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# SOME ALGEBRAIC CHARACTERISATIONS OF GENERALISED MIDDLE BOL LOOPS

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**Abstract**

In this article, some algebraic characterisations of generalised middle Bol loop (GMBL) using its parastrophes and holomorph were studied. In particular, it was shown that if the generalised map $\alpha$ is bijective such that $\alpha : e \to e$, then the (12)-parastrophe of GMBL is a GMBL. The conditions for (13)- and (123)-parastrophes of a GMBL to be GMBL of exponent two were unveiled. We further established that a commutative (13)- and (123)-parastrophes of GMBL has an inverse properties. (23)-parastrophe of $Q$ was shown to have super $\alpha$-elastic property if it is a middle symmetric while (132)-parastrophe of $Q$ satisfies left $\alpha$-symmetric. It is further shown that a commutative (13)- and (123)-parastrophes of $Q$ are generalised Moufang

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loops of exponent two. Also, commutative (132)- and (23)-parastophes of $Q$ are shown to be Steiner loops. A necessary and sufficient condition for holomorph of generalised middle Bol loop to be GMBL was presented. The holomorph of a commutative loop was shown to be a commutative generalised middle Bol loop if and only if the loop is a GMBL.

**Keywords:** loop, parastrophe, holomorph, generalised middle Bol loop.

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## 1. Introduction

### 1.1. Quasigroups and loops

Let $Q$ be a non-empty set. Define a binary operation " $\cdot$ " on $Q$. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid or magma. If the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$ then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and let there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q, x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. At times, we shall write $xy$ instead of $x \cdot y$ and stipulate that " $\cdot$ " has lower priority than juxtaposition among factors to be multiplied. Let $(Q, \cdot)$ be a groupoid and $a$ be a fixed element in $Q$, then the left and right translations $L_a$ and $R_a$ of $a$ are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid

$(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings $L_a^{-1}$ and $R_a^{-1}$ exist.

Let

$$a \backslash b = bL_a^{-1} = aM_b \quad \text{and} \quad a/b = aR_b^{-1} = bM_a^{-1}$$

and note that

$$a \backslash b = c \Longleftrightarrow a \cdot c = b \quad \text{and} \quad a/b = c \Longleftrightarrow c \cdot b = a.$$

Thus, for any quasigroup $(Q, \cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$. $M_a$ is the middle translation for any fixed $a \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

**Definition 1.1.** A *loop* $(Q, \cdot, /, \backslash, e)$ is a set $Q$ together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation $e$ such that

(i) $a \cdot (a\backslash b) = b$, $(b/a) \cdot a = b$ for all $a, b \in Q$,

(ii) $a\backslash a = b/b$ or $e \cdot a = a \cdot e = a$ for all $a, b \in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $a \cdot b/c$ and $a \cdot b\backslash c$ stand for $a(b/c)$ and $a(b\backslash c)$, respectively.

In a loop $(Q, \cdot)$ with identity element $e$, the *left inverse element* of $x \in Q$ is the element $x\lambda J = x^{\lambda} \in Q$ such that

$$x^{\lambda} \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $x\rho J = x^{\rho} \in G$ such that

$$x \cdot x^{\rho} = e.$$

It is well known that every quasigroup $(Q\cdot)$ belongs to a set of six quasigroups, called adjugates by (Fisher, Yates [6] 1934), conjugates by (Stein, 1957) and parastrophes by (Belousov [4], 1967).

A binary groupoid $(Q, A)$ with a binary operation "$A$" such that in the equality $A(x_1, x_2) = x_3$ knowledge of any 2 elements of $x_1, x_2, x_3$ uniquely specifies remaining one is called a binary quasigroup. It follows that any quasigroup $(Q, A)$, associate $(3! - 1)$ quasigroups called parastrophes of quasigroup $(Q, A); A(x_1, x_2) = x_3 \iff A^{(12)}(x_2, x_1) = x_3 \iff A^{(13)}(x_3, x_2) = x_1 \iff A^{(23)}(x_1, x_2) = x_2 \iff A^{(123)}(x_2, x_3) = x_1 \iff A^{(132)}(x_3, x_1) = x_2$. [See (Shcherbacov [29], 2008)]. For more on quasigroups and loops, check [28, 30].

### 1.2. Middle Bol loop and its generalisation

**Definition 1.2.** A loop $(Q, \cdot)$ is called a middle Bol loop if

$$(x/y)(z\backslash x) = (x/(zy))x \text{ or } (x/y)(z\backslash x) = x((zy)\backslash x) \tag{1}$$

for all $x, y \in Q$.

Middle Bol loop were first studied in the work of Belousov [4], where he gave identity (1) characterizing loops that satisfy the universal anti-automorphic inverse property. After this beautiful characterisation by Belousov and the laying of foundations for a classical study of this structure, Gwaramija in [10] gave isostrophic connection between right (left) Bol loop and middle Bol loop.

Grecu [7] showed that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. After that, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [32, 33] considered them in relation to the universality of the elasticity law. In 2003, Kuznetsov [19], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup.

In 2010, Syrbu [34] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right, middle) nuclei, the set of Moufang elements, the center of a middle Bol loop and left Bol loop were established. In 2012, Grecu and Syrbu [8] proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic.

In 2012, Drapal and Shcherbacov [5] rediscovered the middle Bol identities in a new way. In 2013, Syrbu and Grecu [31] established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In 2014, Grecu and Syrbu [9] established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between middle Bol loop and the corresponding right Bol loop and the same authors presented a study of loops with invariant flexibility law under the isostrophy of loop [35].

In 2017, Jaiyéọlá *et al.* [12] presented the holomorphic structure of middle Bol loop and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Adeniran et al. [1], Jaiyéọlá and Popoola [17] studied generalised Bol loops. It was revealed in [14] that isotopy-isomorphy is a necessary and sufficient condition for any distinct quasigroups to be parastrophic invariance relative to the associative law.

(Osoba *et al.* [20] and [21]) investigate further the multiplication group of middle Bol loop in relation to left Bol loop and the relationship of multiplication groups and isostrophic quasigroups respectively while Jaiyéọlá [15, 16] studied second Smarandache Bol loops. The Smarandache nuclei of second Smarandache Bol loops was further studied by Osoba [22].

(Jaiyéọlá *et al.* [11], 2015) in furtherance to their exploit obtained new algebraic identities of middle Bol loop, where necessary and sufficient conditions for a bi-variate mapping of a middle Bol loop to have RIP, LIP, RAIP, LAIP and flexible property were presented. Additional algebraic properties of middle Bol loop were announced in (Jaiyéọlá *et al.* [13], 2021).

The new algebraic connections between right and middle Bol loops and their cores were unveiled by (Osoba and Jaiyéọlá (2022), [23]). More results on the algebraic properties of middle Bol loops using its parastrophes was presented by (Oyebo and Osoba, [27]). The paper revealed some of the algebraic properties the parastrophic structures of middle Bol loop shared with its underline structure. The connections between middle Bol loop and right Bol loop with their crypto-automorphism features were unveiled in [26] by Oyebo *et al.* In [18], Bryant-Schneider group of middle Bol loop with some of the isostrophy-group invariance results was linked. It was further shown that some subgroups of the Bryant-Schneider group of a middle Bol loop were isomorphic to the automorphism and pseudo-aumorphism groups of its corresponding right (left) Bol loop.

A generalised middle Bol loop characterised by

$$(x/y)(z^{\alpha}\backslash x^{\alpha}) = x(z^{\alpha}y\backslash x^{\alpha}) \tag{2}$$

was first introduced in [2], as a consequence of a generalised Moufang loop with universal $\alpha$-elastic property where the map $\alpha : Q \mapsto Q$ is a homomorphism. Thus, if $\alpha : x \mapsto x$, then identity of generalised middle Bol loop reduces to the identity of middle Bol loop. The authors in [3], presented the basic algebraic properties of generalised middle Bol loop, where it revealed the necessary and sufficient conditions for the identity to satisfies left (right) inverse and $\alpha$-alternative property was also presented.

In 2024, Osoba and Oyebo [25] studied the isostrophic invariance of $\alpha$-elastic property and showed that a commutative loop $(Q, \cdot)$ with invariant $\alpha$-elastic property under the isostrophy of loops is a generalized Moufang loop while Osoba *et al.* in [24] established the subgroup of Bryant-Schneider group of a generalised middle Bol loop.

Furtherance to earlier studies, this paper investigates some structural characterisation of generalised middle Bol loop using its parastrophes and holomorph. The second section provides preliminaries for necessary background of the study. Section 3 contains the main results where the parastrophic characterisation of generalised middle Bol loop is presented. It is shown that a (12)-parastrophe of a generalised middle Bol is also a generalised middle Bol loop and further established the conditions for (13)- and (123)-parastrophes of $Q$ to be GMBL. We further investigate the algebraic properties of the parastrophes to obtain some of the related properties and identities they share with the underline structure. Interestingly, some new identities are found. In the fourth section, the holomorphic characterisations of generalised middle Bol loop is studied and the necessary and sufficient condition is found.

## 2. PRELIMINARIES

**Definition 2.1.** A loop $(Q, \cdot, /, \backslash)$ is called a generalised middle Bol loop if is satisfies the identity

$$(x/y)(z^{\alpha}\backslash x^{\alpha}) = (x/(z^{\alpha}y))x^{\alpha}. \tag{3}$$

**Definition 2.2.** For any non-empty set $Q$, the set of all permutations on $Q$ forms a group $SYM(Q)$ called the symmetric group of $Q$. Let $(Q, \cdot)$ be a loop and let

$A, B, C \in SYM(Q)$. If

$$xA \cdot yB = (x \cdot y)C \ \forall \ x, y \in Q,$$

then the triple $(A, B, C)$ is called an autotopism (ATP) and such triples form a group $AUT(Q, \cdot)$ called the autotopism group of $(Q, \cdot)$. Also, suppose that

$$xA \cdot yB = (y \cdot x)C \ \forall \ x, y \in Q$$

then the triple $(A, B, C)$ is called anti-autotopism (AATP). If $A = B = C$, then $A$ is called an automorphism of $(Q, \cdot)$ which form a group $AUM(Q, \cdot)$ called the automorphism group of $(Q, \cdot)$.

