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ON A CLASS OF SEMI-NORMAL MONOIDAL FUNCTORS

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Abstract

In this paper, we introduce and study an intermediate class, termed semi-normal monoidal functors, between the classes of monoidal and normal monoidal functors. We show that any left, or right, rigid braided category admits a contravariant semi-normal (co)monoidal endofunctor. Several examples are presented, showing the non triviality of this class. Moreover, it is shown that semi-normal monoidal functors from a monoidal category to a braided monoidal category, form a braided monoidal category.

Keywords: monoidal category, braiding, normal monoidal functor, natural transformation.

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1. Introduction

Monoidal categories play important roles not only in mathematics, where they serve as structures for grouping various classes of mathematical objects like, among others, groups, linear representations and linear differential matrix equations. They also bear importance in theoretical and mathematical physics, particularly within the context of quantum information theory and topological field theory [10, 11].

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Given two monoidal categories, a functor $F : C \longrightarrow D$ can be either a monoidal or a comonoidal functor. The composite of functors of one of these two types is again a functor of the same type. This implies that a covariant monoidal (resp., comonoidal) functor, sends a monoid (resp., comonoid) into a monoid (resp., comonoid). A monoidal functor is a triplet $(F; \varphi_0; \varphi_2)$, where $F : C \longrightarrow D$ is a functor, $\varphi_2 : F(U) \otimes' F(V) \to F(U \otimes V)$ and $\varphi_0 : I' \to F(I)$ are two maps satisfying the associativity, left and right unitality constraints [11, page 15], for all objects U and V of C. $(F; \varphi_0; \varphi_2)$ is called strong when φ_0 and φ_2 are isomorphisms [8, 9, 11], and it is called normal when only φ_0 is required to be an isomorphism.

In a strict rigid monoidal category C, the square of the duality functor is not generally isomorphic to the identity functor, this is referred to as non involutivity. If it is involutive, we then have that any object is canonically isomorphic to its bidual (reflexivity), in particular, this will imply that the unit object I of C is isomorphic to its dual: $I^* \simeq I \otimes I^* \simeq (I^* \otimes I)^* \simeq I^{**} \simeq I$. This situation is guaranteed if for example C was a ribbon *Ab*-category [2]. As a result, and using the fact that the ground ring $\Bbbk_C$, which is the endomorphism ring $\mathrm{End}_C(I)$ of C, is commutative, the duality maps d_I and b_I are inverse isomorphisms of each other. In general, the duality structures $(I^*; d_I; b_I)$ on I provide a semi invertibilty $d_I \circ b_I = id_I$. An additional structure of a braiding on C seems to equip C with a contravariant monoidal endofunctor, which is not generally normal.

In this paper, we slightly weaken normality of F and study the restricted resulting class of *semi-normal* functors, namely (co)monoidal functors $(F; \varphi_0; \varphi_2)$, such that φ_0 is only semi invertible, i.e., there exists a map φ_0^- in D, such that $\varphi_0^- \circ \varphi_0 = id$. Such a functor sends a monoid (M, m, η) with an additional similar structure, i.e., the existence of a map η^- such that, $\eta^- \circ \eta = id$, which we call augmented, to a monoid with the same additional structure. This holds dually for comonoids, where we shall call this time a coaugmented comonoid. Consequently, monoidal and comonoidal semi-normal functors correspond to augmented monoids and coaugmented comonoids respectively. We show in a main example that any monoidal *Ab*-category [10] admits a contravariant semi-normal monoidal functor to the category of modules over its commutative ground ring. Moreover, we show that any left, or right, rigid braided (monoidal) category, admits a contravariant semi-normal monoidal and comonoidal endofunctor, which is not necessarily normal, unless the category is for example ribbon [2, 10]. We also provide illustrating examples of monoidal categories admitting semi-normal (co)monoidal functors towards other ones. By means of these examples, the introduced class is shown to be distinguished from those of normal and strong monoidal functors.

Semi-normal monoidal functors between monoidal categories C and D are shown to constitute a braided category whenever D is braided, which admits itself, under some assumption, a semi-normal monoidal functor to some functors category, Section 4.

2. Preliminaries

In this section, we briefly recall the necessary basic notions from the theory of monoidal categories. For more details, we refer to [4, 9, 10, 11].

A monoidal category is a quintuplet $C = (C; \otimes; I; \alpha; l; r)$ consisting of a category C, an object I of C (called the unit object), a bifunctor (called tensor product) $\otimes : C \times C \longrightarrow C$ and natural isomorphisms $\alpha : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$ (called associativity constraint), $l : I \otimes A \longrightarrow A$ (called left unitality constraint) and $r : A \otimes I \longrightarrow A$ (called right unitality constraint) such that the pentagon and triangle axioms hold. The class of objects of C will be denoted by $\mathbf{Ob}(C)$. C is said to be strict provided that α, l and r are identities.

From now on, all the considered categories are assumed to be strict, according to a result of Mac-Lane [9] claiming that every monoidal category is equivalent to a strict one.

Recall from [10] that a category C is called an *Ab*-category (also called a pre-additive or a pre-abelian category) if the hom-set $\mathrm{Hom}_C(U, V)$ is an additive abelian group, for any objects U and V of C, and the composition and tensor product are bilinear.

Let now C be a monoidal *Ab*-category. The hom-set $\mathrm{Hom}_C(I, I)$ is denoted by $\Bbbk_C$ and referred to as the ground ring of C. $(\Bbbk_C, +, \circ)$ is a commutative ring and the composition coincides with the tensor product in it. Moreover, for all $U, V \in \mathbf{Ob}(C)$, the hom-set $\mathrm{Hom}_C(U, V)$ becomes a left $\Bbbk_C$-module, and the composition is $\Bbbk_C$-bilinear [10, Chapter II, 1.1, page 72], hence C is a $\Bbbk_C$-linear category [11, Chapter 4, 4.1.1].

A monoid in a monoidal category C is an object M equipped with a morphism $m : M \otimes M \longrightarrow M$ (called multiplication) and a morphism $\eta : I \longrightarrow M$ (called unit) satisfying associativity and unitality axioms [9, page 70]. A morphism $(M, m, \eta) \longrightarrow (M', m', \eta')$ is just a morphism $M \longrightarrow M'$ which commutes with m, m', and η, η'. Dually is defined a comonoid (N, Δ, ε) and a morphism of comonoids (by reversing the arrows), where morphisms are called now respectively comultiplication and counit. A bimonoid $(B, m, \eta, \Delta, \varepsilon)$ is an object B, such that (B, m, η) is a monoid, (B, Δ, ε) is a comonoid, and m, η are morphisms of comonoids (equivalently, Δ, ε are morphisms of monoids) [1, Proposition 1.11].

