therefore, $\gamma(\mathbb{AG}(R)) = 1$ if $|MNP(R)| = 1$ and $\gamma_t(\mathbb{AG}(R)) \leq |D| = |\Lambda|$ if $|MNP(R)| \geq 2$.

(2) Let $D_1 = \{r_\alpha \mid \alpha \in \Lambda\}$. Note that $Z(R) = \bigcup_{\alpha \in \Lambda} \mathfrak{p}_\alpha$ and $|D_1| = |\Lambda|$. Let $r \in Z(R)^*$. Then $r \in \mathfrak{p}_\alpha$ for some $\alpha \in \Lambda$. Note that $r_\alpha r_\beta = 0$ for all distinct $\alpha, \beta \in \Lambda$ by [4, Lemma 3.6]. Hence, it can be shown as in the proof of Proposition 2.12(2) that D_1 is a dominating set of $\Gamma(R)$ if $|\Lambda| = 1$ and D_1 is a total dominating set of $\Gamma(R)$ if $|\Lambda| \geq 2$. Therefore, $\gamma(\Gamma(R)) = 1$ if $|MNP(R)| = 1$ and $\gamma_t(\Gamma(R)) \leq |MNP(R)|$ if $|MNP(R)| \geq 2$. ■

Lemma 2.23. *Let R be a ring such that $|MNP(R)| \geq 3$. If $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$, then $|D| \geq |MNP(R)|$ for each dominating set D of $\mathbb{AG}(R)$.*

Proof. Let $MNP(R) = \{\mathfrak{p}_\alpha \mid \alpha \in \Lambda\}$. Observe that $|MNP(R)| = |\Lambda|$. By hypothesis, $|\Lambda| \geq 3$. Assume that $\mathfrak{p}_\alpha \in \mathbb{A}(R)$ for each $\alpha \in \Lambda$. Since distinct members of $MNP(R)$ are not comparable under inclusion, it follows that $\mathfrak{p}_\alpha^2 \neq (0)$ for each $\alpha \in \Lambda$. From $|MNP(R)| \geq 3$, it follows that $\mathfrak{p}_\alpha \mathfrak{p}_\beta \neq (0)$ for any distinct $\alpha, \beta \in \Lambda$. As any annihilating ideal of a ring is contained in its set of zero-divisors, it follows that $\mathfrak{p}_\alpha + \mathfrak{p}_\beta \notin \mathbb{A}(R)$ for any distinct $\alpha, \beta \in \Lambda$. Let $\alpha \in \Lambda$. Let D be any dominating set of $\mathbb{AG}(R)$. Let $A_\alpha = \{\mathfrak{p}_\alpha\}$ and let $B_\alpha = \{I \in \mathbb{A}(R)^* \mid I\mathfrak{p}_\alpha = (0)\}$. Proceeding as in the proof of Proposition 2.13(1), it can be shown that $|D \cap (A_\alpha \cup B_\alpha)| \geq 1$ for each $\alpha \in \Lambda$ and $(D \cap (A_\alpha \cup B_\alpha)) \cap (D \cap (A_\beta \cup B_\beta)) = \emptyset$ for any distinct $\alpha, \beta \in \Lambda$. Hence, it can be shown as in the proof of Proposition 2.13(1) that $|D| \geq |\Lambda| = |MNP(R)|$. ■

Let R be a ring such that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$. If $\gamma(\mathbb{AG}(R)) \geq 2$, then we prove in Theorem 2.24 that $\gamma(\mathbb{AG}(R)) = |MNP(R)| = \gamma_t(\mathbb{AG}(R)|$.

Theorem 2.24. *Let R be a ring such that $\gamma(\mathbb{AG}(R)) > 1$. If $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$, then $\gamma(\mathbb{AG}(R)) = |MNP(R)| = \gamma_t(\mathbb{AG}(R))$.*

Proof. With the help of Lemmas 2.22(1) and 2.23, this theorem can be proved using arguments similar to those that are used in the proof of Theorem 2.15. Hence, we omit the proof of this theorem. ■

We provide Example 2.25 to illustrate Theorems 2.24.

Example 2.25. Let $R = \mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ be the ring obtained by using Nagata's principle of idealization. Let us denote the set of all positive prime numbers by $\mathbb{P}$. Then $Min(R) = \{(0)(+)\frac{\mathbb{Q}}{\mathbb{Z}}\}$, $Max(R) = \{p\mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}} \mid p \in \mathbb{P}\} = MNP(R)$, $p\mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}} \in \mathbb{A}(R)$ for each $p \in \mathbb{P}$, and $\gamma(\mathbb{AG}(R)) = |MNP(R)| = \gamma_t(\mathbb{AG}(R))$.

Proof. Let us denote $(0)(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ by $\mathfrak{p}$. It is clear that $\mathfrak{p}$ is an ideal of R and since $\frac{R}{\mathfrak{p}} \cong \mathbb{Z}$ as rings and $\mathbb{Z}$ being an integral domain, it follows that $\mathfrak{p} \in Spec(R)$. Note that $\mathfrak{p}^2 = (0)(+)(0 + \mathbb{Z})$ and so, any prime ideal of R will contain $\mathfrak{p}$. This shows that $Min(R) = \{\mathfrak{p}\}$. Let $p \in \mathbb{P}$. It is clear that $\mathfrak{m}_p = p\mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ is an ideal of R. Since $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is a field and $\frac{R}{\mathfrak{m}_p} \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$ as rings, we get that $\mathfrak{m}_p \in Max(R)$. If $\mathfrak{p}' \in Spec(R)$ is such that $\mathfrak{p}' \supset \mathfrak{p}$, then $\mathfrak{p}' = p\mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ for some $p \in \mathbb{P}$. Therefore, $Max(R) = \{\mathfrak{m}_p \mid p \in \mathbb{P}\}$. Let $p \in \mathbb{P}$. Let $r \in \mathfrak{m}_p$. Then $r = (pn, \alpha + \mathbb{Z})$ for some $n \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$. Since $\frac{1}{p} + \mathbb{Z}$ is a nonzero element of $\frac{\mathbb{Q}}{\mathbb{Z}}$ and $(pn, \alpha + \mathbb{Z})(0, \frac{1}{p} + \mathbb{Z})$ equals the zero element of R, it follows that $r = (pn, \alpha + \mathbb{Z}) \in Z(R)$. This proves that $\mathfrak{m}_p$ is a subset of $Z(R)$ and so, $\mathfrak{m}_p \in MNP(R)$. Thus $Max(R) \subseteq MNP(R)$ and since $Spec(R) = \{\mathfrak{p}, \mathfrak{m}_p \mid p \in \mathbb{P}\}$, it follows that $MNP(R) = Max(R)$. Let $p \in \mathbb{P}$. As $\mathfrak{m}_p(0, \frac{1}{p} + \mathbb{Z})$ equals the zero ideal of R, we obtain that $\mathfrak{m}_p \in \mathbb{A}(R)$. This proves that $MNP(R)$ is infinite and each member of $MNP(R)$ belongs to $\mathbb{A}(R)$. It follows from Lemma 2.23 that $\gamma(\mathbb{AG}(R)) > n$ for each $n \in \mathbb{N}$ and we obtain from Theorem 2.24 that $\gamma(\mathbb{AG}(R)) = |MNP(R)| = \gamma_t(\mathbb{AG}(R))$. ∎

Let R be a ring. Let I be a proper ideal of R. A *primary decomposition* of I in R is an expression of I as a finite intersection of primary ideals of R, say $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is a primary ideal of R for each $i \in \{1, \ldots, n\}$ [2, see page 51]. If $\mathfrak{q}$ is a primary ideal of R, then it is known that $\sqrt{\mathfrak{q}}$ is a prime ideal of R [2, Proposition 4.1]. Let $\sqrt{\mathfrak{q}} = \mathfrak{p}$. In such a case, $\mathfrak{q}$ is said to be $\mathfrak{p}$-*primary* [2, page 51]. A $\mathfrak{p}$-primary ideal $\mathfrak{q}$ is said to be *strongly primary* if there exists $n \in \mathbb{N}$ such that $\mathfrak{p}^n \subseteq \mathfrak{q}$. A *strong primary decomposition* of I in R is an expression of I as a finite intersection of strongly primary ideals of R.