**Definition 2.3.** A groupoid (quasigroup) $(Q, \cdot)$ is said to have the

1. left inverse property $(LIP)$ if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x, y \in Q$,
2. right inverse property $(RIP)$ if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x, y \in Q$,
3. inverse property (IP) if it has both the LIP and RIP,
4. flexibility or elasticity if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$,
5. $\alpha$-elastic if $xy \cdot x^\alpha = x \cdot yx^\alpha$ holds for all $x, y \in Q$,
6. super $\alpha$-elastic if $(x \cdot y^\alpha) \cdot x^\alpha = x \cdot (y^\alpha \cdot x^\alpha)$ holds for all $x, y \in Q$,
7. cross inverse property $(CIP)$ if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot yx = y$ or $x^\lambda y \cdot x = y$ for all $x, y \in Q$.

**Definition 2.4.** A loop $(Q, \cdot)$ is said to be

1. commutative loop if $R_x = L_x$ and a commutative square loop if $R_x^2 = L_x^2$ for all $x, y \in Q$,
2. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$,
3. an anti- automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$.

**Definition 2.5** [28]**.** Moufang loops are loops satisfying the identities $(xy \cdot z)y = x(y \cdot zy), yz \cdot xy = y(zx \cdot y)$ and $(yz \cdot y)x = y(z \cdot yx)$.

**Definition 2.6.** A groupoid (quasigroup) $(Q, \cdot)$ is

1. right symmetric if $yx \cdot x = y$ for all $x, y \in Q$,
2. left symmetric if $x \cdot xy = y$ for all $x, y \in Q$,

3. middle symmetric if $x \cdot yx = y$ or $xy \cdot x = y$ for all $x, y \in Q$,
4. idempotent if $x \cdot x = x$ for all $x \in Q$,
5. right $\alpha$-symmetric if $y^\alpha x \cdot x = y^\alpha$ for all $x, y \in Q$,
6. left $\alpha$-symmetric if $x \cdot xy^\alpha = y^\alpha$ for all $x, y \in Q$,
7. middle $\alpha$-symmetric if $x \cdot y^\alpha x = y^\alpha$ or $xy^\alpha \cdot x = y$ for all $x, y \in Q$,
8. super middle $\alpha$-symmetric if $x \cdot (y^\alpha \cdot x^\alpha) = y^\alpha$ or $(x \cdot y^\alpha) \cdot x^\alpha = y^\alpha$ for all $x, y \in Q$.

**Definition 2.7.** A quasigroup $(Q, \cdot)$ is totally symmetric if any relation $xy = z$ implies any other such relation can be obtained by permuting $x$, $y$ and $z$.

**Definition 2.8** [28]**.** If a totally symmetric quasigroup $(Q, \cdot)$ is a loop, then it is called Steiner loop.

**Theorem 2.1** [28]**.** *A quasigroup $(Q, \cdot)$ is totally symmetric if and only if it is commutative $(xy = yx)$ for all $x, y \in Q)$ and is right or left symmetric.*

**Theorem 2.2** [28]**.** *A loop $(Q, \cdot)$ is totally symmetric if and only if $(Q, \cdot)$ is an IP loop of exponent 2.*

**Corollary 2.1** [28]**.** *Every T.S. quasigroup is a commutative I.M. quasigroup.*

**Definition 2.9.** Let $(Q, \cdot)$ be a loop. The pair $(H, \circ) = H(Q, \cdot)$ given by

$$H = A(Q) \times Q, \text{ where } A(Q) \leq AUT(Q, \cdot) \text{ such that } (\phi, x) \circ (\psi, y) = (\phi\psi, x\psi \cdot y)$$

for all $(\phi, x), (\psi, y) \in H$ is called the $A(H)$-Holomorph of $(Q, \cdot)$.

**Lemma 2.1** [12]**.** *Let $(L, \cdot, /, \backslash)$ be a loop with holomorph $G(L, \cdot)$. Then, $G(L, \cdot)$ is a commutative if and only if $A(L, \cdot)$ is an abelian group and $(\psi, \phi^{-1}, I_e) \in AATP(L, \cdot)$ for all $\phi, \psi \in A(L)$.*

**Definition 2.10** [29]**.** Let $(Q, \cdot)$ be a quasigroup with left $e_l$ and right $e_r$ identity elements. $(Q, \cdot)$ is called:

1. a left loop if $e_l \cdot x = x$ $\forall x \in Q$,
2. a right loop if $x \cdot e_r = x$ $\forall x \in Q$,
3. a loop if $e_l \cdot x = x \cdot e_r = x$ $\forall x \in Q$.

A quasigroup $(Q, \cdot)$, for which $e_l = e_r$ is called a loop. In more general note $e_l = e_r = e$.

## 3. MAIN RESULTS

### 3.1. Some algebraic connections between identities (2) and (3)

Here, we uncovered some characterisations of the two identities of GBML: (2) and (3), and further established that they are equivalent.

**Lemma 3.1.** *Let $(Q, \cdot)$ be a loop. Let $x, y, z$ be arbitrary elements in $Q$.*

1. *If $(Q, \cdot)$ obeys identity (2) such that $\alpha : e \mapsto e$, then*
   (a) $(x/y) \cdot x^\alpha = x \cdot (y \backslash x^\alpha)$,
   (b) $y^\lambda \cdot (z^\alpha)^\rho = (z^\alpha \cdot y)^\rho$,
   (c) $y^\lambda = y^\rho$.
2. *If $(Q, \cdot)$ obeys identity (3) such that $\alpha : e \mapsto e$, then*
   (a) $x \cdot (z^\alpha \backslash x^\alpha) = (x/z^\alpha) \cdot x^\alpha$,
   (b) $y^\lambda \cdot (z^\alpha)^\rho = (z^\alpha \cdot y)^\lambda$,
   (c) $(z^\alpha)^\lambda = (z^\alpha)^\rho$.
3. *If $(Q, \cdot)$ obeys identity (2) such that $\alpha$ is bijective and $\alpha : e \mapsto e$, then*
   (a) $(x/y) \cdot x^\alpha = x \cdot (y \backslash x^\alpha)$,
   (b) $y^\lambda \cdot z^\rho = (z \cdot y)^\rho$,
   (c) $y^\lambda = y^\rho$.
4. *If $(Q, \cdot)$ obeys identity (3) such that $\alpha$ is bijective and $\alpha : e \mapsto e$, then*
   (a) $x \cdot (z \backslash x^\alpha) = (x/z) \cdot x^\alpha$.
   (b) $y^\lambda \cdot z^\rho = (z \cdot y)^\lambda$,
   (c) $z^\lambda = z^\rho$.
5. *Let $\alpha : e \mapsto e$. Then, $(Q, \cdot)$ obeys identity (2) if and only if $(Q, \cdot)$ obeys identity (3) and $(x/y) \cdot x^\alpha = x \cdot (y \backslash x^\alpha)$.*
6. *Let $\alpha$ be bijective such that $\alpha : e \mapsto e$. Then, $(Q, \cdot)$ obeys identity (2) if and only if $(Q, \cdot)$ obeys identity (3).*

***Proof.*** 1. Assume that $(Q, \cdot)$ obeys the identity (2) such that $\alpha : e \mapsto e$.

(a) Put $z = e$ in (2) to get $(x/y) \cdot (e^\alpha \backslash x^\alpha) = x \cdot ((e^\alpha \cdot y) \backslash x^\alpha)$ which gives $(x/y) \cdot x^\alpha = x \cdot (y \backslash x^\alpha)$.

(b) In (2), put $x = e$ to get $(e/y) \cdot (z^\alpha \backslash e^\alpha) = e \cdot ((z^\alpha \cdot y) \backslash e^\alpha)$ to get $y^\lambda \cdot (z^\alpha)^\rho = (z^\alpha \cdot y)^\rho$.

(c) In (b), put $z = e$ to get $y^\lambda = y^\rho$.

2. Assume that $(Q, \cdot)$ obeys the identity (3) such that $\alpha : e \mapsto e$. Do similarly step as 1 to prove (a), (b) and (c).

3. Assume that $(Q, \cdot)$ obeys identity (2) such that $\alpha$ is bijective and $\alpha : e \mapsto e$. Then the proofs of (a), (b) and (c) follow up from 1.

4. Assume that $(Q, \cdot)$ obeys identity (3) such that $\alpha$ is bijective and $\alpha : e \mapsto e$. Then the proofs of (a), (b) and (c) follow up from 2.

5. Let $\alpha : e \mapsto e$. If $(Q, \cdot)$ obeys identity (2), then it obeys identity (3) because it satisfies $(x/y) \cdot x^\alpha = x \cdot (y \backslash x^\alpha)$ by 1. The converse follows by reversing the process.

6. This follows from 5. ∎

Henceforth, we shall assume that in a generalised middle Bol loop identity (2) or (3), the map $\alpha : Q \to Q^i$, where $i = (12), (13), (23), (123), (132)$, is a bijective map such that $\alpha : e \mapsto e$. Note that $J : x \mapsto x^{-1}$.

### 3.2. Parastrophes of generalised middle Bol loop

We now look at characterisation of the parastrophe of identity 2.