A monoidal functor $F : (C; \otimes; I) \longrightarrow (D; \otimes'; I')$ between monoidal categories is a triplet $(F; \varphi_0; \varphi_2)$, where $\varphi_2 : F(U) \otimes' F(V) \to F(U \otimes V)$, and $\varphi_0 : I' \to F(I)$ are morphisms in D, satisfying the following associativity, left and right unitality constraints respectively, for any objects U and V of C [11, 1.4.1, page 15]:

$$\begin{tikzcd} F(U) \otimes' F(V) \otimes' F(W) \arrow[r, "1\otimes\varphi_2"] \arrow[d, "\varphi_2\otimes 1"'] & F(U) \otimes' F(V \otimes W) \arrow[d, "\varphi_2"] \\ F(U \otimes V) \otimes' F(W) \arrow[r, "\varphi_2"] & F(U \otimes V \otimes W) \end{tikzcd}$$

$$\begin{tikzcd} I' \otimes' F(U) \arrow[r, "1"] \arrow[d, "\varphi_0\otimes 1"'] & F(U) \\ F(I) \otimes' F(U) \arrow[ur, "\varphi_2"'] & \end{tikzcd} \quad ;; \quad \begin{tikzcd} F(U) \otimes' I' \arrow[r, "1"] \arrow[d, "1\otimes\varphi_0"'] & F(U) \\ F(U) \otimes' F(I) \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

for all objects U, V and W of C.

F is called normal if φ_0 is an isomorphism and strong if both φ_0 and φ_2 are isomorphisms. Dually, one can define a comonoidal functor by reversing the arrows in the above diagrams.

A braiding c [7] for a monoidal category C is a natural isomorphism, consisting of a family of isomorphisms

$$c_{U;V} : U \otimes V \longrightarrow V \otimes U$$

in C, for any objects U and V of C, such that

$$c_{U;V\otimes W} = (id_V \otimes c_{U;W})(c_{U;V} \otimes id_W) \tag{1}$$

$$c_{U\otimes V;W} = (c_{U;W} \otimes id_V)(id_U \otimes c_{V;W}) \tag{2}$$

for any third object W of C.

Naturality of c means that for any morphisms $f : V \to V'$ and $g : U \to U'$ in C, we have

$$c_{V';U'}\,(f \otimes g) = (g \otimes f)\,c_{V;U}. \tag{3}$$

Any braiding c satisfies the following identity called the Yang-Baxter equation

$$(c_{V;W}\otimes id_U)(id_V\otimes c_{U;W})(c_{U;V}\otimes id_W) = (id_W\otimes c_{U;V})(c_{U;W}\otimes id_V)(id_U\otimes c_{V;W}) \tag{4}$$

for any objects U, V and W of C.

A braiding c is called a symmetry if $c^{-1}_{V;U} = c_{U;V}$, for any $U, V \in \mathbf{Ob}(C)$.

A symmetric monoidal category is a monoidal category equipped with a symmetry.

An object V of a monoidal category $(C;\otimes;I)$ admits a left dual if there exists an object V^* of C and morphisms $b_V : I \longrightarrow V\otimes V^*$ (coevaluation) and $d_V : V^*\otimes V \longrightarrow I$ (evaluation) in C such that

$$(id_V\otimes d_V)(b_V\otimes id_V) = id_V \qquad ;; \qquad (d_V\otimes id_{V^*})(id_{V^*}\otimes b_V) = id_{V^*}.$$

Right duality is defined dually and we say that V admits a dual if it admits a left and a right dual.

A monoidal category $(C;\otimes;I)$ is said to be rigid (resp., left, right rigid) if every object of C admits a dual (resp., left, right dual) [5, 6].

For any morphism $f : U \to V$ between left dualizable objects of C, one defines its dual morphism $f^* : V^* \to U^*$ by

$$f^* = (d_V\otimes id_{U^*})(id_{V^*}\otimes f\otimes id_{U^*})(id_{V^*}\otimes b_U).$$

The morphism $\lambda_{U;V} : V^*\otimes U^* \longrightarrow (U\otimes V)^*$ defined by

$$\lambda_{U;V} = (d_V\otimes id_{(U\otimes V)^*})(id_{V^*}\otimes d_U\otimes id_{V\otimes(U\otimes V)^*})(id_{V^*\otimes U^*}\otimes b_{U\otimes V}) \tag{5}$$

is an isomorphism for any two objects U and V of C, see [8, page 344] for more details. For any objects U, V and W of C, the isomorphism $\lambda_{U;V}$ satisfies the following identity

$$\lambda_{U;V\otimes W}(\lambda_{V;W}\otimes id_{U^*}) = \lambda_{U\otimes V;W}(id_{W^*}\otimes\lambda_{U;V}). \tag{6}$$

Indeed, we have

$$\begin{aligned}\lambda(\lambda\otimes 1) &= (d\otimes 1)(1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes d\otimes 1)(1\otimes 1\otimes b\otimes 1\otimes 1)\\ &\quad (1\otimes 1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b)\\ &= (d\otimes 1)(1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b)\\ &= (d\otimes 1)(1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes d\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\lambda(1\otimes\lambda) &= (d\otimes 1)(1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes d\otimes 1\otimes 1\otimes 1)\\ &\quad (1\otimes 1\otimes 1\otimes 1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b)\\ &= (d\otimes 1)(1\otimes d\otimes 1\otimes 1)(1\otimes 1\otimes d\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes b).\end{aligned}$$

Throughout the sequel, by C we mean a strict monoidal category $(C;\otimes;I)$ with unit object I, $\Bbbk$ denotes a base field and R a base commutative ring both are supposed to have a unit 1. Sometimes, we do not distinguish unit objects I and $I^{'}$ when no confusion may appear and we also write 1 to designate the identity map id.

3. Semi-normal functors

Definition. Let $(C;\otimes;I)$ and $(D;\otimes^{'};I^{'})$ be two monoidal categories.

A semi-normal monoidal functor from C to D is a triple $\big(F;(\varphi_0,\varphi_0^{-});\varphi_2\big)$, where $(F;\varphi_0;\varphi_2)$ is a monoidal functor and $\varphi_0^{-}:F(I)\longrightarrow I^{'}$ is a morphism in D such that $\varphi_0^{-}\varphi_0=id_{I'}$.

A semi-normal comonoidal functor from C to D is a triple $\big(F;(\varphi_0,\varphi_0^{-});\varphi_2^{\sim}\big)$ where, $(F;\varphi_0^{-};\varphi_2^{\sim})$ is a comonoidal functor and $\varphi_0:I^{'}\longrightarrow F(I)$ is a morphism in D, such that $\varphi_0^{-}\varphi_0=id_{I'}$.