A primary decomposition $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is said to be *minimal* if (i) $\sqrt{\mathfrak{q}_i}$ are all distinct and (ii) for each $i \in \{1, \ldots, n\}$, $\mathfrak{q}_i \not\supseteq \bigcap_{j \in A_i} \mathfrak{q}_j$, where $A_i = \{1, \ldots, n\} \backslash \{i\}$ [2, page 52].

Let $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition of the zero ideal in R with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each $i \in \{1, \ldots, n\}$. Then $Z(R) = \bigcup_{i=1}^{n} \mathfrak{p}_i$ by [2, Proposition 4.7]. Thus if the zero ideal of R admits a primary decomposition, then it follows from [2, Proposition 1.10(i)] that $MNP(R)$ is finite and $MNP(R)$ equals the set of maximal members of $\{\mathfrak{p}_i \mid i \in \{1, \ldots, n\}\}$.

Let $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal strong primary decomposition of (0) in R with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each $i \in \{1, \ldots, n\}$. Then proceeding as in the proof of [2, Proposition 7.17], it can be shown that there exists $x_i \in R \backslash \{0\}$ such that $\mathfrak{p}_i = ((0) :_R x_i)$ for each $i \in \{1, \ldots, n\}$. Hence, $MNP(R)$ is finite and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$.

Recall that R is said to be *Laskerian* (respectively, *strongly Laskerian*) if each proper ideal of R admits a primary (respectively, a strong primary) decomposition [10]. It follows from [2, Theorem 7.13 and Proposition 7.14] that any Noetherian ring is strongly Laskerian. For a detailed account of Laskerian rings, the reader is referred to [10].

Proposition 2.26. *Let R be a ring such that the zero ideal of R admits a strong primary decomposition. If $\gamma(\mathbb{AG}(R)) > 1$, then $\gamma(\mathbb{AG}(R)) = |MNP(R)| =$*

$\gamma_t(\mathbb{AG}(R))$.

Moreover, for any $T \in [R, R[X]]$ *(respectively,* $T \in [R, R[[X]]]$*),* $\gamma(\mathbb{AG}(T)) = |MNP(T)| = |MNP(R)| = \gamma_t(\mathbb{AG}(T))$.

Proof. By hypothesis, R is a ring such that the zero ideal of R admits a strong primary decomposition and $\gamma(\mathbb{AG}(R)) > 1$. Let $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal strong primary decomposition of (0) in R with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each $i \in \{1, \ldots, n\}$. Let $r_i \in R\backslash\{0\}$ be such that $\mathfrak{p}_i = ((0) :_R r_i)$ for each $i \in \{1, \ldots, n\}$. After a suitable renaming of the $\mathfrak{p}_i, i \in \{1, \ldots, n\}$, we can assume without loss of generality that $MNP(R) = \{\mathfrak{p}_i \mid i \in \{1, \ldots, k\}\}$. Now, it follows from Theorem 2.24 that $\gamma(\mathbb{AG}(R)) = |MNP(R)| = k = \gamma_t(\mathbb{AG}(R))$. Let $i \in \{1, \ldots, n\}$. Note that $\mathfrak{q}_i[X]$ is a $\mathfrak{p}_i[X]$-primary ideal in $R[X]$ by [2, Exercise 7(iii), page 55]. It follows from [2, Exercise 7(iv), page 55] that $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i[X]$ is a minimal primary decomposition of the zero ideal in $R[X]$. For each $i \in \{1, \ldots, n\}$, there exists $t_i \in \mathbb{N}$ such that $\mathfrak{p}_i^{t_i} \subseteq \mathfrak{q}_i$ and so, $(\mathfrak{p}_i[X])^{t_i} \subseteq \mathfrak{q}_i[X]$. Hence, $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i[X]$ is a minimal strong primary decomposition of the zero ideal in $R[X]$. It is clear that $\mathfrak{p}_i[X] = ((0) :_{R[X]} r_i)$ for each $i \in \{1, \ldots, n\}$ and $MNP(R[X]) = \{\mathfrak{p}_i[X] \mid i \in \{1, \ldots, k\}\}$. Let $T \in [R, R[X]]$. Then it is not hard to verify that for each $i \in \{1, \ldots, n\}$, $\mathfrak{q}_i[X] \cap T$ is a $\mathfrak{p}_i[X] \cap T$-primary ideal of T and is a strongly primary ideal of T, $(0) = \bigcap_{i=1}^{n} (\mathfrak{q}_i[X] \cap T)$ is a minimal strong primary decomposition of the zero ideal in T, $\mathfrak{p}_i[X] \cap T = ((0) :_T r_i)$ for each $i \in \{1, \ldots, n\}$, and $MNP(T) = \{\mathfrak{p}_i[X] \cap T \mid i \in \{1, \ldots, k\}\}$. Hence, $|MNP(T)| = |MNP(R)| = k$.

Let $i \in \{1, \ldots, n\}$. Note that $\mathfrak{q}_i[[X]]$ is a $\mathfrak{p}_i[[X]]$-primary ideal of $R[[X]]$ by [7, Corollary 4] and it is clear that $(\mathfrak{p}_i[[X]])^{t_i} \subseteq \mathfrak{q}_i[[X]]$. Therefore, $(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i[[X]]$ is a minimal strong primary decomposition of (0) in $R[[X]]$. Let $T \in [R, R[[X]]]$. Observe that $(0) = \bigcap_{i=1}^{n} (\mathfrak{q}_i[[X]] \cap T)$ is a minimal strong primary decomposition of (0) in T. Observe that $\mathfrak{p}_i[[X]] \cap T = ((0) :_T r_i)$ for each $i \in \{1, \ldots, n\}$ and $MNP(T) = \{\mathfrak{p}_i[[X]] \cap T \mid i \in \{1, \ldots, k\}\}$. Therefore, $|MNP(T)| = k = |MNP(R)|$.

By assumption, $\gamma(\mathbb{AG}(R)) > 1$. It follows from Lemma 2.22(1) that $|MNP(R)| \geq 2$. Thus $k \geq 2$. Note that $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathbb{A}(R)^*$ are such that $\mathfrak{p}_1 + \mathfrak{p}_2 \notin \mathbb{A}(R)$. Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). It now follows from Lemma 2.14 that $\gamma(\mathbb{AG}(T)) > 1$. Hence, it follows from Theorem 2.24 that $\gamma(\mathbb{AG}(T)) = |MNP(T)| = |MNP(R)| = k = \gamma_t(\mathbb{AG}(T))$. ∎

Corollary 2.27. *Let R be an Artinian ring. If* $\gamma(\mathbb{AG}(R)) > 1$*, then for any* $T \in [R, R[X]]$ (*respectively,* $T \in [R, R[[X]]]$), $\gamma(\mathbb{AG}(T)) = |MNP(T)| = |Min(R)| = \gamma_t(\mathbb{AG}(T)) = |Min(T)|$.