**Lemma 3.2.** *Let $(Q, \cdot)$ be a quasigroup with $e_l$ and $e_r$ be the identity elements:*

(a)
1. (12)-*parastrophe of a left loop is a right loop,*
2. (12)-*parastrophe of a right loop is a left loop,*
3. (12)-*parastrophe of a loop is also a loop,*

(b)
1. (13)-*parastrophe of a left loop is a not loop,*
2. (13)-*parastrophe of right loop is a right loop,*
3. (13)-*parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$,*

(c)
1. (23)-*parastrophe of a left loop is a left loop,*
2. (23)-*parastrophe of right loop is not a loop,*
3. (23)-*parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$,*

(d)
1. (123)-*parastrophe of a left loop is a not loop,*
2. (123)-*parastrophe of right loop is a left loop,*
3. (123)-*parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$,*

(e)
1. (132)-*parastrophe of a left loop is a right loop,*
2. (132)-*parastrophe of right loop is not a loop,*
3. (132)-*parastrophe of loop is a loop if and only if $|x| = 2$ for all $x \in Q$.*

***Proof.*** (a) " $\circ_{(12)}$ " denotes the operation of (12)-parastrophe of $Q$. If $(Q,\cdot)$ is a left loop, then $e_l \cdot x = x$ this implies that (12)-parastrophe of $Q$ is $x \circ_{(12)} e_r = x$ for all $x \in Q$. $(Q,\cdot)$ is right loop if $x \circ_{(12)} e_r = x \Rightarrow$ (12)-parastrophe of $Q$ is $e_l \circ_{(12)} x = x$ for all $x \in Q$. Therefore, (12)-parastrophe of $Q$ is a loop.

(b) (13)-parastrophe of a left loop is given as $x \circ_{(13)} x = e_l$. This is only possible iff $|x| = 2$ for all $x \in Q$. Conversely, suppose that (13)-parastrophe of a left loop is of exponent 2, this implies that $x^\lambda = x$, then $x^\lambda \cdot x = e_l$. Also, if $(Q,\cdot)$ is right loop, then (13)-parastrophe of $Q$ is also loop, that is $x \circ_{(13)} e_r = x$. Thus, $x^\lambda = x^\rho = x$ Therefore, (13)-parastrophe of $Q$ is a loop if and only if $|x| = 2$. Similar results are obtained for (c), (d) and (e). ∎

**Theorem 3.1.** *Let $(Q,\cdot,/,\backslash)$ be a generalised middle Bol loop. Then,* (12)-*parastrophe of $Q$ is also a generalised middle Bol loop.*

***Proof.*** Let

$$a \cdot b = x(z^\alpha y \backslash x^\alpha) \tag{4}$$

in equation (2) where $a = x/y \Rightarrow x = ay \underbrace{\Rightarrow}_{\text{by (12)-permuation}} y \circ_{(12)} a = x \Rightarrow a = y\backslash^{(12)} x$. And $b = z^\alpha \backslash x^\alpha \Rightarrow z^\alpha b = x^\alpha \underbrace{\Rightarrow}_{\text{take (12)-permuation}} b \circ_{(12)} z^\alpha = x^\alpha \Rightarrow b = x^\alpha /^{(12)} z^\alpha$.

Substitute for $a$ and $b$ into equation (4), give

$$\big(y\backslash^{(12)}x\big) \cdot \big(x^\alpha/^{(12)}z^\alpha\big) = x(z^\alpha y\backslash x^\alpha). \tag{5}$$

Applying (12)-permutation on equation (5), to get

$$\big(x^\alpha/^{(12)}z^\alpha\big) \circ_{(12)} \big(y\backslash^{(12)}x\big) = \big((y \circ_{(12)} z^\alpha)\backslash x^\alpha\big) \circ_{(12)} x. \tag{6}$$

Let

$$(y \circ_{(12)} z^\alpha)\backslash x^\alpha = c \Rightarrow (y \circ_{(12)} z^\alpha) \cdot c = x^\alpha \underbrace{\Rightarrow}_{\text{take (12)-permuation}} c \circ_{(12)} (y \circ_{(12)} z^\alpha) = x^\alpha$$

$$\Rightarrow c = x^\alpha/^{(12)}(y \circ_{(12)} z^\alpha).$$

Put $c$ into equation (6) and make the substitution $x \leftrightarrow x^\alpha$, $z^\alpha \leftrightarrow y$, one obtains

$$\big(x/^{(12)}y\big) \circ_{(12)} \big(z^\alpha\backslash^{(12)}x^\alpha\big) = \big(x/^{(12)}(z^\alpha \circ_{(12)} y)\big) \circ_{(12)} x^\alpha.$$ ∎

**Lemma 3.3.** *Let $(Q,\cdot,/,\backslash)$ be a generalised middle Bol loop. Then, the* (13)-*parastrophe of $Q$ is given by*

$$(x \circ_{(13)} y)/^{(13)}\big(x^\alpha\backslash^{(13)}z^\alpha\big) = x/^{(13)}\big[x^\alpha\backslash^{(13)}\big(z^\alpha/^{(13)}y\big)\big]. \tag{7}$$

***Proof.*** Let

$$a \cdot b = x\big(z^\alpha y \backslash x^\alpha\big) \tag{8}$$

in equation (2), where

$$a = x/y \Rightarrow x = ay \underbrace{\Rightarrow}_{\text{taking (13)-permuation}} a = x \circ_{(13)} y \tag{9}$$

and

$$b = z^\alpha \backslash x^\alpha \Rightarrow z^\alpha b = x^\alpha \underbrace{\Rightarrow}_{\text{take (13)-permuation}} z^\alpha = x^\alpha \circ_{(13)} b \Rightarrow x^\alpha \backslash^{(13)} z^\alpha = b. \tag{10}$$

Let $c = z^\alpha y$ in identity (2), this implies that $z^\alpha = \underbrace{c \circ_{(13)} y}_{\text{(13)-permutation}} \Rightarrow c = z^\alpha /^{(13)} y$.

Also, let $d = c \backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \underbrace{\Rightarrow}_{\text{by taking (13)-permuation}} x^\alpha \circ_{(13)} d = c \Rightarrow d = x^\alpha \backslash^{(13)} c$. Then, substituting $c$ into $d$, we have

$$d = x^\alpha \backslash^{(13)} (z^\alpha /^{(13)} y). \tag{11}$$

Let $s = x \cdot d \Rightarrow x = s \circ_{(13)} d \Rightarrow s = x /^{(13)} d \underbrace{\Rightarrow}_{\text{substitute } d \text{ into } s}$

$$s = x /^{(13)} \big[x^\alpha \backslash^{(13)} (z^\alpha /^{(13)} y)\big]. \tag{12}$$

Now, according to identity (2), we have $a \cdot b = s \Rightarrow \underbrace{s \circ_{(13)} b}_{\text{(13)-permutaion}} = a \Rightarrow a /^{(13)} b = s$. Substituting (9), (10) and (12) into the last equality, we have

$$(x \circ_{(13)} y) /^{(13)} \big(x^\alpha \backslash^{(13)} z^\alpha\big) = x /^{(13)} \big[x^\alpha \backslash^{(13)} (z^\alpha /^{(13)} y)\big]$$

which is the (13)-parastrophe of $Q$ as required. ■

**Theorem 3.2.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following hold in (13)-parastrophe of $Q$*

1. $(L_x, L_{x^\alpha}^{-1}, L_{x^\alpha}^{-1} M_x^{-1}) \in AATP(Q, /^{(13)})$,
2. $t^\lambda \circ_{(13)} (t \circ_{(13)} y) = y$ *that is left inverse property for all $t \in Q$,*
3. $L_x R_{(x^\alpha)^\rho}^{-1} = \lambda J L_{x^\alpha}^{-1} M_x^{-1}$
4. $L_x M_x = L_{x^\alpha}^{-1} M_{x^\alpha}^{-1}$,
5. $x /^{(13)} (x^\alpha)^\rho = (x \cdot_{(13)} y) /^{(13)} (x \backslash^{(13)} y)$ *for all $x, y \in Q$,*

6. $y = (y^\lambda)^\lambda$ *for all* $y \in Q$,
7. $L_x R^{-1}_{(x^\alpha)^\lambda} = \lambda J L_{(x^\alpha)^\lambda} M_x^{-1}$.

***Proof.*** 1. From equation (7) of Lemma 3.3, we have

$$yL_x/^{(13)}z^\alpha L^{-1}_{x^\alpha} = \big(z^\alpha/^{(13)}y\big)L^{-1}_{x^\alpha}M_x^{-1} \Rightarrow \big(L_x, L^{-1}_{x^\alpha}, L^{-1}_{x^\alpha}M_x^{-1}\big) \in AATP\big(Q, /^{(13)}\big).$$

2. Let $x = e \Rightarrow e^\alpha \to e$ the identity element in $Q$, in equation (7), we have

$$\begin{aligned} ey/^{(13)}z^\alpha &= e/^{(13)}\big(z^\alpha/^{(13)}y\big) \Rightarrow y/^{(13)}z^\alpha \\ &= \big(z^\alpha/^{(13)}y\big)^\lambda \Rightarrow y = \big(z^\alpha/^{(13)}y\big)^\lambda \circ_{(13)} z^\alpha. \end{aligned} \tag{13}$$

Let $t = z^\alpha/^{(13)}y \Rightarrow z^\alpha = t\circ_{(13)}y$, put $z^\alpha$ and $t$ in (13), give $y = t^\lambda\circ_{(13)}(t\circ_{(13)}y)$ for all $t \in Q$.