Example 1. A strong monoidal (resp., comonoidal) functor is a semi-normal monoidal (resp., comonoidal) functor.

Recall that a Frobenius monoidal functor $\big(F;(r_0,i_0);(r,i)\big)$ is a functor $F:C\longrightarrow D$ between monoidal categories, such that $(F;r_0;r)$ is monoidal and $(F;i_0;i)$ is comonoidal, subject to adequate coherence axioms [3].

Example 2. A Frobenius monoidal functor $\big(F;(r_0,i_0);(r,i)\big)$, such that $i_0r_0=id$, is a semi-normal monoidal and comonoidal functor.

Remark 3. Let $F:C\longrightarrow D$ be a contravariant semi-normal monoidal functor between monoidal categories. For every split monomorphism $f:I\longrightarrow V$ (equivalently, split epimorphism $f:V\longrightarrow I$), $V\in\mathbf{Ob}(C)$; the following short sequence is left and right split

$$0\longrightarrow I\xrightarrow{\varphi_0}F(I)\xrightarrow{F(f)}F(V)\longrightarrow 0.$$

Example 4. Let $F:(\mathbf{vect}_{\Bbbk};\otimes_{\Bbbk};\Bbbk)\longrightarrow(\mathbf{Set};\times;\{*\})$ be the underlying (forgetful) functor between the category of finite dimensional vector spaces over a field $\Bbbk$ and the category of sets with cartesian product as tensor product, and the unit object is given by the set $\{*\}$ of one element. Consider the following monoidal structures on F

$$\begin{array}{cccc}\varphi_{2V;W}: & F(V)\times F(W) & \longrightarrow & F(V\otimes_{\Bbbk}W)\\ & (v,w) & \longmapsto & v\otimes w\end{array}$$

$$\begin{array}{cccc}\varphi_0: & \{*\} & \longrightarrow & F(\Bbbk)=\Bbbk\\ & * & \longmapsto & 1\end{array}\quad ;;\quad \begin{array}{cccc}\varphi_0^{-}: & \Bbbk & \longrightarrow & \{*\}\\ & x & \longmapsto & *\end{array}$$

Then, $\big(F;(\varphi_0,\varphi_0^{-});\varphi_2\big)$ is a semi-normal monoidal functor, and which is neither normal, nor strong monoidal functor. In fact, the associativity diagram:

$$\begin{tikzcd} F(U)\times F(V)\times F(W) \arrow[r, "{(1,\varphi_2)}"] \arrow[d, "{(\varphi_2,1)}"'] & F(U)\times F(V\otimes_{\Bbbk}W) \arrow[d, "\varphi_2"] \\ F(U\otimes_{\Bbbk}V)\times F(W) \arrow[r, "\varphi_2"] & F(U\otimes_{\Bbbk}V\otimes_{\Bbbk}W) \end{tikzcd}$$

and the unitality diagrams:

$$\begin{tikzcd} \{*\}\times F(U) \arrow[r, "1"] \arrow[d, "(\varphi_0,1)"'] & F(U) \\ \Bbbk\times F(U) \arrow[ur, "\varphi_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(U)\times\{*\} \arrow[r, "1"] \arrow[d, "(1,\varphi_0)"'] & F(U) \\ F(U)\times\Bbbk \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

are clearly commutative. Moreover, we have $\varphi_0^-\varphi_0 = id$.

Definition. Let C be a monoidal category and M and N objects of C.

An augmented monoid is a triple $\big(M; m; (\eta,\eta^-)\big)$ where, $(M;m;\eta)$ is a monoid and $\eta^- : M \longrightarrow I$ is a map in C such that $\eta^-\circ\eta = id_I$.

A coaugmented comonoid is a triple $\big(N;\Delta;(\varepsilon^-,\varepsilon)\big)$ where, $(M;\Delta;\varepsilon^-)$ is a comonoid and $\varepsilon : I\longrightarrow N$ is a map in C such that $\varepsilon^-\circ\varepsilon = id_I$.

A morphism of augmented monoids $\big(M;m;(\eta,\eta^-)\big)\longrightarrow\big(M';m';(\eta',\eta'^-)\big)$ is a morphism of monoids $(M;m;\eta)\longrightarrow(M';m';\eta')$ (given by a map $f : M\longrightarrow M'$), such that $\eta'^-\circ f = \eta^-$.

Similarly, a morphism of coaugmented comonoids $\big(N;\Delta;(\varepsilon^-,\varepsilon)\big)\longrightarrow\big(N';\Delta'; (\varepsilon'^-,\varepsilon')\big)$ is a morphism of comonoids which commutes with ε and ε'.

Example 5. Every bimonoid is an augmented (resp. coaugmented) monoid (resp. comonoid).

It is well known that (covariant) monoidal (resp., comonoidal) functors send monoids (resp., comonoids) to monoids (resp., comonoids) [1]. We get the next result.

Proposition 6. (a) *A covariant semi-normal monoidal functor between monoidal categories sends augmented monoids to augmented monoids.*

(b) *A covariant semi-normal comonoidal functor between monoidal categories sends coaugmented comonoids to coaugmented comonoids.*

(c) *A contravariant semi-normal monoidal functor between monoidal categories sends augmented monoids to coaugmented comonoids.*

(d) *A contravariant semi-normal comonoidal functor between monoidal categories sends coaugmented comonoids to augmented monoids.*

Proof. Straightforward. ∎

Corollary 7. *Let F be a semi-normal monoidal (resp., comonoidal) functor between monoidal categories. Then, $F(I)$ is an augmented monoid (resp., coaugmented comonoid).*

Corollary 8. *A (covariant) semi-normal monoidal (resp., comonoidal) functor sends a morphism of augmented monoids (resp., coaugmented comonoids) to a morphism of augmented monoids (resp., coaugmented comonoids).*

Theorem 9. *Every left (resp., right) rigid braided monoidal category admits a semi-normal monoidal and comonoidal endofunctor.*