Proof. As R is an Artinian ring by hypothesis, R is Noetherian and $dim R = 0$. Therefore, the zero ideal of R admits a strong primary decomposition in R and $Spec(R) = Max(R) = Min(R) = MNP(R)$. Assume that $\gamma(\mathbb{AG}(R)) > 1$. From Proposition 2.26, it follows that $\gamma(\mathbb{AG}(R)) = |MNP(R)| = |Min(R)| =$

$\gamma_t(\mathbb{AG}(R))$. Let $MNP(R) = Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, \ldots, k\}\}$. Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). We know from the proof of Proposition 2.26 that $MNP(T) = \{\mathfrak{p}_i[X] \cap T \mid i \in \{1, 2, \ldots, k\}\}$ (respectively, $MNP(T) = \{\mathfrak{p}_i[[X]] \cap T \mid i \in \{1, 2, \ldots, k\}\}$). Observe that $\sqrt{(0)}$ in T equals $\bigcap_{i=1}^k (\mathfrak{p}_i[X] \cap T)$ (respectively, $\bigcap_{i=1}^k (\mathfrak{p}_i[[X]] \cap T)$). It follows that $Min(T) = \{\mathfrak{p}_i[X] \cap T \mid i \in \{1, 2, \ldots, k\}\}$(respectively, $Min(T) = \{\mathfrak{p}_i[[X]] \cap T \mid i \in \{1, 2, \ldots, k\}\}$).

Let $T \in [R, R[X]]$ (respectively, $T \in [R, R[[X]]]$). Now, we obtain from the moreover part of Proposition 2.26 that $\gamma(\mathbb{AG}(T)) = |MNP(T)| = |MNP(R)| = |Min(R)| = \gamma_t(\mathbb{AG}(T)) = |Min(T)|$. ■

In Example 2.28, we provide a ring R to illustrate that the conclusion of Proposition 2.26 can fail to hold if the hypothesis, the zero ideal of R admits a strong primary decomposition is omitted.

Example 2.28. Let $(V, \mathfrak{m})$ be a rank one valuation domain which is not discrete. Let $m \in \mathfrak{m}, m \neq 0$. Let $R = \frac{V}{Vm}$. Then $|MNP(R)| = 1$ but $\mathbb{AG}(R)$ (respectively, $\Gamma(R)$) does not admit any finite dominating set.

Proof. Note that R is quasi-local with $\frac{\mathfrak{m}}{Vm}$ as its unique maximal ideal. For convenience, let us denote $\frac{\mathfrak{m}}{Vm}$ by $\mathfrak{p}$. Since $dim V = 1$, it follows that $dim R = 0$. Therefore, $Spec(R) = Max(R) = Min(R) = \{\mathfrak{p}\}$. It is clear that $Z(R) \subseteq \mathfrak{p}$. As $nil(R) = \mathfrak{p}$ by [2, Proposition 1.8], it follows that $\mathfrak{p} \subseteq Z(R)$ and so, $Z(R) = \mathfrak{p}$. Hence, $MNP(R) = \{\mathfrak{p}\}$. This shows that $|MNP(R)| = 1$. Note that $\sqrt{(0 + Vm)} = \mathfrak{p} \in Max(R)$ and hence, $(0 + Vm)$ is a $\mathfrak{p}$-primary ideal of R by [2, Proposition 4.2]. Since R is not an integral domain, it follows that $\mathbb{A}(R)^* \neq \emptyset$. Note that R is not Noetherian, since V is not Noetherian. Hence, R admits a strictly increasing sequence of finitely generated ideals of R. Since the set of ideals of V is linearly ordered by inclusion, the set of ideals of R is linearly ordered by inclusion and so, any finitely generated proper ideal of R is principal generated by a zero-divisor. Hence, any finitely generated proper ideal of R belongs to $\mathbb{A}(R)$. Therefore, $\mathbb{A}(R)^*$ is infinite.

First, we verify that $\mathfrak{p} \notin \mathbb{A}(R)$. Let $w \in V \backslash Vm$. Then $m = wy$ for some $y \in \mathfrak{m}$. Since V is a rank one non-discrete valuation domain, $\mathfrak{m}$ is not principal and so, $\mathfrak{m} \not\subseteq Vy$ and so, $\mathfrak{m}w \not\subseteq Vm$. Since $\mathfrak{p} = \frac{\mathfrak{m}}{Vm}$, we get that $\mathfrak{p}(w+Vm) \neq (0+Vm)$ for any $w \in V$ such that $w + Vm \neq 0 + Vm$. This shows that $\mathfrak{p} \notin \mathbb{A}(R)$.

Let D be any finite subset of $\mathbb{A}(R)^*$ with $|D| \geq 2$. Let $D = \{I_i \mid i \in \{1, 2, \ldots, n\}\}$. We can find $k, t \in \{1, 2, \ldots, n\}$ such that $I_k \subset I_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{k\}$ and $I_t \supset I_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{t\}$. Since $\mathfrak{p} \notin \mathbb{A}(R)$, it follows that $\mathfrak{p} \not\subseteq I_t \cup ((0 + Vm) :_R I_k)$. Hence, there exists $r \in \mathfrak{p}$ such that $r \notin I_t$ and $rI_k \neq (0+Vm)$. Let $J = Rr$. Then it is clear that $J \in \mathbb{A}(R)^*$. By the choice of k and t, it follows that $J \notin D$ and $JI_i \neq (0 + Vm)$ for any $i \in \{1, 2, \ldots, n\}$. This shows that $\mathbb{AG}(R)$ does not admit any finite dominating set.

Let D be any finite subset of $Z(R)^*$ with $|D| \geq 2$. Let $D = \{r_i \mid i \in \{1, 2, \ldots, n\}\}$. It is possible to find $k, t \in \{1, 2, \ldots, n\}$ such that $Rr_k \subseteq Rr_j$ for all $j \in \{1, 2, \ldots, n\}$ and $Rr_t \supseteq Rr_j$ for all $j \in \{1, 2, \ldots, n\}$. Since $\mathfrak{p} \notin \mathbb{A}(R)$, it follows that $\mathfrak{p} \not\subseteq Rr_t \cup ((0 + Vm) :_R Rr_k)$. Hence, there exists $r \in \mathfrak{p}$ such that $r \notin Rr_t$ and $rr_k \neq 0 + Vm$. It is clear that $r \in Z(R)^*$, $r \notin D$ and $rr_i \neq 0 + Vm$ for any $i \in \{1, 2, \ldots, n\}$. This shows that $\Gamma(R)$ does not admit any finite dominating set. ∎

Let R be a ring such that $dimR = 0$. Then $Spec(R) = Max(R) = Min(R)$. It is known that any minimal prime ideal of a ring is contained in its set of zero-divisors [12, Theorem 84]. Therefore, $Spec(R) = MNP(R)$. In Proposition 2.29, we provide a necessary condition in order that $\gamma(\mathbb{AG}(R))$ (respectively, $\gamma(\Gamma(R))$) to be finite.

Proposition 2.29. *Let R be a zero-dimensional ring. If $\gamma(\mathbb{AG}(R)) < \infty$ (respectively, $\gamma(\Gamma(R)) < \infty$), then $|MNP(R)| < \infty$.*

Proof. Note that $Spec(R) = Max(R) = Min(R) = MNP(R)$, since $dimR = 0$ by hypothesis. Assume that $\gamma(\mathbb{AG}(R)) < \infty$. Let $n \in \mathbb{N}$ be such that $\gamma(\mathbb{AG}(R)) = n$. We claim that $|Max(R)| \leq n + 1$. If $|Max(R)| \geq n + 2$, then there exist zero-dimensional rings $R_1, R_2, R_3, \ldots, R_{n+2}$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_{n+2}$ as rings by [17, Lemma 2.2]. Hence, by Lemma 2.21(1), we get that $\gamma(\mathbb{AG}(R)) \geq n + 2$. This is a contradiction and so, $|MNP(R)| = |Max(R)| \leq n + 1$.