3. Let $z = e$ and $e^\alpha \mapsto e$ in equation (7), we have $(x \circ_{(13)} y)/(x^\alpha)^\rho = x/^{(13)}(x^\alpha\backslash^{(13)}y^\lambda) \Rightarrow yL_xR^{-1}_{(x^\alpha)^\rho} = y\lambda L^{-1}_{x^\alpha}M_x^{-1} \Rightarrow L_xR^{-1}_{(x^\alpha)^\rho} = \lambda J L^{-1}_{x^\alpha}M_x^{-1}$.

4. Let $z = x$ in equation (7), we have $x \circ_{(13)} y = x/^{(13)}\big(x^\alpha\backslash^{(13)}(x^\alpha/^{(13)}y) \Rightarrow yL_x = yM^{-1}_{x^\alpha}L^{-1}_{x^\alpha}M_x^{-1} \Rightarrow L_xM_x = L^{-1}_{x^\alpha}M^{-1}_{x^\alpha}$.

5. Let $z = y$ and $y^\alpha \mapsto y$ in equation (7), give $x/^{(13)}(x^\alpha)^\rho = (x \circ_{(13)} y)/^{(13)}(x\backslash^{(13)}y)$.

6. Let $z = x = e$ in equation (7), we obtain $(y^\lambda)^\lambda = y$.

7. Apply 2 to equation (7), to get $(x\circ_{(13)}y)/^{(13)}((x^\alpha)^\lambda\circ_{(13)}z^\alpha) = x/^{(13)}((x^\alpha)^\lambda \circ_{(13)}(z^\alpha/^{(13)}y))$. Let $z^\alpha \mapsto e$ to get $x \circ_{(13)} y/^{(13)}(x^\alpha)^\lambda = x/^{(13)}((x^\alpha)^\lambda \circ_{(13)} y^\lambda) \Rightarrow yL_xR^{-1}_{(x^\alpha)^\lambda} = y\lambda JL_{(x^\alpha)^\lambda}M_x^{-1} \Rightarrow L_xR^{-1}_{(x^\alpha)^\lambda} = \lambda JL_{(x^\alpha)^\lambda}M_x^{-1}$. ■

**Corollary 3.1.** *In* (13)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$, *the following hold:*

1. $(x^\alpha)^\rho = (x^\alpha)^\lambda\ \forall x \in Q$,
2. $x^\rho = x^\lambda\ \forall x \in Q$.

***Proof.*** From 7 of Theorem 3.2, we have $L_xR^{-1}_{(x^\alpha)^\lambda} = \lambda JL^{-1}_{x^\alpha}M_x^{-1}$. Recall from 3 of Theorem 3.2, $L_xR^{-1}_{(x^\alpha)^\rho} = \lambda JL^{-1}_{x^\alpha}M_x^{-1}$. This implies that $L_xR^{-1}_{(x^\alpha)^\rho} = L_xR^{-1}_{(x^\alpha)^\lambda} \Rightarrow R^{-1}_{(x^\alpha)^\rho} = R^{-1}_{(x^\alpha)^\lambda} \Rightarrow (x^\alpha)^\rho = (x^\alpha)^\lambda$. Since $\alpha$ is bijective, we have $x^\rho = x^\lambda\ \forall x \in Q$. ■

**Remark 3.1.** The above Corollary shows that in (13)-parastrophe of a generalised middle Bol loop $(Q, \cdot, /, \backslash)$, the right and the left inverse properties coincide. So, the (13)-parastrophe satisfies IP if it is commutative. Also, if $|Q^{(13)}| = 2$, then $x^\rho = x^\lambda = x\ \forall x \in Q$. Thus, (13)-parastrophe of $Q$ is a loop.

**Corollary 3.2.** *A commutative* (13)*-parastrophe of a generalised middle Bol loop* $(Q,\cdot,/,\backslash)$*, satisfies AAIPL if* $|x| = 2\ \forall\ x \in Q$.

***Proof.*** Based on the Remark (3.1), the identity (7) become $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = x \circ_{(13)} \left[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\right]^{-1}$. Let $x = e$ to get $(e \circ_{(13)} y) \circ_{(13)} (e^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = e \circ_{(13)} \left[(e^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\right]^{-1}$, then $y \circ_{(13)} (z^\alpha)^{-1} = (z^\alpha \circ_{(13)} y^{-1})^{-1} \Rightarrow y^{-1} \circ_{(13)} (z^\alpha)^{-1} = (z^\alpha \circ_{(13)} y)^{-1}$. ■

**Corollary 3.3.** *A commutative* (13)*-parastrophe of a generalised middle Bol loop* $(Q,\cdot,/,\backslash)$ *is a Steiner loop if it is a loop of exponent two.*

***Proof.*** This is a consequence of 2 of theorem 3.2 and the Corollary 3.1. ■

**Theorem 3.3.** *A commutative* (13)*-parastrophe, of exponent two, of a generalised middle Bol loop* $(Q,\cdot,/,\backslash)$ *is a Moufang loop.*

***Proof.*** From Remark (3.1), we have the identity (7) to be $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha)^{-1} \circ_{(13)} (z^\alpha)^{-1} = x \circ_{(13)} \left[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\right]^{-1}$. Since $|Q^{(13)}| = 2$, we have $(x \circ_{(13)} y) \circ_{(13)} (x^\alpha \circ_{(13)} z^\alpha) = x \circ_{(13)} \left[x^\alpha \circ_{(13)} (z^\alpha \circ_{(13)} y)\right] \Rightarrow z^\alpha L_{x^\alpha} L_{xy} = z^\alpha R_y L_{x^\alpha} L_x \Rightarrow z^\alpha R_{x^\alpha} L_{xy} = z^\alpha L_y L_{x^\alpha} L_x \Rightarrow (x \circ_{(13)} y) \circ_{(13)} (z^\alpha \circ_{(13)} x^\alpha) = x \circ_{(13)} ((y \circ_{(13)} z^\alpha) \circ_{(13)} x^\alpha)$. ■

**Corollary 3.4.** *In* (13)*-parastrophe, of exponent two, of a generalised middle Bol loop* $(Q,\cdot,/,\backslash)$ *is a GMBL.*

***Proof.*** Follow from Theorem 3.3, we have $(x \circ_{(13)} y) \circ_{(13)} [(x^\alpha)^{-1} \circ_{(13)} (z^\alpha)]^{-1} = x \circ_{(13)} \left[(x^\alpha)^{-1} \circ_{(13)} (z^\alpha \circ_{(13)} y^{-1})\right]^{-1}$. Use $y^{-1} = y$ and Corollary 3.2 to get $(x \circ_{(13)} y^{-1}) \circ_{(13)} ((z^\alpha)^{-1} \circ_{(13)} x^\alpha) = x \circ_{(13)} \left[(z^\alpha \circ_{(13)} y)\right]^{-1} \circ_{(13)} x^\alpha \Rightarrow (x/^{(13)} y) \circ_{(13)} (z^\alpha \backslash^{(13)} x^\alpha) = x \circ_{(13)} [(z^\alpha \circ_{(13)} y) \backslash^{(13)} x^\alpha]$. ■

**Lemma 3.4.** *Let* $(Q,\cdot,/,\backslash)$ *be a generalised middle Bol loop. Then, the* (23)*-parastrophe of* $Q$ *is given by*

$$\big(y/^{(23)}x\big)\backslash^{(23)}(z^\alpha \circ_{(23)} x^\alpha) = x\backslash^{(23)}\big[(z^\alpha\backslash^{(23)}y) \circ_{(23)} x^\alpha\big]. \tag{14}$$

***Proof.*** Let

$$a \cdot b = x(z^\alpha y\backslash x^\alpha) \tag{15}$$

in an identity (2), where

$$a = x/y \Rightarrow x = a \cdot y \quad \underbrace{\Rightarrow}_{\text{(23)-permutation}} \quad y = a \circ_{(23)} x \Rightarrow a = y/^{(23)}x \tag{16}$$

and

$$b = z^{\alpha}\backslash x^{\alpha} \underbrace{\Rightarrow}_{(23)\text{-permutation}} z^{\alpha} \circ_{(23)} b = x^{\alpha} \Rightarrow z^{\alpha} \circ_{(23)} x^{\alpha} = b. \tag{17}$$

Let $c = z^{\alpha}y$ in identity (2), then $\underbrace{z^{\alpha} \circ_{(23)} c}_{(23)\text{-permutation}} = y \Rightarrow c = z^{\alpha}\backslash^{(23)}y$ . Let $d = c\backslash x^{\alpha} \Rightarrow c \circ_{(23)} d = x^{\alpha} \Rightarrow c \circ_{(23)} x^{\alpha} = d$, put $c$ into $d$ to get

$$d = \big(z^{\alpha}\backslash^{(23)}y\big) \circ_{(23)} x^{\alpha}. \tag{18}$$

Also, let $t = x \cdot d \underbrace{\Rightarrow}_{(23)\text{-permutation}} x \circ_{(23)} t = d \Rightarrow t = x\backslash^{(23)}d$. Substitute $d$ into $t$

$$t = x\backslash^{(23)}\big[(z^{\alpha}\backslash^{(23)}y) \circ_{(23)} x^{\alpha}\big]. \tag{19}$$

Now, going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{(23)\text{-permutation}} a \circ_{(23)} t = b \Rightarrow a\backslash^{(23)}b = t$. Then, substituting equations (16), (17) and (19) in the equality $a\backslash^{(23)}b = t$, gives

$$\big(y/^{(23)}x\big)\backslash^{(23)}(z^{\alpha} \circ_{(23)} x^{\alpha}) = x\backslash^{(23)}\big[(z^{\alpha}\backslash^{(23)}y) \circ_{(23)} x^{\alpha}\big] \tag{20}$$

which is the (23)-parastrophe of $Q$. ■

**Theorem 3.4.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following holds in (23)-parastrophe of $Q$*

1. $(L_x^{-1}, R_{x^{\alpha}}, R_{x^{\alpha}}L_x^{-1}) \in AATP(Q, \backslash^{(23)})$ *for all* $x \in Q$,
2. $(z \circ_{(23)} t) \circ_{(23)} t = z$ *for all* $z, t \in Q$,
3. *if* $Q^{(23)}$ *is middle symmetric then,* $x \circ_{(23)} (z^{\alpha} \circ_{(23)} x^{\alpha}) = (x \circ_{(23)} z^{\alpha}) \circ_{(23)} x^{\alpha}$ *that is, super $\alpha$-elastic,*
4. $R_x^{-1}M_{x^{\alpha}} = R_{x^{\alpha}}L_x^{-1}$,
5. $\rho J R_{x^{\alpha}}L_x^{-1} = R_{x^{\alpha}}L_{x^{\lambda}}^{-1}$,
6. $\rho J R_{x^{\alpha}}^{-1}M_{x^{\alpha}} = R_{x^{\alpha}}L_{x^{\lambda}}^{-1}$.