Proof. We prove the result only for left rigidity, since it holds similarly for right rigidity. Assume that every object V of C admits a left dual V^*. Let $F : C \longrightarrow C$ be the left duality functor, i.e., the functor defined by $F(V) = V^*$ and $F(f) = f^*$ for every object V of C and every morphism f of C and let

$$\varphi_0 : I \longrightarrow I^* \quad ;; \quad \varphi_0^- : I^* \longrightarrow I \quad ;; \quad \varphi_{2\,U,V} : U^* \otimes V^* \longrightarrow (U \otimes V)^*$$

be the morphisms defined by

$$\varphi_{2\,U,V} = \lambda_{U,V} \circ c_{U^*,V^*} \quad ;; \quad \varphi_0 = b_I \quad ;; \quad \varphi_0^- = d_I$$

where, c is the braiding on C, d_I and b_I are the corresponding evaluation and coevaluation maps of the unit object, and $\lambda_{U,V}$ is the isomorphism defined in (5). Then, we have $\varphi_0^- \varphi_0 = d_I b_I = (id_I \otimes d_I)(b_I \otimes id_I) = id_I$, by strictness of C as it is assumed throughout the paper. Moreover, the following (associativity) diagram commutes

$$\begin{tikzcd}
U^* \otimes V^* \otimes W^* \arrow[r, "id_{U^*}\otimes c_{V^*;W^*}"] \arrow[d, "c_{U^*;V^*}\otimes id_{W^*}"] & U^* \otimes W^* \otimes V^* \arrow[r, "1\otimes\lambda"] \arrow[d, "c_{U^*;W^*\otimes V^*}"] & U^* \otimes (V \otimes W)^* \arrow[d, "c_{U^*;(U\otimes W)^*}"] \\
V^* \otimes U^* \otimes W^* \arrow[r, "c_{V^*\otimes U^*;W^*}"] \arrow[d, "\lambda\otimes 1"] & W^* \otimes V^* \otimes U^* \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\lambda"] & (V \otimes W)^* \otimes U^* \arrow[d, "\lambda"] \\
(U \otimes V)^* \otimes W^* \arrow[r, "c_{(U\otimes V)^*;W^*}"] & W^* \otimes (U \otimes V)^* \arrow[r, "\lambda"] & (U \otimes V \otimes W)^*
\end{tikzcd}$$

In fact, for the commutativity of the upper left square: by the first and second axioms of the braiding c as displayed in (1) and (2), we have

$$c_{U^*;W^*\otimes V^*} = (id_{W^*} \otimes c_{U^*;V^*})(c_{U^*;W^*} \otimes id_{V^*})$$

$$c_{V^*\otimes U^*;W^*} = (c_{V^*;W^*} \otimes id_{U^*})(id_{V^*} \otimes c_{U^*;W^*}).$$

Then, commutativity of this square is equivalent to prove that

$$\begin{aligned}(id_{W^*} \otimes c_{U^*;V^*})&(c_{U^*;W^*}\otimes id_{V^*})(id_{U^*} \otimes c_{V^*;W^*}) \\ &= (c_{V^*;W^*} \otimes id_{U^*})(id_{V^*} \otimes c_{U^*;W^*})(c_{U^*;V^*} \otimes id_{W^*})\end{aligned}$$

which holds since this is exactly the Yang-Baxter equation as displayed in (4).

For the commutativity of the upper right and lower left squares, this is due to the naturality of the braiding (3). For the lower right square, this holds by (6). Now, the left unitality diagram:

$$\begin{tikzcd} U^* \otimes I \arrow[r, "1_{U^*}"] \arrow[d, "1_{U^*}\otimes b_I"'] & U^* \\ U^* \otimes I^* \arrow[ur, "\lambda_{U,I} \circ c_{U^*,I^*}"'] & \end{tikzcd}$$

is commutative. In fact, we have

$$\begin{aligned}
\lambda_{U,I} \ \circ \ c_{U^*,I^*} \circ (1_{U^*} \otimes b_I) &= (d_I \otimes 1_{U^*}) \circ c_{U^*,I^*} \circ (1_{U^*} \otimes b_I)\\
&= (d_I \otimes 1_{U^*}) \circ (b_I \otimes 1_{U^*}) \circ c_{U^*,I}\\
&= 1_{U^*} \circ c_{U^*,I}\\
&= 1_{U^*}.
\end{aligned}$$

Similarly, the following right unitality diagram is commutative:

$$\begin{tikzcd} I \otimes U^* \arrow[r, "1_{U^*}"] \arrow[d, "b_I\otimes 1_{U^*}"'] & U^* \\ I^* \otimes U^* \arrow[ur, "\lambda_{I,U} \circ c_{I^*,U^*}"'] & \end{tikzcd}$$

Hence, $\big(F;(\varphi_0,\varphi_0^-);\varphi_2\big)$ is a semi-normal monoidal functor. Furthermore, for any objects U and V of C, the morphism $\lambda_{U,V} \circ c_{U,V}$ is invertible with inverse $c_{U,V}^{-1} \circ \lambda_{U,V}^{-1}$, then, in a similar way, one can easily check that $\big(F;(\varphi_0,\varphi_0^-);\varphi_2^{-1}\big)$ is a semi-normal comonoidal functor. ∎

Remark 10. Note that the above defined semi-normal monoidal structures on the left duality functor do not turn out, in general, to normal monoidal structures on it. If C is moreover a ribbon *Ab*-category, then this turns out to a normal (resp., strong) monoidal functor. In fact, I being isomorphic in this case to its bidual I^{**}, implies that we also have $\varphi_0\varphi_0^- = id_{I^*}$, see [10, Corollary 2.6.2].

The composite of semi-normal (co)monoidal functors is again a semi-normal (co)monoidal functor. More exactly, we have

Proposition 11. *Let $F : C \longrightarrow D$ and $G : D \longrightarrow E$ be two functors between monoidal categories.*

(a) *If G is covariant, we have*
   (i) *if F and G are both semi-normal monoidal functors, then, $G \circ F$ is a semi-normal monoidal functor as well;*
   (ii) *if F and G are both semi-normal comonoidal functors, then, $G \circ F$ is a semi-normal comonoidal functor as well.*

(b) *If G is contravariant, we have*

(i) *if F is a semi-normal comonoidal functor and G is a semi-normal monoidal functor, then, $G \circ F$ is a semi-normal monoidal functor;*

(ii) *if F is a semi-normal monoidal functor and G is a semi-normal comonoidal functor, then, $G \circ F$ is a semi-normal comonoidal functor.*

Proof. Denote by F_0, F_0^- and F_2, the monoidal structures of F and by G_0, G_0^- and G_2 those of G. Hence, the monoidal structures of $G \circ F$ are denoted and given as follows.

If G is covariant

$$(G \circ F)_0 = G(F_0)G_0 \quad ;; \quad (G \circ F)_0^- = G_0^- G(F_0^-).$$

If G is contravariant

$$(G \circ F)_0 = G(F_0^-)G_0 \quad ;; \quad (G \circ F)_0^- = G_0^- G(F_0).$$

In both cases we have

$$(G \circ F)_0^-(G \circ F)_0 = id.$$

In the first and third cases (a), (i) and (b), (i) of the Proposition, $(G \circ F)_2$ is given by

$$(G \circ F)_{2A;B} = G(F_{2A;B})G_{2F(A);F(B)}.$$

In the second and fourth cases (a), (ii) and (b), (ii), $(G \circ F)_2$ is given by

$$(G \circ F)_{2A;B} = G_{2F(A);F(B)}G(F_{2A;B}),$$

for any objects A and B of C. ∎

Proposition 12. *Let F and F' be semi-normal monoidal functors between monoidal categories. Then, $F \times F'$ is as well a semi-normal monoidal functor.*

Proof. Let $F : C \longrightarrow D$ and $F' : C' \longrightarrow D'$ be functors as assumed. Then, $F \times F' : C \times C' \longrightarrow D \times D'$ is a semi-normal monoidal functor in the canonical way, namely via the structures defined as follows. For any $U, V \in \mathbf{Ob}(C)$ and $U', V' \in \mathbf{Ob}(C')$:

(1) $(F \times F')_{2(U,U');(V,V')} := F_{2U;V} \times F'_{2U';V'}$.