Assume that $\gamma(\Gamma(R)) < \infty$. Let $t \in \mathbb{N}$ be such that $\gamma(\Gamma(R)) = t$. Then with the help of [17, Lemma 2.2] and Lemma 2.21(2), it can be shown as in the previous paragraph that $|MNP(R)| = |Max(R)| \leq t + 1$. ∎

The zero-dimensional R provided in Example 2.28 illustrates that the converse of Proposition 2.29 can fail to hold.

Let R be a ring such that $dimR = 0$. In Proposition 2.30, we provide a necessary and sufficient condition in order that $\gamma(\mathbb{AG}(R))$ (respectively, $\gamma(\Gamma(R))$) to be equal to 1.

Proposition 2.30. *Let R be a ring such that $dimR = 0$. Then the following statements hold:*

(1) $\gamma(\mathbb{AG}(R)) = 1$ *if and only if either $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)^*$ or $|Max(R)| = 2$ and in such a case, $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.*

(2) $\gamma(\Gamma(R)) = 1$ *if and only if either $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)^*$ or $|Max(R)| = 2$ and in such a case, $R \cong \mathbb{Z}_2 \times F$ as rings, where F is a field.*

Proof. (1) Assume that $\gamma(\mathbb{AG}(R)) = 1$. Let $I \in \mathbb{A}(R)^*$ be such that $\{I\}$ is a dominating set of $\mathbb{AG}(R)$. Note that $|Max(R)| \leq 2$ by the proof of Proposition

2.29. Suppose that $|Max(R)| = 1$. Let $\mathfrak{m}$ be the unique maximal ideal of R. Note that $Z(R) = \mathfrak{m}$. If $I = \mathfrak{m}$, then it is clear that $\mathfrak{m} \in \mathbb{A}(R)^*$. Suppose that $I \neq \mathfrak{m}$. Then for any $x \in \mathfrak{m}\backslash I$, $Rx \in \mathbb{A}(R)^*$ and $IRx = (0)$. This shows that $\mathfrak{m} \subseteq I \cup ((0) :_R I)$ and so, $\mathfrak{m} \subseteq ((0) :_R I) \subseteq \mathfrak{m}$. Thus $\mathfrak{m} = ((0) :_R I) \in \mathbb{A}(R)^*$. Suppose that $|Max(R)| = 2$. Note that R has at least one non-trivial idempotent element (see the proof of [17, Lemma 2.2]). Let e be an idempotent element of R with $e \notin \{0, 1\}$. Then $1 - e$ is also an idempotent element of R with $1 - e \notin \{0, 1\}$. It is clear that $Re, R(1 - e) \in \mathbb{A}(R)^*$ and $Re + R(1 - e) = R$. If $I \notin \{Re, R(1 - e)\}$, then from $Ie = I(1 - e) = (0)$, we get that $I = (0)$. This is a contradiction. Therefore, $I \in \{Re, R(1 - e)\}$. Without loss of generality, we can assume that $I = Re$. Let $Max(R) = \{\mathfrak{m}_i \mid i \in \{1, 2\}\}$. Without loss of generality, we can assume that $e \in \mathfrak{m}_1$. Then $1 - e \in \mathfrak{m}_2$. If $y \in \mathfrak{m}_1\backslash Re$, then $ye = 0$ and so, $y \in R(1 - e) \subseteq \mathfrak{m}_2$. This proves that $\mathfrak{m}_1 \subseteq Re \cup \mathfrak{m}_2$. Hence, $\mathfrak{m}_1 \subseteq Re$ and so, $\mathfrak{m}_1 = Re$. If $x \in \mathfrak{m}_2\backslash Re$, then $xe = 0$ and so, $x \in R(1 - e)$. This implies that $\mathfrak{m}_2 \subseteq \mathfrak{m}_1 \cup R(1 - e)$. Hence, $\mathfrak{m}_2 \subseteq R(1 - e)$ and so, $\mathfrak{m}_2 = R(1 - e)$. Now, $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. Hence, it follows from [2, Proposition 1.10(ii) and (iii)] that $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$ as rings. Let $i \in \{1, 2\}$. Let $F_i = \frac{R}{\mathfrak{m}_i}$. Then F_i is a field and $R \cong F_1 \times F_2$ as rings.

If $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)^*$, then $MNP(R) = \{\mathfrak{m}\}$. Hence, $\gamma(\mathbb{AG}(R)) = 1$ by Lemma 2.22(1). If $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$, then $\mathbb{AG}(R)$ is a complete graph with two vertices and so, $\gamma(\mathbb{AG}(R)) = 1$.

(2) Assume that $\gamma(\Gamma(R)) = 1$. Then $\gamma(\mathbb{AG}(R)) = 1$ (for a proof, see the paragraph which follows immediately after Remark 2.18). Therefore, we obtain from (1) that either $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)^*$ or $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. Let $T = F_1 \times F_2$. Note that $\Gamma(T)$ is a complete bipartite graph with vertex partition $Z(T)^* = (F_1^* \times (0)) \cup ((0) \times F_2^*)$. From $\gamma(\Gamma(T)) = 1$, it follows that $|F_i^*| = 1$ for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that $|F_1^*| = 1$. Therefore, $R \cong \mathbb{Z}_2 \times F$ as rings with $F = F_2$ is a field.

If $(R, \mathfrak{m})$ is quasi-local with $\mathfrak{m} \in \mathbb{A}(R)^*$, then $MNP(R) = \{\mathfrak{m}\}$. Hence, $\gamma(\Gamma(R)) = 1$ by Lemma 2.22(2). If $R \cong \mathbb{Z}_2 \times F$ as rings, where F is a field, then $\Gamma(R)$ is a star graph and so, $\gamma(\Gamma(R)) = 1$. ∎

Let R be a zero-dimensional ring such that $|MNP(R)| < \infty$. Then it is not hard to verify that (0) admits a primary decomposition. Assume that $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$. We do not know whether in such a ring R, (0) admits a strong primary decomposition. However, we are able to determine $\gamma(\mathbb{AG}(R))$(respectively, $\gamma(\Gamma(R))$) in Proposition 2.31.

Proposition 2.31. *Let R be a ring such that $dim R = 0$, $|MNP(R)| < \infty$ and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$. Then the following statements hold:*

(1) *If* $R \ncong F_1 \times F_2$ *as rings, where* F_1 *and* F_2 *are fields, then* $\gamma(\mathbb{AG}(R)) = |MNP(R)| = |Min(R)|$ *and if* $|MNP(R)| \geq 2$*, then* $\gamma(\mathbb{AG}(R)) = |Min(R)| = \gamma_t(\mathbb{AG}(R))$.

(2) *If* $R \ncong \mathbb{Z}_2 \times F$ *as rings, where* F *is a field, then* $\gamma(\Gamma(R)) = |MNP(R)| = |Min(R)|$ *and if* $|MNP(R)| \geq 2$*, then* $\gamma(\Gamma(R)) = |Min(R)| = \gamma_t(\Gamma(R))$.

Proof. Assume that $dim R = 0$, $|MNP(R)| < \infty$, and $\mathfrak{p} \in \mathbb{A}(R)$ for each $\mathfrak{p} \in MNP(R)$.

(1) Assume that $R \ncong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. If $|MNP(R)| = 1$, then it follows from Lemma 2.22(1) that $\gamma(\mathbb{AG}(R)) = 1$. Assume that $|MNP(R)| \geq 2$. Then $\gamma(\mathbb{AG}(R)) > 1$ by Proposition 2.30(1). It then follows from Theorem 2.24 that $\gamma(\mathbb{AG}(R)) = |MNP(R)| = |Min(R)| = \gamma_t(\mathbb{AG}(R))$.

(2) Assume that $R \ncong \mathbb{Z}_2 \times F$ as rings, where F is a field. We consider the following cases:

Case (i). $|MNP(R)| = 1$. Then we obtain from Lemma 2.22(2) that $\gamma(\Gamma(R)) = 1$. Hence, $\gamma(\Gamma(R)) = 1 = |MNP(R)| = |Min(R)|$.