***Proof.*** 1. This follows from equation (14),

$$yR_x^{-1}\backslash^{(23)}z^{\alpha}R_{x^{\alpha}} = \big(z^{\alpha}\backslash^{(23)}y\big)R_{x^{\alpha}}L_x^{-1} \Rightarrow \big(R_x^{-1}, R_{x^{\alpha}}, R_{x^{\alpha}}L_x^{-1}\big) \in AATP(Q, \backslash).$$

2. Put $x = e$ such that $e^{\alpha} \mapsto e$ is the identity map in (14), give $y\backslash^{(23)}z^{\alpha} = z^{\alpha}\backslash^{(23)}y \Rightarrow y \circ_{(23)} (z^{\alpha}\backslash^{(23)}y) = z^{\alpha}$. Let $t = z^{\alpha}\backslash^{(23)}y \Rightarrow z^{\alpha} \circ_{(23)} t = y$. Put $y$ into the last equality to get $(z^{\alpha} \circ_{(23)} t) \circ_{(23)} t = z^{\alpha}$ for any $t \in Q$.

3. Put $y = x$ in (14), we have

$$z^\alpha \circ_{(23)} x^\alpha = x\backslash^{(23)}\left[\left(z^\alpha\backslash^{(23)}x\right) \circ_{(23)} x^\alpha\right] \Rightarrow$$
$$x^\alpha \circ_{(23)} (z^\alpha \circ_{(23)} x) = \left(z^\alpha\backslash^{(23)}x\right) \circ_{(23)} x^\alpha \Rightarrow$$
$$z^\alpha R_{x^\alpha} L_x = z^\alpha M_x R_{x^\alpha} \quad \underbrace{\Rightarrow}_{\text{Use middle symmetric as } L_x = M_x \text{ to get}} \quad z^\alpha R_{x^\alpha} L_x = z^\alpha L_x R_{x^\alpha}$$

or $x \circ_{(23)} (z^\alpha \circ_{(23)} x^\alpha) = (x \circ_{(23)} z^\alpha) \circ_{(23)} x^\alpha$.

4. Put $z = e$ and $e^\alpha \mapsto e$, the identity element in (14), we have $(y/^{(23)}x)\backslash^{(23)}x^\alpha = x\backslash^{(23)}(y \circ_{(23)} x^\alpha) \Rightarrow yR_x^{-1}M_{x^\alpha} = yR_{x^\alpha}L_x^{-1} \Rightarrow R_x^{-1}M_{x^\alpha} = R_{x^\alpha}L_x^{-1}$.

5. $y = e$ in (14), we have

$$x^\lambda\backslash^{(23)}(z^\alpha \circ_{(23)} x^\alpha) = x\backslash^{(23)}((z^\alpha)^\rho \circ_{(23)} x^\alpha) \Rightarrow$$
$$z^\alpha \rho J R_{x^\alpha} L_x^{-1} = z^\alpha R_{x^\alpha} L_{x^\lambda}^{-1} \Rightarrow \rho J R_{x^\alpha} L_x^{-1} = R_{x^\alpha} L_{x^\lambda}^{-1}.$$

6. Use 4 and 5. ■

**Corollary 3.5.** *A commutative* (23)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *is totally symmetric.*

***Proof.*** This is a consequence of the right symmetric property 2 of Theorem 3.4. ■

**Theorem 3.5.** *Let the* (23)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *be commutative and of exponent two, then* $L_{x^\alpha} L_x = R_x R_{x^\alpha}$ *for all* $x \in Q$.

***Proof.*** Recall (6) in Theorem 3.4, we have $\rho J R_x^{-1} M_{x^\alpha} = R_{x^\alpha} L_{x^\lambda}^{-1}$. Since $Q^{(23)}$ is commutative, then it implies that it has a middle symmetric property as $L_x = M_x$. Applying the middle symmetric identity gives $\rho J R_x^{-1} L_{x^\alpha} = R_{x^\alpha} L_{x^\lambda}^{-1}$. Then, for all $t \in Q$, we have

$$t^\rho R_x^{-1} L_{x^\alpha} = t R_{x^\alpha} L_{x^\lambda}^{-1} \Rightarrow x^\alpha \circ_{(23)} (t^\rho/x) = x^\lambda\backslash^{(23)}(t \circ_{(23)} x^\alpha)$$
$$\Rightarrow x^\lambda \circ_{(23)} \left[x^\alpha \circ_{(23)} (t^\rho/x)\right] = t \circ_{(23)} x^\alpha.$$

Let $t^\rho/^{(23)}x = s \Rightarrow t^\rho = s \circ_{(23)} x$. Then, $x^\lambda \circ_{(23)} (x^\alpha \circ_{(23)} s) = (s \circ_{(23)} x) \circ_{(23)} x^\alpha$. Using the fact that $|Q^{(23)}| = 2$ for all $x \in Q$, one obtains $sL_{x^\alpha}L_x = sR_xR_{x^\alpha} \Rightarrow L_{x^\alpha}L_x = R_xR_{x^\alpha}$ for all $x \in Q$. ■

**Corollary 3.6.** *If* (23)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *is commutative and* $x^\alpha \mapsto x$*, then* $L_x^2 = R_x^2$ *for all* $x \in Q$.

***Proof.*** Consequence of Theorem 3.5. ■

**Lemma 3.5.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol. Then, the* (123)*-parastrophe of $Q$ is given by*

$$\big(z^\alpha/^{(123)}x^\alpha\big)\backslash^{(123)}(y \circ_{(123)} x) = \big[(y\backslash^{(123)}z^\alpha)/^{(123)}x^\alpha\big]\backslash^{(123)}x. \tag{21}$$

***Proof.*** Let $a \cdot b = x \cdot (z^\alpha y\backslash x^\alpha)$ in equation (2) where

$$a = x/y \Rightarrow a \cdot y = x \underbrace{\Rightarrow}_{(123)\text{-permutation}} y \circ_{(123)} x = a \tag{22}$$

$$b = z^\alpha\backslash x^\alpha \Rightarrow z^\alpha \circ b = x^\alpha \underbrace{\Rightarrow}_{(123)\text{-permutation}} b \circ_{(123)} x^\alpha = z^\alpha \Rightarrow b = z^\alpha/^{(123)}x^\alpha. \tag{23}$$

Let $c = z^\alpha \cdot y$ in equation (2), then, we have $y \circ_{(123)} c = z^\alpha \underbrace{\Rightarrow}_{(123)\text{-permutation}} c = y\backslash^{(123)}z^\alpha$. Also, let $d = c\backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \Rightarrow d \circ_{(123)} x^\alpha = c \Rightarrow d = c/^{(123)}x^\alpha$. Substitute $c$ into $d$, give

$$d = \big(y\backslash^{(123)}z^\alpha\big)/^{(123)}x^\alpha. \tag{24}$$

Next, let $t = x \cdot d \underbrace{\Rightarrow}_{(123)\text{-permutation}} d \circ_{(123)} t = x \Rightarrow t = d\backslash^{(123)}x$. Substitute (24) into $t$ give

$$t = \big[(y\backslash^{(123)}z^\alpha)/^{(123)}x^\alpha\big]\backslash^{(123)}x. \tag{25}$$

Going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{(123)\text{-permutation}} b \circ_{(123)} t = a \Rightarrow b\backslash^{(123)}a = t$. Substitute (22), (23) and (25) into the equality $b\backslash^{(123)}a = t$, gives the (123)-parastrophe as

$$\big(z^\alpha/^{(123)}x^\alpha\big)\backslash^{(123)}(y \circ_{(123)} x) = \big[(y\backslash^{(123)}z^\alpha)/^{(123)}x^\alpha\big]\backslash^{(123)}x.$$

∎

**Theorem 3.6.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following hold in* (123)*-parastrophe of $Q$*

1. $(L_x^{-1}, R_x, R_x^{-1}M_x) \in AATP(Q, \backslash^{(123)})$,
2. $(y \circ_{(123)} t) \circ_{(123)} t^\rho = y$, *i.e., right inverse property,*
3. $(z^\alpha/^{(123)}x^\alpha)[(x^\alpha)^\lambda\backslash^{(123)}x] = z^\alpha \circ_{(123)} x$,
4. $R_x L^{-1}_{(x^\alpha)^\lambda} = \rho J R^{-1}_{x^\alpha} M_x$,
5. $R_x M_x^{-1} = M_{x^\alpha} R^{-1}_{x^\alpha}$,

6. $(x \circ_{(123)} t) \circ_{(123)} x = (x\backslash^{(123)}t)\backslash^{(123)}x$ *for all* $x, t \in Q$.