(2) $(F \times F')_0 := F_0 \times F'_0$.

(3) $(F \times F')_0^- := F_0^- \times F_0^{\prime -}$. ∎

We give now some examples of monoidal categories admitting semi-normal, which are not necessarily normal, monoidal functors to other ones.

Proposition 13. *Let* $\mathbf{bialg_R}$ *be the category of finitely generated and projective bialgebras over R (see* [11, page 101] *for a definition). Then*

(i) $\mathbf{bialg_R}$ *admits a covariant semi-normal monoidal functor to the category* $\mathbf{Mod_R}$ *of modules over* R*, via the forgetful functor* $F : \mathbf{bialg_R} \longrightarrow \mathbf{Mod_R}$.

(ii) *Let* $(B; m; \eta; \Delta; \varepsilon)$ *be an object of* $\mathbf{bialg_R}$*. Then,* $\mathbf{bialg_R}$ *admits a contravariant semi-normal monoidal functor to* $\mathbf{Mod_R}$ *via the functor defined by*

$$\begin{aligned} F_B := (-)^* \otimes B : \mathbf{bialg_R} &\longrightarrow \mathbf{Mod_R} \\ H &\longmapsto H^* \otimes B \\ f &\longmapsto F_B(f) = f^* \otimes id_B \end{aligned}$$

where, $H^* = \mathrm{Hom}_R(H, R)$*, and* f^* *is the dual morphism of* f.

Proof. (i) Straightforward.

(ii) The category $\mathbf{Mod_R}$ is a monoidal category $(\mathbf{Mod_R}; \otimes_R; R)$, but not strict. The associativity constraint is $\alpha_{U,V,W} : U \otimes (V \otimes W) \longrightarrow (U \otimes V) \otimes W$, defined by $\alpha(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$, for any $u \in U$, $v \in V$ and $w \in W$, for any R-modules U, V and W. The left unitality constraint is defined by $l_U : R \otimes U \longrightarrow U$, $r \otimes u \mapsto r.u$, with "." the R-module structure product of U in $\mathbf{Mod_R}$. The right unitality constraint r_U is defined similarly. The monoidal structures on F_B are defined by

(a) For any objects N and M of $\mathbf{bialg_R}$, $(F_B)_{2\ N,M}$ is the following composite:

$$\begin{tikzcd}
(N^* \otimes B) \otimes (M^* \otimes B) \arrow[r, "(F_B)_{2\ N,M}"] \arrow[d, "\alpha"'] & (N \otimes M)^* \otimes B \\
{[(N^* \otimes B) \otimes M^*] \otimes B} \arrow[d, "\alpha^{-1} \otimes id_B"'] & (M^* \otimes N^*) \otimes B \arrow[u, "\lambda \otimes id_B"'] \\
{[N^* \otimes (B \otimes M^*)] \otimes B} \arrow[d, "(id_{N^*} \otimes \tau) \otimes id_B"'] & (M^* \otimes N^*) \otimes (R \otimes B) \arrow[u, "(id_{M^*} \otimes id_{N^*}) \otimes l_B"'] \\
{[N^* \otimes (M^* \otimes B)] \otimes B} \arrow[d, "\alpha \otimes id_B"'] & (M^* \otimes N^*) \otimes (B \otimes B) \arrow[u, "(id_{M^*} \otimes id_{N^*}) \otimes (\varepsilon \otimes id_B)"'] \\
{[(N^* \otimes M^*) \otimes B] \otimes B} \arrow[r, "(\tau \otimes id_B) \otimes id_B"] & {[(M^* \otimes N^*) \otimes B] \otimes B} \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

where, τ is the flip map, and λ is the isomorphism (5).

(b) $(F_B)_0 = (\beta \otimes id_B)\ l_B^{-1}\ \eta : R \longrightarrow B \longrightarrow R \otimes B \longrightarrow R^* \otimes B$, where $\beta : R \to R^*$ is the canonical isomorphism.

(c) $(F_B)_0^- = \varepsilon\, l_B\, (\beta^{-1} \otimes id_B) : R^* \otimes B \longrightarrow R \otimes B \longrightarrow B \longrightarrow R.$

F_B is contravariant by definition and we have $(F_B)_0^-(F_B)_0 = \varepsilon\eta = id_R$ by the compatibility bialgebra structures of B. Furthermore, the following left unitality diagram

$$\begin{tikzcd} R \otimes (U^* \otimes B) \arrow[r, "l_{(U^*\otimes B)}"] \arrow[d, "(F_B)_0 \otimes id_{(U^*\otimes B)}"'] & U^* \otimes B \\ (R^* \otimes B) \otimes (U^* \otimes B) \arrow[r, "(F_B)_{2\ R,U}"'] & (R \otimes U)^* \otimes B \arrow[u, "F_B(l_U^{-1})"'] \end{tikzcd}$$

and the right unitality diagram:

$$\begin{tikzcd} (U^* \otimes B) \otimes R \arrow[r, "r_{(U^*\otimes B)}"] \arrow[d, "id_{(U^*\otimes B)} \otimes (F_B)_0"'] & U^* \otimes B \\ (U^* \otimes B) \otimes (R^* \otimes B) \arrow[r, "(F_B)_{2\ U,R}"'] & (U \otimes R)^* \otimes B \arrow[u, "F_B(r_U^{-1})"'] \end{tikzcd}$$

are commutative. In fact, for the first diagram: for any $r \in R$, $u \in U$, $b \in B$, we have

$$\begin{aligned} F_B(l_U^{-1}) \circ (F_B)_{2\ R,U} \circ & \big((F_B)_0 \otimes id_{(U^*\otimes B)}\big)\big(r \otimes (u \otimes b)\big) \\ &= F_B(l_U^{-1}) \circ (F_B)_{2\ R,U}\big(\big(\beta(1_R) \otimes \eta(r)\big) \otimes (u \otimes b)\big) \\ &= F_B(l_U^{-1})\Big(\lambda\big(u \otimes \beta(1_R)\big) \otimes \big(\varepsilon\eta(r).b\big)\Big) \\ &= F_B(l_U^{-1})\Big(\lambda\big(u \otimes \beta(1_R)\big) \otimes (r.b)\Big) \\ &= \lambda\big(u \otimes \beta(1_R)\big) \circ l_U^{-1} \otimes (r.b) \\ &= r.\Big(\lambda\big(u \otimes \beta(1_R)\big) l_U^{-1} \otimes b\Big). \end{aligned}$$