Case (ii). $|MNP(R)| \geq 2$. Since $R \ncong \mathbb{Z}_2 \times F$ as rings, where F is a field, it follows from Proposition 2.30(2) that $\gamma(\Gamma(R)) \geq 2$. We know from Lemma 2.22(2) that $\gamma_t(\Gamma(R)) \leq |MNP(R)|$. If $|MNP(R)| = 2$, then it is clear that $\gamma(\Gamma(R)) = 2 = \gamma_t(\Gamma(R)) = |MNP(R)| = |Min(R)|$. Suppose that $|MNP(R)| \geq 3$. Let $|MNP(R)| = n$. Then there exist zero-dimensional rings $R_1, R_2, R_3, \ldots, R_n$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings by [17, Lemma 2.2]. It now follows from Lemmas 2.21(2) and 2.22(2) that $n \leq \gamma(\Gamma(R)) \leq \gamma_t(\Gamma(R)) \leq n$. Therefore, $\gamma(\Gamma(R)) = n = |MNP(R)| = |Min(R)| = \gamma_t(\Gamma(R))$. ∎

It is known that a ring R is Artinian if and only if R is Noetherian and $dim R = 0$ [2, Theorem 8.5]. Let R be an Artinian ring. Then $Spec(R) = Max(R) = Min(R) = MNP(R)$ and $|MNP(R)| < \infty$. It is not hard to verify that each proper ideal of R is an annihilating ideal of R. Hence, if R is an Artinian ring with $R \ncong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$, then we obtain from Proposition 2.31(1) that $\gamma(\mathbb{AG}(R)) = |Min(R)|$. Thus Proposition 2.31(1) generalizes [14, Theorem 2.8]. Moreover, if $|Min(R)| \geq 2$, then $\gamma_t(\mathbb{AG}(R)) = |Min(R)|$. If $R \ncong \mathbb{Z}_2 \times F$ as rings, where F is a field, then we obtain from Proposition 2.31(2) that $\gamma(\Gamma(R)) = |Min(R)|$ and if $|Min(R)| \geq 2$, then $\gamma_t(\Gamma(R)) = |Min(R)|$.

We provide an example to illustrate Proposition 2.31. Let K be a field and let V be an infinite dimensional vector space over K. Let $R = K(+)V$ be the ring obtained by using Nagata's principle of idealization. Then $\dim R = 0$ and R is quasi-local with $\mathfrak{m} = (0)(+)V$ as its unique maximal ideal and $\mathfrak{m}^2$ is the zero ideal of R. Thus $MNP(R) = \{\mathfrak{m}\}$ and $\mathfrak{m} \in \mathbb{A}(R)$. Since $dim_K V$ is infinite by assumption, it follows that R is not Noetherian. Let $n \geq 1$ and let $T = R_1 \times$

$\cdots \times R_n$ with $R_i = R$ for each $i \in \{1, \ldots, n\}$. Then $dimT = 0$, $|MNP(T)| = n$, T is not reduced and each proper ideal of T is an annihilating ideal of T. Note that T is not Noetherian. And $\gamma(\mathbb{AG}(T)) = \gamma(\Gamma(T)) = |MNP(T)|$ if $n = 1$ and $\gamma(\mathbb{AG}(T)) = \gamma_t(\mathbb{AG}(T)) = |MNP(T)| = \gamma(\Gamma(T)) = \gamma_t(\Gamma(T))$ if $n \geq 2$.

Let R be a von Neumann regular ring which is not a field. Note that R is reduced and $dimR = 0$. Assume that $\gamma(\mathbb{AG}(R)) < \infty$ (respectively, $\gamma(\Gamma(R)) < \infty$). Then $|MNP(R)| < \infty$ by Proposition 2.29. Let $Max(R) = MNP(R) = \{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$. Since $\mathfrak{m}_i + \mathfrak{m}_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$ and $\bigcap_{i=1}^{n} \mathfrak{m}_i = (0)$, it follows that $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \cdots \times \frac{R}{\mathfrak{m}_n}$ as rings by [2, Proposition 1.10(ii) and (iii)]. Let $i \in \{1, 2, \ldots, n\}$ and let $F_i = \frac{R}{\mathfrak{m}_i}$. Then F_i is a field and $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. Note that if $n = 2$, then $\gamma(\mathbb{AG}(R)) = 1$ and if $n \geq 3$, then $\gamma(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = n$. There is no von Neumann regular ring R with $\gamma(\mathbb{AG}(R)) = 2$.

Let $n = 2$. If $|F_i| = 2$ for at least one $i \in \{1, 2\}$, then $\gamma(\Gamma(R)) = 1$. If $|F_i| > 1$ for each $i \in \{1, 2\}$, then $\gamma(\Gamma(R)) = 2 = \gamma_t(\Gamma(R))$. If $n \geq 3$, then $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = n$.

For the sake of convenient reference, the above derived facts are included in the form of the following proposition.

Proposition 2.32. *Let R be a von Neumann regular ring which is not a field. Then the following statements hold:*

(1) *Let $n \in \mathbb{N}$ be such that $\gamma(\mathbb{AG}(R)) = n$. Then $n \geq 3$ and $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, 3, \ldots, n\}$ and in such a case, $\gamma_t(\mathbb{AG}(R)) = n$.*

(2) *Let $n \in \mathbb{N}$ be such that $\gamma(\Gamma(R)) = n$. If $n = 1$, then $R \cong \mathbb{Z}_2 \times F$ as rings, where F is a field. If $n = 2$, then $R \cong F_1 \times F_2$ as rings, where F_i is a field with $|F_i| > 2$ for each $i \in \{1, 2\}$. In such a case, $\gamma_t(\Gamma(R)) = 2$. If $n \geq 3$, then $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ as rings, where F_i is a field for each $i \in \{1, 2, 3, \ldots, n\}$ and in such a case, $\gamma(\Gamma(R)) = \gamma_t(\Gamma(R)) = n$.*

3. Some results on two supergraphs of $\mathbb{AG}(R)$

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. The aim of this section is to discuss some results on the domination number of two spanning supergraphs of $\mathbb{AG}(R)$. First, we discuss some results on the domination number of $SAG(R)$, the strongly annihilating-ideal graph of R. It is already noted in Section 1 that $SAG(R)$ is a spanning supergraph of $\mathbb{AG}(R)$. As several results on the domination number of $\mathbb{AG}(R)$ (where R is a reduced ring) are already discussed in Section 2, we focus on determining the domination number of $SAG(R)$, where R is a reduced ring such that $SAG(R) \neq \mathbb{AG}(R)$. For such a ring R, it is known that $|Min(R)| \geq 3$ by

(1) $\Rightarrow$ (2) of [16, Theorem 4.1] and we prove in Proposition 3.3 that $\gamma(SAG(R)) > 1$ and in Theorem 3.4, we prove that $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$. We use Lemmas 3.1 and 3.2 in the proof of Proposition 3.3.