***Proof.*** 1. From equation (21), we have

$z^\alpha R_{x^\alpha}^{-1}\backslash^{(123)} yR_x = \big(y\backslash^{(123)}z^\alpha\big)R_{x^\alpha}^{-1}M_x \Rightarrow \big(R_{x^\alpha}^{-1}, R_x, R_{x^\alpha}^{-1}M_x\big) \in AATP\big(Q, \backslash^{(123)}\big).$

2. Let $x^\alpha \mapsto x$ and put $x = e$, the identity element in equation (21), we have

$$\big((z^\alpha/^{(123)}e^\alpha)\backslash^{(123)}(y \circ_{(123)} e) = \big((y\backslash^{(123)}z^\alpha\big)/^{(123)}e\big)\backslash^{(123)}e^\alpha \Rightarrow z^\alpha\backslash^{(123)}y$$
$$= \big(y\backslash^{(123)}z^\alpha\big)^\rho \Rightarrow z^\alpha \circ_{(123)} \big(y\backslash^{(123)}z^\alpha\big)^\rho = y.$$

Let $t = y\backslash^{(123)}z^\alpha \Rightarrow y \circ_{(123)} t = z^\alpha$ for any $t \in Q$, this implies that $(y \circ_{(123)} t) \circ_{(123)} t^\rho = y$.

3. Set $y = z^\alpha$ in equation (21), we have $(z^\alpha/^{(123)}x^\alpha)\backslash^{(123)}(z^\alpha \circ_{(123)} x) = (x^\alpha)^\lambda\backslash^{(123)}x \Rightarrow (z^\alpha/^{(123)}x^\alpha)[(x^\alpha)^\lambda\backslash^{(123)}x] = z^\alpha \circ_{(123)} x$.

4. Put $z \to e$ in equation (21), to get $(x^\alpha)^\lambda\backslash^{(123)}y \circ_{(123)} x) = (y^\rho/^{(123)}x^\alpha)\backslash^{(123)}x \Rightarrow yR_xL_{(x^\alpha)^\lambda}^{-1} = y\rho JR_{x^\alpha}^{-1}M_x \Rightarrow R_xL_{(x^\alpha)^\lambda}^{-1} = \rho JR_{x^\alpha}^{-1}M_x$.

5. Set $z = x$ in equation (21), give $y \circ_{(123)} x = \big((y\backslash^{(123)}x^\alpha)/^{(123)}x^\alpha\big)\backslash^{(123)}x \Rightarrow yR_x = yM_{x^\alpha}R_{x^\alpha}^{-1}M_x \Rightarrow R_xM_x^{-1} = M_{x^\alpha}R_{x^\alpha}^{-1}$. ■

**Corollary 3.7.** *A commutative* (123)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *has an inverse property.*

***Proof.*** This is a consequence of 2 of Theorem 3.6. ■

**Corollary 3.8.** *A commutative* (123)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *has AAIP if* $|Q^{(123)}| = 2$.

***Proof.*** Applying Corollary 3.7 to (21) $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \big[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\big]^{-1} \circ_{(123)} x$. Put $x = e$ and $y = y^{-1}$ to get $(z^\alpha)^{-1} \circ_{(123)} y^{-1} = (y^{-1} \circ_{(123)} z^\alpha)^{-1}$. ■

**Corollary 3.9.** *A commutative* (123)*-parastrophe, of exponent 2, of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *is Steiner loop.*

***Proof.*** Follows from Corollary 3.7. ■

**Theorem 3.7.** *Let* $Q^{(123)}$ *be a commutative* (123)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *of exponent two, then* $Q^{(123)}$ *is a Moufang loop.*

***Proof.*** Using the Corollary 3.7 on identity (21), we have $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \big[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\big]^{-1} \circ_{(123)} x$. Since $|Q^{(123)}| = 2$, we have $(z^\alpha \circ_{(123)} x^\alpha) \circ_{(123)} (y \circ_{(123)} x) = \big[(y \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)\big] \circ_{(123)} x \Rightarrow z^\alpha L_{x^\alpha}R_{yx} = z^\alpha L_yR_{x^\alpha}R_x \Rightarrow z^\alpha L_{x^\alpha}R_{yx} = z^\alpha L_yL_{x^\alpha}R_x \Rightarrow (x^\alpha \circ_{(123)} z^\alpha) \circ (y \circ_{(123)} x) = (x^\alpha \circ_{(123)} (z^\alpha \circ_{(123)} y)) \circ_{(123)} x \Rightarrow (x^\alpha \circ_{(123)} z^\alpha) \circ (y \circ_{(123)} x) = x^\alpha \circ_{(123)} ((z^\alpha \circ_{(123)} y) \circ_{(123)} x)$. ■

**Corollary 3.10.** *A commutative* (123)*-parastrophe of a generalised middle Bol loop* $(Q, \cdot, /, \backslash)$ *is a GMBL of exponent two.*

***Proof.*** Follow from Corollaries 3.7 and 3.8 and (21), we get $(z^\alpha \circ_{(123)} (x^\alpha)^{-1})^{-1} \circ_{(123)} (y \circ_{(123)} x) = \big[(y^{-1} \circ_{(123)} z^\alpha) \circ_{(123)} (x^\alpha)^{-1}\big]^{-1} \circ_{(123)} x$. So, use $y^{-1} = y$ and take the following steps: $x \leftrightarrow x^\alpha$, $z^\alpha \leftrightarrow y$, one obtains

$$\big(x/^{(12)}y\big) \circ_{(12)} \big(z^\alpha \backslash^{(12)} x^\alpha\big) = \big(x/^{(12)}(z^\alpha \circ_{(12)} y)\big) \circ_{(12)} x^\alpha$$

which is the same as (3) ■

**Lemma 3.6.** *Let* $(Q, \cdot, /, \backslash)$ *be a generalised middle Bol loop. Then, the* (132)*-parastrophe of* $Q$ *is given by*

$$\big(x^\alpha \circ_{(132)} z^\alpha\big)/^{(132)}\big(x\backslash^{(132)}y\big) = \big[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\big]/^{(132)}x. \tag{26}$$

***Proof.*** Let $a \cdot b = x \cdot (z^\alpha y \backslash x^\alpha)$ in equation (2) where

$$x/y = a \Rightarrow x = a \cdot y \quad \underbrace{\Rightarrow}_{\text{(132)-permutation}} \quad x \circ_{132} a = y \Rightarrow a = x\backslash^{(132)}y \tag{27}$$

and

$$z^\alpha \backslash x^\alpha = b \Rightarrow z^\alpha \cdot b = x^\alpha \quad \underbrace{\Rightarrow}_{\text{(132)-permutation}} \quad x^\alpha \circ_{(132)} z^\alpha = b. \tag{28}$$

Let $c = z^\alpha \cdot y \underbrace{\Rightarrow}_{\text{(132)-permutation}} c \circ_{(132)} z^\alpha = y \Rightarrow c = y/^{(132)}z^\alpha$. Also, let $d = c\backslash x^\alpha \Rightarrow c \cdot d = x^\alpha \underbrace{\Rightarrow}_{\text{(132)-permutation}} x^\alpha \circ_{(132)} c = d$. Substitute $c$ into to $d$ to get $d = x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)$. Let $t = x \cdot d \underbrace{\Rightarrow}_{\text{taking (132)-permutation}} t \circ_{(132)} x = d \Rightarrow t = d/^{(132)}x$. Hence, putting $d$ into $t$, we have

$$t = \big[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\big]/^{(132)}x. \tag{29}$$

Now, going by the identity (2), we have $a \cdot b = t \underbrace{\Rightarrow}_{\text{taking (132)-permutation}} t \circ_{(132)} a = b \Rightarrow b/^{(132)}a = t$. Substitute equations (27), (28) and (29) into the equality $b/^{(132)}a = t$, we have

$$(x^\alpha \circ_{(132)} z^\alpha)/^{(132)}(x\backslash^{(132)}y) = \big[x^\alpha \circ_{(132)} (y/^{(132)}z^\alpha)\big]/^{(132)}x$$

which is the (132)- parastrophe of $Q$. ■

**Theorem 3.8.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, the following holds in* $(132)$*-parastrophe of* $Q$

1. $(L_{x^\alpha}, L_x^{-1}, L_{x^\alpha}R_x^{-1}) \in AATP(Q, /^{(132)})$ *for all* $x \in Q$,
2. $z^\alpha = t \circ_{132} (t \circ_{132} z^\alpha)$ *i.e.,* $\alpha$*-left symmetric property,*
3. $(x^\alpha \circ_{(132)} z^\alpha) \circ_{(132)} x = x^\alpha \circ_{(132)} (x/^{(132)} z^\alpha)$ *or* $M_x^{-1}L_{x^\alpha} = L_{x^\alpha}R_x$,
4. $L_{x^\alpha}R_{x^\rho}^{-1} = \lambda J L_{x^\alpha}R_x^{-1}$,
5. $L_x^{-1}M_{x^\alpha}^{-1} = L_{x^\alpha}R_x^{-1}$.

***Proof.*** 1. From equation (26), we have $z^\alpha L_{x^\alpha}/^{(132)} y L_x^{-1} = (y/^{(132)} z^\alpha)L_{x^\alpha}R_x^{-1} \Rightarrow (L_{x^\alpha}, L_x^{-1}, L_{x^\alpha}R_x^{-1}) \in AATP(Q, /^{(132)})$ for all $x \in Q$.

2. Let $x^\alpha \mapsto e$ in (26), give $z^\alpha /^{(132)} y = y/^{(132)} z^\alpha$, by setting $t = y/^{(132)} z^\alpha \Rightarrow y = (z^\alpha \circ_{(132)} t) \Rightarrow z^\alpha = t \circ_{(132)} (t \circ_{(132)} z^\alpha)$.