On the other hand, for every $a \in U$ we have

$$\lambda\big(u \otimes \beta(1_R)\big) l_U^{-1}(a) = \lambda\big(u \otimes \beta(1_R)\big)(1_R \otimes a) = u(a)$$

where, the isomorphism λ as in (5) is given in this case by

$$\lambda_{N;M} : M^* \otimes N^* \longrightarrow (N \otimes M)^*,\ f \otimes g \mapsto \Big(g \otimes f : n \otimes m \mapsto g(n)f(m) \in R\Big).$$

Hence

$$r.\Big(\lambda\big(u \otimes \beta(1_R)\big) l_U^{-1} \otimes b\Big) = r.(u \otimes b) = l_{(U^*\otimes B)}\big(r \otimes (u \otimes b)\big),$$

which completes the proof. Similarly, commutativity of the second (right unitality) diagram holds.

For the commutativity of the associativity diagram of F_B : let N, M and P be three objects of $\mathbf{bialg_R}$, and let $n \in N^*$, $m \in M^*$, $p \in P^*$ and $b, b', b'' \in B$. In order to simplify the computations, we will omit the associativity constraint α. Then, on the one hand, we have

$$\begin{aligned}&(F_B)_2\ \big((F_B)_2 \otimes 1\big)(n \otimes b \otimes m \otimes b' \otimes p \otimes b'')\\&= (F_B)_2\big(\varepsilon(b).\lambda(n \otimes m) \otimes b' \otimes p \otimes b''\big)\\&= \varepsilon(b)\varepsilon(b').\lambda\big(\lambda(n \otimes m) \otimes p\big) \otimes b''.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&(F_B)_2\ \big(1 \otimes (F_B)_2\big)(n \otimes b \otimes m \otimes b' \otimes p \otimes b'')\\&= (F_B)_2\big(\varepsilon(b').n \otimes b \otimes \lambda(m \otimes p) \otimes b''\big)\\&= \varepsilon(b)\varepsilon(b').\lambda\big(n \otimes \lambda(m \otimes p)\big) \otimes b''.\end{aligned}$$

Hence, the main step consists of showing that

$$\lambda_{P;M\otimes N}(\lambda_{M;N} \otimes id_{P^*}) = \lambda_{P\otimes M;N}(id_{N^*} \otimes \lambda_{P;M}),$$

which holds generally as in (6), and in particular, this holds for any three objects of $\mathbf{bialg_R}$. Hence, $\big(F_B; ((F_B)_0, (F_B)_0^-), (F_B)_2\big)$ is a contravariant semi-normal monoidal functor. ■

Remark 14. Note that since the dual of a finitely generated and projective bialgebra is also a finitely generated projective bialgebra, then the category $\mathbf{bialg_R}$ admits also a contravariant semi-normal monoidal functor to Mod_R via the functor defined by

$$\begin{aligned}F_{B^*} := (-)^* \otimes B^* : \mathbf{bialg_R} &\longrightarrow \mathbf{Mod_R}\\ H &\longmapsto H^* \otimes B^*\\ f &\longmapsto F_{B^*}(f) = f^* \otimes id_{B^*}\end{aligned}$$

with monoidal structures defined this time as follows.

(a) For any objects N and M of $\mathbf{bialg_R}$, $(F_{B^*})_{2\ N,M}$ is the following composite

$$\begin{tikzcd}
(N^* \otimes B^*) \otimes (M^* \otimes B^*) \arrow[r, "(F_{B^*})_{2\ N,M}"] \arrow[d, "\alpha"'] & (N \otimes M)^* \otimes B^* \\
{[(N^* \otimes B^*) \otimes M^*] \otimes B^*} \arrow[d, "\alpha^{-1}\otimes id_{B^*}"'] & (M^* \otimes N^*) \otimes (R \otimes B^*) \arrow[u, "\lambda\otimes\, l_{B^*}"'] \\
{[N^* \otimes (B^* \otimes M^*)] \otimes B^*} \arrow[d, "(id_{N^*}\otimes\tau)\otimes id_{B^*}"'] & (M^* \otimes N^*) \otimes (R^* \otimes B^*) \arrow[u, "(id_{M^*}\otimes id_{N^*})\otimes\, (\beta^{-1}\otimes id_{B^*})"'] \\
{[N^* \otimes (M^* \otimes B^*)] \otimes B^*} \arrow[d, "\alpha\otimes id_{B^*}"'] & (M^* \otimes N^*) \otimes (B^* \otimes B^*) \arrow[u, "(id_{M^*}\otimes id_{N^*})\otimes\, (\eta^*\otimes id_{B^*})"'] \\
{[(N^* \otimes M^*) \otimes B^*] \otimes B^*} \arrow[r, "(\tau\otimes id_{B^*})\otimes id_{B^*}"] & {[(M^* \otimes N^*) \otimes B^*] \otimes B^*} \arrow[u, "\alpha^{-1}"']
\end{tikzcd}$$

(b) $(F_{B^*})_0 = (\beta \otimes id_{B^*})\, l_{B^*}^{-1}\, \varepsilon^*\, \beta : R \longrightarrow R^* \longrightarrow B^* \longrightarrow R \otimes B^* \longrightarrow R^* \otimes B^*$,

(c) $(F_{B^*})_0^- = \beta^{-1}\, \eta^*\, l_{B^*}\, (\beta^{-1} \otimes id_{B^*}) : R^* \otimes B^* \longrightarrow R \otimes B^* \longrightarrow B^* \longrightarrow R^* \longrightarrow R$.