Lemma 3.1. *Let R be a reduced ring. If $I, J \in \mathbb{A}(R)^*$ are such that $I \subset J$, then I and J are not adjacent in $SAG(R)$.*

Proof. Assume that $I, J \in \mathbb{A}(R)^*$ with $I \subset J$. Note that if $AB = (0)$ for some ideals A, B of R, then for any $x \in A \cap B$, $x^2 \in AB = (0)$ and so, $x = 0$, since R is reduced. Hence, $A \cap B = (0)$. As $IAnn_R(I) = (0)$, it follows that $I \cap Ann_R(I) = (0)$. Observe that $I \cap Ann_R(J) \subseteq I \cap Ann_R(I) = (0)$. Hence, $I \cap Ann_R(J) = (0)$ and so, I and J are not adjacent in $SAG(R)$. ■

Lemma 3.2. *Let R be a reduced ring. Then for any $I \in \mathbb{A}(R)^*$, $I + Ann_R(I) \notin \mathbb{A}(R)$.*

Proof. Let $r \in R$ be such that $(I + Ann_R(I))r = (0)$. Then $Ir = Ann_R(I)r = (0)$. Hence, $r \in Ann_R(I)$ and so, $r^2 = 0$. This implies that $r = 0$, since R is reduced. Therefore, $I + Ann_R(I) \notin \mathbb{A}(R)$. ■

Proposition 3.3. *Let R be a reduced ring with $|Min(R)| \geq 3$. Then $\gamma(SAG(R)) > 1$.*

Proof. By hypothesis, R is a reduced ring with $|Min(R)| \geq 3$. Let $I \in \mathbb{A}(R)^*$. It is convenient to denote $Ann_R(I)$ by J. It is clear that $J \in \mathbb{A}(R)^*$. We claim that there exists $A \in \mathbb{A}(R)^* \backslash \{I\}$ such that A and I are not adjacent in $SAG(R)$. Since $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$, there exists a member of $Min(R)$, say $\mathfrak{p}_1$ such that $I \not\subseteq \mathfrak{p}_1$. From $IJ = (0)$, it follows that $J \subseteq \mathfrak{p}_1$. Similarly, there exists a member of $Min(R)$, say $\mathfrak{p}_2$ such that $J \not\subseteq \mathfrak{p}_2$. Hence, $I \subseteq \mathfrak{p}_2$. It is clear that $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Since $|Min(R)| \geq 3$, there exists a member of $Min(R)$, say $\mathfrak{p}_3$ such that $\mathfrak{p}_3 \notin \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. Since distinct minimal prime ideals of a ring are not comparable under inclusion, it follows that $\mathfrak{p}_3 \not\subseteq \bigcup_{i=1}^{2} \mathfrak{p}_i$ and so, $\mathfrak{p}_3 \not\subseteq I \cup J$. Let $r \in \mathfrak{p}_3$ be such that $r \notin I \cup J$. Hence, $r \notin I$ and $Ir \neq (0)$. Since $Z(R) = \bigcup_{\mathfrak{p} \in Min(R)} \mathfrak{p}$, it follows that $\mathfrak{p}_3 \subset Z(R)$. Therefore, $r \in Z(R)$ and so, there exists $s \in R \backslash \{0\}$ such that $rs = 0$. As $I + J \notin \mathbb{A}(R)$ by Lemma 3.2, either $Is \neq (0)$ or $Js \neq (0)$. Suppose that $Is \neq (0)$. Then $Is \in \mathbb{A}(R)^*$. As $rs = 0$, whereas $Ir \neq (0)$, we get that $Is \subset I$. It then follows from Lemma 3.1 that Is and I are not adjacent in $SAG(R)$. If $Is = (0)$, then $Js \neq (0)$. In such a case, $(I + Rr)Js = (0)$. Hence, $I + Rr \in \mathbb{A}(R)^*$. Since $r \notin I$, it follows that $I \subset I + Rr$ and hence, $I + Rr$ and I are not adjacent in $SAG(R)$.

Thus given $I \in \mathbb{A}(R)^*$, there exists $A \in \mathbb{A}(R)^* \backslash \{I\}$ such that A and I are not adjacent in $SAG(R)$. Therefore, $\gamma(SAG(R)) > 1$. ■

Theorem 3.4. *Let R be a reduced ring such that $|Min(R)| \geq 3$. Then $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$.*

Proof. Assume that R is a reduced ring with $|Min(R)| \geq 3$. It follows from Proposition 3.3 that $\gamma(SAG(R)) \geq 2$. Let $I \in \mathbb{A}(R)^*$. It is clear that $Ann_R(I) \in \mathbb{A}(R)^*$ and $I \neq Ann_R(I)$. We claim that $D = \{I, Ann_R(I)\}$ is a dominating set of $SAG(R)$. Let $A \in \mathbb{A}(R)^* \backslash D$. Suppose that A and I are not adjacent in $SAG(R)$. Then either $I \cap Ann_R(A) = (0)$ or $A \cap Ann_R(I) = (0)$. Suppose that $I \cap Ann_R(A) = (0)$. As both $I + Ann_R(I)$ and $A + Ann_R(A)$ do not belong to $\mathbb{A}(R)$ by Lemma 3.2, it follows that $Ann_R(I) \cap Ann_R(A) \neq (0)$ and $I \cap A \neq (0)$. As $I \subseteq Ann_R(Ann_R(I))$, we obtain that $A \cap Ann_R(Ann_R(I)) \neq (0)$. Thus $A \cap Ann_R(Ann_R(I)) \neq (0)$ and $Ann_R(I) \cap Ann_R(A) \neq (0)$. Hence, A and $Ann_R(I)$ are adjacent in $SAG(R)$. If $A \cap Ann_R(I) = (0)$, then A and $Ann_R(I)$ are adjacent in $\mathbb{AG}(R)$ and so, they are adjacent in $SAG(R)$. This proves that $D = \{I, Ann_R(I)\}$ is a dominating set of $SAG(R)$. As I and $Ann_R(I)$ are adjacent in $\mathbb{AG}(R)$, it follows that I and $Ann_R(I)$ are adjacent in $SAG(R)$. Therefore, D is a total dominating set of $SAG(R)$. This proves that $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$. ∎

Let R be the reduced ring given in Example 2.2. In the notation of the proof of Example 2.2, $Min(R) = \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$. It follows from Theorem 3.4 that $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$. It is already verified in Section 2 of this article that $\gamma(\mathbb{AG}(R)) = |Min(R)| = \gamma_t(\mathbb{AG}(R))$.

For any von Neumann regular ring R with $|Min(R)| \geq 3$, we obtain from Theorem 3.4 that $\gamma(SAG(R)) = \gamma_t(SAG(R)) = 2$. Some results on the domination number of $\mathbb{AG}(R)$ are already discussed in Section 2 of this article.

Let R be a non-reduced ring such that $|MNP(R)| = 1$. Let $MNP(R) = \{\mathfrak{p}\}$. If $\mathfrak{p} \in \mathbb{A}(R)$, then $\gamma(\mathbb{AG}(R)) = 1$ by Lemma 2.22(1). Since $\mathbb{AG}(R)$ is a spanning subgraph of $SAG(R)$, we get that $\gamma(SAG(R)) = 1$.

In Example 3.5, we provide a non-reduced ring R such that $|MNP(R)| = 1$, $MNP(R) \cap \mathbb{A}(R) = \emptyset$, and $\gamma(SAG(R)) = 1$.

Recall that a nonzero ideal I of a ring T is said to be an *essential ideal* of T if $I \cap J \neq (0)$ for each nonzero ideal J of T [1]. For convenience, we denote the set of all essential ideals of T by $\mathbb{E}(T)$.

Example 3.5. Let R be the ring considered in Example 2.28. Then R is a non-reduced ring, $|MNP(R)| = 1$, $MNP(R) \cap \mathbb{A}(R) = \emptyset$, and $\gamma(SAG(R)) = 1$.