3. Put $y = x$ in (26), to get $(x^\alpha \circ_{(132)} z^\alpha) \circ_{(132)} x = x^\alpha \circ_{(132)} (x/^{(132)} z^\alpha) \Rightarrow z^\alpha M_x^{-1}L_{x^\alpha} = z^\alpha L_{x^\alpha}R_x \Rightarrow M_x^{-1}L_{x^\alpha} = L_{x^\alpha}R_x$ for all $x \in Q$.

4. Put $y = e$ in (26), we have $(x^\alpha \circ_{(132)} z^\alpha)/^{(132)} x^\rho = (x^\alpha \circ_{(132)} (z^\alpha)^\lambda)/^{(132)} x \Rightarrow z^\alpha L_{x^\alpha}R_{x^\rho}^{-1} = (z^\alpha)\lambda J L_{x^\alpha}R_x^{-1} \Rightarrow L_{x^\alpha}R_{x^\rho}^{-1} = \lambda J L_{x^\alpha}R_x^{-1}$.

5. Put $z = e$, we have $x/^{(132)}(x\backslash^{(132)} y) = (x \circ_{(132)} y)/^{(132)} x \Rightarrow yL_xM_x^{-1} = yL_xR_x^{-1} \Rightarrow L_x^{-1}M_{x^\alpha}^{-1} = L_{x^\alpha}R_x^{-1}$ for all $x \in Q$. ■

**Corollary 3.11.** *Let $(Q, \cdot, /, \backslash)$ be a generalised middle Bol loop. Then, a commutative* $(132)$*-parastrophe of* $Q$ *is totally symmetric.*

***Proof.*** This is a consequence, of 2, of Theorem 3.8. ■

### 3.3. Holomorphic structure of generalised middle Bol loop

**Theorem 3.9.** *$(Q, \cdot, /, \backslash)$ is a generalised middle Bol loop if and only if $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x)$ is an autotopism.*

***Proof.*** Suppose $(Q, \cdot)$ is a generalised middle Bol loop, then

$$\begin{aligned} x(y^\alpha z\backslash x^\alpha) = (x/z)(y^\alpha \backslash x^\alpha) &\Leftrightarrow zM_x^{-1} \cdot y^\alpha M_{x^\alpha} = (y^\alpha \cdot z)M_{x^\alpha}L_x \\ &\Leftrightarrow zM_x^{-1} \cdot y^\alpha M_{x^\alpha} = (zJ \cdot y^\alpha J)JM_{x^\alpha}L_x \\ &\Leftrightarrow zJM_x^{-1} \cdot y^\alpha JM_{x^\alpha} = (z \cdot y^\alpha)JM_{x^\alpha}L_x. \end{aligned}$$

Thus, $(JM_x^{-1}, JM_{x^\alpha}, JM_{x^\alpha}L_x) \in ATP(Q, \cdot)$. ■

**Theorem 3.10.** *Let $(Q, \cdot, /, \backslash)$ be a loop with holomorph $(H(Q), *)$. Then, $(H(Q), *)$ is a generalised middle Bol loop if and only if $(x\tau) \cdot (y \cdot z^\alpha\tau)\backslash x^\alpha = (x^\alpha\tau/z^\alpha\tau) \cdot (y\backslash x)$ for all $x, y, z \in Q, \tau \in A(Q)$.*

***Proof.*** We need to show the necessary and sufficient condition for the holomorph of a generalised middle Bol loop to ba a generalised middle Bol loop.

$$(x^\alpha/z^\alpha)(y\backslash x) = x((y \cdot z^\alpha)\backslash x^\alpha) \tag{30}$$

$$\begin{aligned}
&\text{Let } (\phi, x) * (\psi, y) = (\theta, z) \Rightarrow (\phi, x) = (\theta, z)/(\psi, y), \text{ so,} && (31)\\
&(\phi\psi, x\psi \cdot y) = (\theta, z)\\
&\Rightarrow \phi = \theta\psi^{-1}, x = (z/y)\psi^{-1}.\\
&\Rightarrow (\theta, z)/(\psi, y) = \big(\theta\psi^{-1}, (z/y)\phi^{-1}\big) = (\phi, x). && (32)\\
&\text{Also}, (\phi, x) * (\psi, y) = (\theta, z) \Rightarrow (\psi, y) = (\phi, x)\backslash(\theta, z).\\
&\text{Thus}, (\phi\psi, x\psi \cdot y) = (\theta, z) \Rightarrow \psi = \phi^{-1}\theta, y = (x\phi^{-1}\theta)\backslash z\\
&\Rightarrow (\psi, y) = (\phi^{-1}\theta, (x\phi^{-1}\theta)\backslash z) = (\phi, x)\backslash(\theta, z) && (33)
\end{aligned}$$

$$\begin{gathered}
((\phi, x)/(\psi, y)) * ((\theta, z^\alpha)\backslash(\phi, x^\alpha)) = (\phi, x) * [((\psi, y) * (\theta, z^\alpha))\backslash(\phi, x^\alpha)]\\
\text{RHS} = (\phi, x) * [((\psi, y) * (\theta, z^\alpha))\backslash(\phi, x^\alpha)]\\
= (\phi, x) * \Big((\psi\theta)^{-1}\phi, (y\theta \cdot z^\alpha)\theta^{-1}\psi^{-1}\phi\backslash x^\alpha\Big)\\
= (\phi, x) * \Big((\psi\theta)^{-1}\phi, y\psi^{-1}\phi \cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha\Big)\\
= \big(\phi\theta^{-1}\psi^{-1}\phi, (x\theta^{-1}\psi^{-1}\phi) \cdot (y\psi^{-1}\phi \cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha)\big)\\
\text{LHS } = ((\phi, x)/(\psi, y)) * ((\theta, z^\alpha)\backslash(\phi, x^\alpha))\\
= \big(\phi\theta^{-1}, (x^\alpha/z^\alpha)\theta^{-1}\big) * \big(\psi^{-1}\phi, (y\psi^{-1}\phi)\backslash x\big)\\
\big(\phi\theta^{-1}\psi^{-1}\psi, (x^\alpha/z^\alpha)\theta^{-1}\psi^{-1}\phi \cdot (y\psi^{-1}\phi)\backslash x\\
RHS = LHS\\
\Leftrightarrow \big((x\theta^{-1}\psi^{-1}\phi) \cdot (y\psi^{-1}\phi \cdot z^\alpha\theta^{-1}\psi^{-1}\phi\backslash x^\alpha)\big) = \big((x^\alpha/z^\alpha)\theta^{-1}\psi^{-1}\phi \cdot (y\psi^{-1}\phi)\backslash x\big).
\end{gathered}$$

Let $\tau = \theta^{-1}\psi^{-1}\phi$, then $(x\tau) \cdot (y\theta\tau \cdot z^\alpha\tau)\backslash x^\alpha) = (x^\alpha/z^\alpha)\tau \cdot (y\theta\tau)\backslash x$. Replacing $y$ by $y(\theta\tau)^{-1}$, we have

$$\begin{gathered}
(x\tau) \cdot (y(\theta\tau)^{-1}\theta\tau \cdot z^\alpha\tau)\backslash x^\alpha = (x^\alpha/z^\alpha)\tau \cdot (y(\theta\tau)^{-1}\theta\tau)\backslash x\\
\Leftrightarrow (x\tau) \cdot (y \cdot z^\alpha\tau)\backslash x^\alpha = (x^\alpha/z^\alpha)\tau \cdot (y\backslash x)\\
\Leftrightarrow (x\tau) \cdot (y \cdot z^\alpha\tau)\backslash x^\alpha = (x^\alpha\tau/z^\alpha\tau) \cdot (y\backslash x).
\end{gathered}$$

■

**Corollary 3.12.** *Let $(Q, \cdot, /, \backslash)$ be a loop with holomorph $H(Q, \cdot)$. Then, $H(Q, \cdot)$ is a commutative generalised middle Bol loop if and only if $(\tau^{-1}M_{x^\alpha}^{-1}\tau, M_x, M_x^\alpha L_{x\tau}) \in ATP(Q, \cdot)$.*

***Proof.*** From the consequence of Theorem 3.10, we have

$$z^{\alpha}\tau^{-1}M_{x^{\alpha}}^{-1}\tau \cdot yM_x = (y \cdot z^{\alpha})M_{x^{\alpha}}L_{x\tau} \tag{34}$$

$$\Leftrightarrow (\tau^{-1}M_{x^{\alpha}}^{-1}\tau, M_x, M_{x^{\alpha}}L_{x\tau}) \in ATP(Q, \cdot). \tag{35}$$

■

**Theorem 3.11.** *Let $(Q, \cdot, /, \backslash)$ be a commutative generalised middle Bol loop with a holomorph $(H, *) = H(Q, \cdot)$. If*

1. $\tau = \tau(a, b) = R_{(b\backslash a)}R_b^{-1}$ *for each $\tau \in A(Q)$ and for any $a, b \in Q$,*
2. $M_x^{-1}R_{s\tau} = R_sR_x^{-1}R_{x\tau}$ *for all $s, x \in Q$ and $\tau \in A(Q)$, then $H(Q, \cdot)$ is a GMBL.*

***Proof.*** From Corollary 3.12, observe that $(\tau^{-1}M_{x^{\alpha}}^{-1}\tau, M_x, M_x^{\alpha}L_{x\tau}) = (\tau^{-1}, M_x, I_e) \circ (M_{x^{\alpha}}^{-1}, M_x, M_{x^{\alpha}}L_x) \circ (\tau, M_x^{-1}, L_x^{-1}L_{x\tau})$. Where $I_e$ is an identity map.