Corollary 15. *For every $B \in \mathbf{Ob}(\mathbf{bialg_R})$, the covariant functor F_B of Proposition* 13, *extends to a covariant semi-normal monoidal functor, given by*

$$\begin{aligned}
\mathcal{F} : (\mathbf{bialg_R}; \otimes_R; R) &\longrightarrow \Big(\mathbf{SNFun}(\mathbf{bialg_R}; \mathbf{Mod_R}); \otimes; \mathbb{I}\Big)\\
B &\longmapsto F_B\\
f : B \to B' &\longmapsto \Big(\mathcal{F}(f)\Big)_{H\in bialg_R} = id_{H^*} \otimes f : F_B(H) \to F_{B'}(H)
\end{aligned}$$

from $\mathbf{bialg_R}$ *to the category* $\big(\mathbf{SNFun}(\mathbf{bialg_R}; \mathbf{Mod_R}); \otimes; \mathbb{I}\big)$ *of contravariant semi-normal monoidal functors from* $\mathbf{bialg_R}$ *to* $\mathbf{Mod_R}$*, which is monoidal, see Section 4 for proof of its monoidality, where the monoidal product is the pointwise monoidal product, and the unit object $\mathbb{I}$ is the functor associating to each bialgebra H, the R-module R.*

Proof. The monoidal structures are defined by

(a) $\mathcal{F}_0 : \mathbb{I} \longrightarrow F_R$, subject to

$$(\mathcal{F}_0)_{H\in \mathbf{Ob}(\mathbf{bialg_R})} := r_{H^*}^{-1}\, \varepsilon_H^*\, \beta : R \longrightarrow R^* \longrightarrow H^* \longrightarrow H^* \otimes R.$$

(b) $\mathcal{F}_0^- : F_R \longrightarrow \mathbb{I}$, subject to

$$(\mathcal{F}_0^-)_{H\in \mathbf{Ob}(\mathbf{bialg_R})} := \beta^{-1}\, \eta_H^*\, r_{H^*} : H^* \otimes R \longrightarrow H^* \longrightarrow R^* \longrightarrow R.$$

(c) $\mathcal{F}_{2\ N,M} : F_N \otimes F_M \longrightarrow F_{N\otimes M}$, for any objects N and M of $\mathbf{bialg_R}$, subject to

$$(\mathcal{F}_{2\ N,M})_H : (F_N \otimes F_M)_H = (H^* \otimes N) \otimes (H^* \otimes M) \longrightarrow (F_{N\otimes M})_H = H^* \otimes (N \otimes M)$$

for every $H \in \mathbf{Ob}(\mathbf{bialg_R})$, which is defined by the following composite

$$\begin{tikzcd}
(H^* \otimes N) \otimes (H^* \otimes M) \arrow[r, "(\mathcal{F}_{2\ N,M})_H"] \arrow[d, "\alpha"'] & H^* \otimes (N \otimes M) \\
{[(H^* \otimes N) \otimes H^*] \otimes M} \arrow[d, "\alpha^{-1}\otimes id_M"'] & \\
{[H^* \otimes (N \otimes H^*)] \otimes M} \arrow[d, "(id_{H^*}\otimes\tau)\otimes id_M"'] & (R \otimes H^*) \otimes (N \otimes M) \arrow[uu, "l_{H^*}\otimes(id_N\otimes id_M)"'] \\
{[H^* \otimes (H^* \otimes N)] \otimes M} \arrow[d, "\alpha\otimes id_M"'] & (R^* \otimes H^*) \otimes (N \otimes M) \arrow[u, "(\beta^{-1}\otimes id_{H^*})\otimes(id_N\otimes id_M)"'] \\
{[(H^* \otimes H^*) \otimes N] \otimes M} \arrow[r, "\alpha^{-1}"] & (H^* \otimes H^*) \otimes (N \otimes M) \arrow[u, "(\eta_H^*\otimes id_{H^*})\otimes(id_N\otimes id_M)"']
\end{tikzcd}$$

The following left unitality diagram

$$\begin{tikzcd}
\mathbb{I} \otimes F_B \arrow[r, "l_{F_B}"] \arrow[d, "\mathcal{F}_0\otimes id"'] & F_B \\
F_R \otimes F_B \arrow[r, "\mathcal{F}_{2\ R,B}"'] & F_{R\otimes B} \arrow[u, "\mathcal{F}(l_B)"']
\end{tikzcd}$$

and the right unitality diagram

$$\begin{tikzcd}
F_B \otimes \mathbb{I} \arrow[r, "r_{F_B}"] \arrow[d, "id\otimes\mathcal{F}_0"'] & F_B \\
F_B \otimes F_R \arrow[r, "\mathcal{F}_{2\ B,R}"'] & F_{B\otimes R} \arrow[u, "\mathcal{F}(r_B)"']
\end{tikzcd}$$

are commutative, where for every $B \in \mathbf{Ob}(\mathbf{bialg_R})$, the left and right unitality constraints on objects F_B are also denoted by l_{F_B} and r_{F_B} respectively. In fact, for the first diagram, one should prove that for every $H \in \mathbf{Ob}(\mathbf{bialg_R})$, the following diagram commutes

$$\begin{tikzcd}
R \otimes (H^* \otimes B) \arrow[r, "l_{(H^*\otimes B)}"] \arrow[d, "(\mathcal{F}_0)_H\otimes id_{(U^*\otimes B)}"'] & H^* \otimes B \\
(H^* \otimes R) \otimes (H^* \otimes B) \arrow[r, "(\mathcal{F}_{2\ R,B})_H"'] & H^* \otimes (R \otimes B) \arrow[u, "id_{H^*}\otimes l_B"']
\end{tikzcd}$$

Let us proceed by elementary calculus, where we will also omit the associativity constraint α. For every $r \in R$, $h \in H^*$, $b \in B$, we have

$$\begin{aligned}
\Big(id_{H^*} \otimes l_B\Big) \circ \big(\mathcal{F}_{2\ R,B}\big)_H \circ \Big((\mathcal{F}_0)_H \otimes id_{(U^* \otimes B)}\Big)\big(r \otimes (h \otimes b)\big) \\
&= \Big(id_{H^*} \otimes l_B\Big) \circ \big(\mathcal{F}_{2\ R,B}\big)_H \big(\beta(r)\varepsilon \otimes 1_R \otimes h \otimes b\big) \\
&= \Big(id_{H^*} \otimes l_B\Big)\big(\beta^{-1}\beta(r).h \otimes 1_R \otimes b\big) \\
&= \Big(id_{H^*} \otimes l_B\Big)\big(r.h \otimes 1_R \otimes b\big) \\
&= r.h \otimes b \\
&= l_{(H^* \otimes B)}(r \otimes h \otimes b).
\end{aligned}$$

Commutativity of the second (right unitality) diagram holds in a similar way.