Proof. In the notation of Example 2.28, $R = \frac{V}{Vm}$, where $(V, \mathfrak{m})$ is a rank one non-discrete valuation domain and $m \in \mathfrak{m}\backslash\{0\}$. It is noted in the proof of Example 2.28 that $dim R = 0$, $Spec(R) = Max(R) = MNP(R) = \{\mathfrak{p}\}$, where $\mathfrak{p} = \frac{\mathfrak{m}}{Vm}$. It is observed in the proof of Example 2.28 that $\mathfrak{p} = nil(R)$. Hence, R is not reduced. It is verified in the proof of Example 2.28 that $\mathfrak{p} \notin \mathbb{A}(R)$. Therefore, $MNP(R) \cap \mathbb{A}(R) = \emptyset$. Since the set of ideals of V is linearly ordered by inclusion, it follows that the set of ideals of R is linearly ordered by inclusion. Hence, each nonzero ideal of R is an essential ideal of R. So, for any distinct $I, J \in \mathbb{A}(R)^*$, $I \cap Ann_R(J) \neq (0 + Vm)$ and $J \cap Ann_R(I) \neq (0 + Vm)$. Hence, I and J are adjacent in $SAG(R)$. This shows that $SAG(R)$ is complete. Therefore, $\gamma(SAG(R)) = 1$. It is verified in the proof of Example 2.28 that $\mathbb{AG}(R)$ does not admit any finite dominating set. ■

Let R be a ring such that R admits at least one non-trivial idempotent. We prove in Proposition 3.6 that $\gamma(SAG(R)) \leq 2$.

Proposition 3.6. *Let R be a ring and let e be an idempotent element of R such that $e \notin \{0, 1\}$. Then $\gamma(SAG(R)) \leq 2$.*

Proof. By hypothesis, e is an idempotent element of R such that $e \notin \{0, 1\}$. Note that $Re, R(1-e) \in \mathbb{A}(R)^*$ with $Re \neq R(1-e)$. Let $D = \{Re, R(1-e)\}$. We claim that D is a dominating set of $SAG(R)$. Let $I \in \mathbb{A}(R)^* \backslash D$. There exists $r \in R\backslash\{0\}$ such that $Ir = (0)$. From $r = re + r(1-e)$, it follows that either $re \neq 0$ or $r(1-e) \neq 0$. Without loss of generality, we can assume that $re \neq 0$. Then $0 \neq re \in Ann_R(I) \cap Re$. If $I(1-e) \neq (0)$, then $I(1-e) \subseteq I \cap Ann_R(Re)$ and so, $I \cap Ann_R(Re) \neq (0)$. Hence, I and Re are adjacent in $SAG(R)$. If $I(1-e) = (0)$, then I and $R(1-e)$ are adjacent in $\mathbb{AG}(R)$ and so, they are adjacent in $SAG(R)$. This proves that D is a dominating set of $SAG(R)$ and so, $\gamma(SAG(R)) \leq 2$. ■

We provide in Example 3.8, a non-reduced ring R such that R has no non-trivial idempotent element but $\gamma(SAG(R)) = 1$. We use Lemma 3.7 in the proof of Example 3.8.

Lemma 3.7. *Let R be a non-reduced ring. Suppose that R admits a nonzero nilpotent ideal I with $Ann_R(I) = I$. Then $\{I\}$ is dominating set of $SAG(R)$.*

Proof. By hypothesis, I is a nonzero nilpotent ideal of R such that $Ann_R(I) = I$. Let $D = \{I\}$. As $Ann_R(I) = I$, it follows from [16, Lemma 2.2] that $I \in \mathbb{E}(R)$. Let $J \in \mathbb{A}(R)^*$ be such that $J \neq I$. As $Ann_R(J) \neq (0)$, it follows that $I \cap Ann_R(J) \neq (0)$ and $J \cap Ann_R(I) = J \cap I \neq (0)$. Hence, J and I are adjacent in $SAG(R)$. This proves that $\{I\}$ is a dominating set of $SAG(R)$. ■

Example 3.8. Let $R = \mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ be the ring obtained by using Nagata's principle of idealization. Then R is not a reduced ring, R has no non-trivial idempotent element, and $\gamma(SAG(R)) = 1$.

Proof. The ring $R = \mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}}$ is already considered in Example 2.25. Let $\mathfrak{p} = (0)(+)\frac{\mathbb{Q}}{\mathbb{Z}}$. Note that $\mathfrak{p}^2 = (0)(+)(0+\mathbb{Z})$ and so, R is not reduced. Let $e = (n, \alpha+\mathbb{Z})$ $(n \in \mathbb{Z}, \alpha \in \mathbb{Q})$ be an idempotent element of R. Then $n^2 = n$, and $2n\alpha+\mathbb{Z} = \alpha+\mathbb{Z}$. Therefore, $n \in \{0, 1\}$. If $n = 0$, then $\alpha + \mathbb{Z} = 0 + \mathbb{Z}$ and so, $e = (0, 0 + \mathbb{Z})$. If $n = 1$, then $\alpha + \mathbb{Z} = 0 + \mathbb{Z}$. Therefore, $e = (1, 0 + \mathbb{Z})$. This shows that R has no non-trivial idempotent element.

It is clear that $\mathfrak{p} \subseteq Ann_R(\mathfrak{p})$. Let $r = (n, \alpha + \mathbb{Z})$ $(n \in \mathbb{Z}, \alpha \in \mathbb{Q})$ be such that $r \in Ann_R(\mathfrak{p})$. Let $\mathbb{P}$ denote the set of all positive prime numbers. Then for any $p \in \mathbb{P}$, $(n, \alpha+\mathbb{Z})(0, \frac{1}{p}+\mathbb{Z}) = (0, 0+\mathbb{Z})$. This implies that $n(\frac{1}{p}+\mathbb{Z}) = 0+\mathbb{Z}$ and so, $n \in p\mathbb{Z}$. Thus $n \in \bigcap_{p\in\mathbb{P}} p\mathbb{Z} = (0)$. Therefore, $r \in \mathfrak{p}$. This proves that $Ann_R(\mathfrak{p}) \subseteq \mathfrak{p}$ and so, $Ann_R(\mathfrak{p}) = \mathfrak{p}$. It now follows from Lemma 3.7 that $\{\mathfrak{p}\}$ is a dominating set of $SAG(R)$. Therefore, $\gamma(SAG(R)) = 1$. It is verified in the proof of Example 2.25 that $MNP(R) = \{p\mathbb{Z}(+)\frac{\mathbb{Q}}{\mathbb{Z}} \mid p \in \mathbb{P}\}$ and $\gamma(\mathbb{AG}(R)) = |MNP(R)|$. ■

In Example 3.9, we provide non-reduced rings R, T such that $\gamma(SAG(R)) = 1$ and $\gamma(SAG(T)) = 2$ to illustrate Proposition 3.6.

Example 3.9. (1) Let $R = \mathbb{Z}_4\times\mathbb{Z}_4$. Then R is not reduced, R admits non-trivial idempotent elements, and $\gamma(SAG(R)) = 1$.

(2) Let $T = \mathbb{Z}_4 \times K$, where K is a field. Then T is not reduced, T admits non-trivial idempotent elements, and $\gamma(SAG(T)) = 2$.

Proof. (1) Since $\mathbb{Z}_4$ is not reduced, it follows that $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ is not reduced. Note that $(1, 0)$ and $(0, 1)$ are non-trivial idempotent elements of R. Let $D = \{2\mathbb{Z}_4 \times 2\mathbb{Z}_4\}$. We claim that D is a dominating set of $SAG(R)$. As $(2\mathbb{Z}_4)^2 = (0)$ and $Ann_{\mathbb{Z}_4}(2\mathbb{Z}_4) = 2\mathbb{Z}_4$, it follows that $Ann_R(2\mathbb{Z}_4 \times 2\mathbb{Z}_4) = 2\mathbb{Z}_4 \times 2\mathbb{Z}_4$. It now follows from Lemma 3.7 that D is a dominating set of $SAG(R)$. Therefore, $\gamma(SAG(R)) = 1$.