$$\begin{aligned}
\text{Consider one hand, } (\tau^{-1}, M_x, I_e) \in ATP(Q, \cdot) &\Leftrightarrow a\tau^{-1} \cdot bM_x = ab \\
&\Leftrightarrow a\tau^{-1} \cdot b\backslash x = ab \\
&\Leftrightarrow a\tau^{-1}R_{b\backslash x} = aR_b \\
\Leftrightarrow \tau^{-1}R_{b\backslash x} = R_b &\Leftrightarrow \tau = \tau(a, b) = R_{b\backslash a}R_b^{-1}.
\end{aligned}$$

Also,

$$\begin{gathered}
(\tau, M_x^{-1}, L_x^{-1}L_{x\tau}) \in ATP(Q, \cdot) \\
\Leftrightarrow s\tau \cdot yM_x^{-1} = (sy)L_x^{-1}L_{x\tau} \\
\Leftrightarrow yM_x^{-1}L_{s\tau} = yL_sL_x^{-1}L_{x\tau} \Leftrightarrow M_x^{-1}R_{s\tau} = R_sR_x^{-1}R_{x\tau}.
\end{gathered}$$

■

**Corollary 3.13.** *Let $(Q, \cdot, /, \backslash)$ be a commutative loop such that $M_x^{-1}R_{s\tau} = R_sR_x^{-1}R_{x\tau}$ for all $x, s \in Q$ and $\tau \in A(Q)$. $(H, *) = H(Q, \cdot)$ is commutative GMBL if and only if*

1. $(Q, \cdot)$ *is a generalised middle Bol loop,*
2. $\tau = \tau(a, b) = R_{b\backslash a}R_b^{-1}$ *for arbitrarily fixed $a, b \in Q$ and for each $\tau \in A(Q)$.*

***Proof.*** It is straightforward. ■

## CONCLUSION

In this research, we have been able to show that the two identities of GMBL are equivalent if the generalising map $\alpha$ is bijective such that it fixes the identity element. Also, among all the five parastrophes of GMBL, (12)-parastrophe of GMBL is a GMBL and (13)- and (123)-parastrophes of $Q$ are GMBL of exponent two. In line with Lemma 3.2, it can be seen that (13)- and (123)-parastrophes of GMBL of exponent two are loop. It is noted that (23)- and (132)-parastrophes of GMBL with commutative property are totally symmetric. The work further reveals that, in (13)-parastrophe of $Q$, the right inverse element coincides with left inverse element if $\alpha$ is bijective such that $\alpha : e \to e$ which is one of the general property of middle Bol loop revealed by Kuznetsov in [19].

## References

[1] J.O. Adeniran, T.G. Jaiyéọlá and K.A. Idowu, *Holomorph of generalized Bol loops*, Novi Sad J. Math. **44(1)** (2014) 37–51.

[2] A.O. Abdulkareem, J.O. Adeniran, A.A.A. Agboola and G.A. Adebayo, *Universal $\alpha$-elasticity of generalised Moufang loops*, Ann. Math. Comput. Sci. **14** (2023) 1–11.

[3] A.O. Abdulkareem and J.O. Adeniran, *Generalised middle Bol loops*, J. Nigerian Math. Soc. **39(3)** (2020) 303–313.

[4] V.D. Belousov, *Foundations of the theory of quasigroups and loops*, (Russian) Izdat. Nauka (Moscow, 1967) pp. 223.

[5] A. Drapal and V. Shcherbacov *Identities and the group of isostrophisms*, Comment. Math. Univ. Carolin. **53(3)** (2012) 347–374.

[6] R.O. Fisher and F. Yates, *The* $6x6$ *latin squares*, Proc. Soc. **30** (1934) 429–507.

[7] I. Grecu, *On multiplication groups of isostrophic quasigroups*, in: Proceedings of the Third Conference of Mathematical Society of Moldova, IMCS-50, 19–23 (Chisinau, Republic of Moldova, 2014) 78–81.

[8] I. Grecu and P. Syrbu, *On Some isostrophy invariants of Bol loops*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics **54(5)** (2012) 145–154.

[9] I. Grecu and P. Syrbu, *Commutants of middle Bol loops*, Quasigroups and Related Systems **22** (2014) 81–88.

[10] A. Gvaramiya, *On a class of loops*, (Russian), Uch. Zapiski MAPL. **375** (1971) 25–34.

[11] T.G. Jaiyéọlá, S.P. David and Y.T. Oyebo, New algebraic properties of middle Bol loops, ROMAI J. **11(2)** (2015) 161–183.

[12] T.G. Jaiyéọlá, S.P. David, E. Ilojide and Y.T. Oyebo, *Holomorphic structure of middle Bol loops*, Khayyam J. Math. **3(2)** (2017) 172–184. https://doi.org/10.22034/kjm.2017.51111

[13] T.G. Jaiyéọlá, S.P. David and O.O. Oyebola, *New algebraic properties of middle Bol loops* II, Proyecciones J. Math. **40(1)** (2021) 85–106. https://doi.org/10.22199/issn.0717-6279-2021-01-0006

[14] T.G. Jaiyéọlá, *Some necessary and sufficient conditions fro parastrophic invarance in the associative law in quasigroups*, Fasciculi Math. **40** (2008) 25–35.

[15] T.G. Jaiyéọlá, *Basic properties of second smarandache Bol loops*, Int. J. Math. Combin. **2** (2009) 11–20.
https://doi.org/10.5281/zenodo.32303

[16] T.G. Jaiyéọlá, *Smarandache isotopy of second smarandache Bol loops*, Scientia Magna J. **7(1)** (2011) 82–93.
https://doi.org/10.5281/zenodo.234114

[17] T.G. Jaiyéọlá and B.A. Popoola, *Holomorph of generalized Bol loops* II, Discuss. Math. Gen. Alg. and Appl. **35(1)** (2015) 59–78.
https://doi.org/10.7151/dmgaa.1234

[18] T.G. Jaiyéọlá, B. Osoba and A. Oyem, *Isostrophy Bryant-Schneider group-invariant of Bol loops*, Buletinul Academiei De S¸ Tiinte¸ A Republicii Moldova, Matematica **2(99)** (2022) 3–18.

[19] E. Kuznetsov, *Gyrogroups and left gyrogroups as transversals of a special kind*, Alg. and Discrete Math. **3** (2005) 54–81.

[20] B. Osoba and Y.T. Oyebo, *On multiplication groups of middle Bol loop related to left Bol loop*, Int. J. Math. and Appl. **6(4)** (2018) 149–155.

[21] B. Osoba and Y.T. Oyebo, *On relationship of multiplication groups and isostrophic quasigroups*, Int. J. Math. Trends and Technology (IJMTT) **58(2)** (2018) 80–84.
https://doi.org/10.14445/22315373/IJMTT-V58P511

[22] B. Osoba, *Smarandache nuclei of second smarandache Bol loops*, Scientia Magna J. **17(1)** (2022) 11–21.

[23] B. Osoba and T.G. Jaiyéọlá, *Algebraic connections between right and middle Bol loops and their Cores*, Quasigroups and Related Systems **30** (2022) 149–160.

[24] B. Osoba, T.G. Jaiyéọlá, Y.T. Oyebo and A.O. Isere, *Some characterization of generalized middle Bol loop*, Buletinul Academiei De Stiinte a Republicii Moldova, paper accepted.

[25] B. Osoba and Y.T. Oyebo, *Isostrophy invariant $\alpha$-elastic property*, Proyecciones **45(6)** (2024) 1373–1388.

[26] T.Y Oyebo, B. Osoba and T.G. Jaiyéọlá, *Crypto-automorphism group of some quasigroups*, Discuss. Math. Gen. Alg. and Appl. **44(1)** (2024) 57–72.

[27] Y.T. Oyebo and B. Osoba, *More results on the algebraic properties of middle Bol loops*, J. Nigerian Math. Soc. **41(2)** (2022) 129–42.

[28] H.O. Pflugfelder, Quasigroups and Loops: Introduction, Sigma Series in Pure Mathematics **7** (Heldermann Verlag, Berlin, 1971) viii+147.

[29] V.A. Shcherbacov, *A-nuclei and A-centers of quasigroup*, Institute of Mathematics and Computer Science Academiy of Science of Moldova Academiei str. 5, Chisinau, MD -2028 (Moldova, 2011).

[30] A.R.T. Solarin, J.O. Adeniran, T.G. Jaiyéọlá, A.O. Isere and Y.T. Oyebo, *Some Varieties of Loops* (*Bol-Moufang and Non-Bol-Moufang Types*), in: M.N. Hounkonnou, M. Mitrović, M. Abbas, M. Khan, (eds) Algebra without Borders – Classical and Constructive Nonassociative Algebraic Structures. STEAM-H: Science, Technology, Engineering, Agriculture, Mathematics & Health (Springer, Cham., 2023). https://doi.org/10.1007/978-3-031-39334-1_3

[31] P. Syrbu and I. Grecu, *On some groups related to middle Bol loops*, Studia Univ. Moldaviae (Seria Stiinte Exacte si Economice) **7(67)** (2013) 10–18.

[32] P. Syrbu, *Loops with universal elasticity*, Quasigroups Related Systems **1** (1994) 57–65.

[33] P. Syrbu, *On loops with universal elasticity*, Quasigroups Related Systems **3** (1996) 41–54.

[34] P. Syrbu, *On middle Bol loops*, ROMAI J. **6(2)** (2010) 229–236.

[35] P. Syrbu and I. Grecu, *Loops with invariant flexibility under the isostrophy*, Bul. Acad. Stiinte Repub. Mold. Mat. **92(1)** (2020) 122–128.

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