(2) Since $\mathbb{Z}_4$ is not reduced, it follows that $T = \mathbb{Z}_4 \times K$ is not reduced. Note that $(1, 0)$ and $(0, 1)$ are non-trivial idempotent elements of T. Hence, $\gamma(SAG(T)) \leq 2$ by Proposition 3.6. Observe that $\mathbb{A}(T)^* = \{v_1 = (0) \times K, v_2 = 2\mathbb{Z}_4 \times K, v_3 = 2\mathbb{Z}_4 \times (0), v_4 = \mathbb{Z}_4 \times (0)\}$. As $v_1v_3 = v_1v_4 = v_2v_3 = (0) \times (0)$, it follows that v_1 is adjacent to both v_3 and v_4 in $\mathbb{AG}(T)$ and so, in $SAG(T)$. The vertices v_2 and v_3 are adjacent in $\mathbb{AG}(T)$ and so, in $SAG(T)$. Now, $Ann_T(v_2) = 2\mathbb{Z}_4 \times (0)$. Hence, $v_1 \cap Ann_T(v_2) = (0) \times (0)$ and so, v_1 and v_2 are not adjacent in $SAG(T)$. Note that $Ann_T(v_2) \cap v_4 \neq (0) \times (0)$ and as $Ann_T(v_4) = (0) \times K$, $v_2 \cap Ann_T(v_4) \neq (0) \times (0)$. Therefore, v_2 and v_4 are adjacent in $SAG(T)$. As $v_3 \cap Ann_T(v_4) = (0) \times (0)$, we get that v_3 and v_4 are not adjacent in $SAG(T)$. Therefore, $SAG(T)$ is a cycle of length 4 given by $v_1 - v_3 - v_2 - v_4 - v_1$. (This fact has been noted in the proof of (2) $\Rightarrow$ (3) of [16, Theorem 2.2].) Hence, $\gamma(SAG(T)) \geq 2$ and so, $\gamma(SAG(T)) = 2$. ■

Let R be a ring such that $\mathbb{A}(R)^* \neq \emptyset$. Next, we discuss some results on the domination number of $\mathcal{AE}_R$, the sum-annihilating essential ideal graph of R. It is already noted in Section 1 that $\mathcal{AE}_R$ is a spanning supergraph of $\mathbb{AG}(R)$. If R is a non-reduced ring, then $\gamma(\mathcal{AE}_R) = 1$ [1, Corollary 8]. For any reduced ring R, we verify in Proposition 3.11 that $\mathbb{AG}(R) = \mathcal{AE}_R$. This fact has not been noted in [1]. We use Lemma 3.10 in the proof of Proposition 3.11.

Lemma 3.10. *Let R be a reduced ring and let I be a nonzero ideal of R. Then $I \in \mathbb{E}(R)$ if and only if $I \notin \mathbb{A}(R)$.*

Proof. Assume that $I \in \mathbb{E}(R)$. Let $r \in R\backslash\{0\}$. Then $I \cap Rr \neq (0)$. If $x = sr$ ($s \in R$) is a nonzero element of $I \cap Rr$, then $sr^2 \in Ir$ and since R is reduced, $sr^2 \neq 0$. Thus $Ir \neq (0)$ for any nonzero $r \in R$. Therefore, $I \notin \mathbb{A}(R)$.

Conversely, assume that $I \notin \mathbb{A}(R)$. Let J be a nonzero ideal of R. Then $IJ \neq (0)$ and so, $I \cap J \neq (0)$. Hence, $I \in \mathbb{E}(R)$. (This part of the proof is true for any non-reduced ring also.) ■

Proposition 3.11. *Let R be a reduced ring. Then $\mathbb{AG}(R) = \mathcal{AE}_R$.*

Proof. For any ring R (reduced or not), it is known that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathcal{AE}_R$. Assume that R is reduced. Let $I, J \in \mathbb{A}(R)^*$ be such that I and J are adjacent in $\mathcal{AE}_R$. Therefore, $Ann_R(I) + Ann_R(J) \in \mathbb{E}(R)$. Hence, $Ann_R(I) + Ann_R(J) \notin \mathbb{A}(R)$ by Lemma 3.10. As $(Ann_R(I) + Ann_R(J))IJ = (0)$, it follows that $IJ = (0)$. Hence, I and J are adjacent in $\mathbb{AG}(R)$. This shows that $\mathcal{AE}_R$ is a spanning subgraph of $\mathbb{AG}(R)$ and so, $\mathbb{AG}(R) = \mathcal{AE}_R$. ■

Acknowledgements

I am very much thankful to the referee for his/her valuable and encouraging comments and support and I am also very much thankful to the Editorial Board members of DM-GAA for their support.

References

[1] A. Alilou and J. Amjadi, *The sum-annihilating essential ideal graph of a commutative ring*, Commun. Comb. Optimization **1(2)** (2016) 117–135. https://doi.org/22049/CCO.2016.13555

[2] M.F. Atiyah and I.G. Macdonald, Introduction to Commutative Algebra (Addison-Wesley, Massachusetts, 1969).

[3] R. Balakrishnan and K. Ranganathan, A Textbook of Graph Theory, Second Edition (Universitext Springer, New York, 2012).

[4] I. Beck, *Coloring of commutative rings*, J. Algebra **116 (1)** (1988) 208–226. https://doi.org/10.1016/0021-8693(88)90202-5

[5] M. Behboodi and Z. Rakeei, *Annihilating-ideal graph of commutative rings* I, J. Algebra Appl. **10 (4)** (2011) 727–739. https://doi.org/10.1142/S0219498811004896

[6] M. Behboodi and Z. Rakeei, *Annihilating-ideal graph of commutative rings* II, J. Algebra Appl. **10 (4)** (2011) 741–753. https://doi.org/10.1142/S0219498811004902

[7] D.E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc. **27 (3)** (1971) 427–433.
https://doi.org/10.1090/S0002-9939-1971-0271100-6

[8] R. Gilmer, Multiplicative Ideal Theory (Marcel Dekker, New York, 1972).

[9] R. Gilmer and W. Heinzer, *The Laskerian property, power series rings and Noetherian spectra*, Proc. Amer. Math. Soc. **79 (1)** (1980) 13–16.
https://doi.org/10.1090/S0002-9939-1980-0560575-6

[10] W. Heinzer and D. Lantz, *The Laskerian property in commutative rings*, J. Algebra **72 (1)** (1981) 101–114.
https://doi.org/10.1016/0021-8693(81)90313-6

[11] W. Heinzer and J. Ohm, *On the Noetherian-like rings of E.G. Evans*, Proc. Amer. Math. Soc. **34 (1)** (1972) 73–74.
https://doi.org/10.1090/S0002-9939-1972-0294316-2

[12] I. Kaplansky, Commutative Rings (The University of Chicago Press, Chicago, 1974).

[13] H.B. Mann, Introduction to Algebraic Number Theory (Ohio State University Press, Columbus, Ohio, 1955).

[14] R. Nikandish and H.R. Maimani, *Dominating sets of Annihilating-ideal graphs*, Electronic Notes in Disc. Math. **45** (2014) 17–22.
https://doi.org/10.1016/j.endm.2013.11.005

[15] R. Nikandish, H.R. Maimani and S. Kiani, *Domination number in the annihilating-ideal graphs of commutative rings*, Publ. de l'Institut Math. Nouvelle serie tome 97 **111** (2015) 225–231.
https://doi.org/10.2298/PIM140222001N

[16] N. Kh. Tohidi, M.J. Nikmehr and R. Nikandish, *On the strongly annihilating-ideal graph of a commutative ring*, Discrete Math. Algorithms Appl. **9(2)** (2017) Art ID:1750028 (13 pages).
https://doi.org/10.1142/S1793830917500288

[17] S. Visweswaran and Premkumar T. Lalchandani, *The exact zero-divisor graph of a reduced ring*, Indian J. Pure Appl. Math. **52(4)** (2021) 1123–1144.
https://doi.org/10.1007/s13226-021-00086-9

Received 30 January 2023
Revised 8 July 2023
Accepted 10 July 2023