For the commutativity of the associativity diagram of $\mathcal{F}$: let H, N, M and P be objects of $\mathbf{bialg_R}$, and let $n \in N$, $m \in M$, and $p \in P$. Then, for every $H \in \mathbf{Ob}(\mathbf{bialg_R})$ and for any $h, h', h'' \in H^*$, on the one hand, we have

$$\begin{aligned}
(\mathcal{F})_2\big((\mathcal{F})_2 \otimes 1\big)(h \otimes n \otimes h' \otimes m \otimes h'' \otimes p) &= (\mathcal{F})_2\Big(\beta^{-1}(h\eta).h' \otimes n \otimes m \otimes h'' \otimes p\Big) \\
&= \beta^{-1}(h\eta)\beta^{-1}(h'\eta).h'' \otimes n \otimes m \otimes p.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\mathcal{F})_2\big(1 \otimes (\mathcal{F})_2\big)(h \otimes n \otimes h' \otimes m \otimes h'' \otimes p) &= (\mathcal{F})_2\Big(h \otimes n \otimes \beta^{-1}(h'\eta).h'' \otimes m \otimes p\Big) \\
&= (\mathcal{F})_2\Big(\beta^{-1}(h'\eta).h \otimes n \otimes h'' \otimes m \otimes p\Big) \\
&= \beta^{-1}(h'\eta)\beta^{-1}(h\eta).h'' \otimes n \otimes m \otimes p.
\end{aligned}$$

Hence, $\mathcal{F}$ is monoidal. Now, for every $H \in \mathbf{Ob}(\mathbf{bialg_R})$, we have

$$\Big(\mathcal{F}_0^- \circ \mathcal{F}_0\Big)_H = \beta^{-1}\ \eta_H^*\ r_{H^*} r_{H^*}^{-1}\ \varepsilon_H^*\ \beta = \beta^{-1}\ (\varepsilon_H \eta_H)^*\ \beta = \beta^{-1}\beta = id_R.$$

Finally, $\big(\mathcal{F}; (\mathcal{F}_0, \mathcal{F}_0^-); \mathcal{F}_2\big)$ is a semi-normal monoidal functor. ∎

Similarly to the previous Corollary and based on Remark 14, we get the next conclusion.

Corollary 16. *For every $B \in \mathbf{Ob}(\mathbf{bialg_R})$, the contravariant functor F_{B^*} of Remark* 14, *extends to a functor between* $\mathbf{bialg_R}$ *and* $\mathbf{SNFun}(\mathbf{bialg_R}; \mathbf{Mod_R})$, *given by*

$$\begin{aligned}\mathcal{F}^* : (\mathbf{bialg_R}; \otimes_R; R) &\longrightarrow \Big(\mathbf{SNFun}(\mathbf{bialg_R}; \mathbf{Mod_R}); \otimes; \mathbb{I}\Big)\\ B &\longmapsto F_{B^*}\\ f : B \to B' &\longmapsto \Big(\mathcal{F}^*(f)\Big)_{H\in\mathbf{Ob}(\mathbf{bialg_R})} = id_{H^*} \otimes f^* : F_{B'^*}(H) \to F_{B^*}(H)\end{aligned}$$

Then, $\mathcal{F}^*$ is again a contravariant semi-normal monoidal functor.

Proof. In fact, the monoidal structures are given this time as follows.

(a) ${\mathcal{F}^*}_0 : \mathbb{I} \longrightarrow F_{R^*}$, subject to $\Big({\mathcal{F}^*}_0\Big)_{H\in\mathbf{Ob}(\mathbf{bialg_R})} := (id_{H^*} \otimes \beta)\, r^{-1}_{H^*}\, \varepsilon^*_H\, \beta$

$$R \longrightarrow R^* \longrightarrow H^* \longrightarrow H^* \otimes R \longrightarrow H^* \otimes R^*.$$

(b) ${\mathcal{F}^*}^-_0 : F_{R^*} \longrightarrow \mathbb{I}$, subject to $\Big({\mathcal{F}^*}^-_0\Big)_{H\in\mathbf{Ob}(\mathbf{bialg_R})} := \beta^{-1}\eta^*_H r_{H^*}(id_{H^*} \otimes \beta^{-1})$

$$H^* \otimes R^* \longrightarrow H^* \otimes R \longrightarrow H^* \longrightarrow R^* \longrightarrow R.$$

(c) ${\mathcal{F}^*}_{2\ N,M} : F_{N^*} \otimes F_{M^*} \longrightarrow F_{(N\otimes M)^*}$, for any $N, M \in \mathbf{Ob}(\mathbf{bialg_R})$, subject to $\Big({\mathcal{F}^*}_{2\ N,M}\Big)_H$

$$(F_{N^*} \otimes F_{M^*})_H = (H^* \otimes N^*) \otimes (H^* \otimes M^*) \longrightarrow (F_{(N\otimes M)^*})_H = H^* \otimes (N \otimes M)^*$$

for every $H \in \mathbf{Ob}(\mathbf{bialg_R})$, which is defined by the following composite

$$\begin{tikzcd}
(H^* \otimes N^*) \otimes (H^* \otimes M^*) \arrow[r, "({\mathcal{F}^*}_{2\ N,M})_H"] \arrow[d, "\alpha"'] & H^* \otimes (N \otimes M)^* \\
{[(H^* \otimes N^*) \otimes H^*] \otimes M^*} \arrow[d, "\alpha^{-1}\otimes id_{M^*}"'] & H^* \otimes (M^* \otimes N^*) \arrow[u, "id_{H^*}\otimes\lambda"'] \\
{[H^* \otimes (N^* \otimes H^*)] \otimes M^*} \arrow[d, "(id_{H^*}\otimes\tau)\otimes id_{M^*}"'] & (R \otimes H^*) \otimes (M^* \otimes N^*) \arrow[u, "l_{H^*}\otimes(id_{M^*}\otimes id_{N^*})"'] \\
{[H^* \otimes (H^* \otimes N^*)] \otimes M^*} \arrow[d, "\alpha\otimes id_{M^*}"'] & (R^* \otimes H^*) \otimes (N^* \otimes M^*) \arrow[u, "(\beta^{-1}\otimes id_{H^*})\otimes\tau"'] \\
{[(H^* \otimes H^*) \otimes N^*] \otimes M^*} \arrow[r, "\alpha^{-1}"] & (H^* \otimes H^*) \otimes (N^* \otimes M^*) \arrow[u, "(\eta^*_H\otimes id_{H^*})\otimes(id_{N^*}\otimes id_{M^*})"']
\end{tikzcd}$$

Thus, proceeding as in the proof of Corollary 15, $\Big(\mathcal{F}^*; ({\mathcal{F}^*}_0, {\mathcal{F}^*}^-_0); {\mathcal{F}^*}_2\Big)$ is a contravariant semi-normal monoidal functor. ∎

Proposition 17. *Let C be a monoidal Ab-category. Then, C admits a contravariant semi-normal monoidal functor to the category* $(\mathbf{Mod}_{\Bbbk_{\mathbf{C}}}; \otimes_{\Bbbk_C}; \Bbbk_C)$.

Proof. This is due to the fact that $\mathrm{Hom}_C(M, N)$ admits the structure of a left $\Bbbk_C$-module, for any objects M and N of C. Consider the functor

$$\begin{aligned} F : C &\longrightarrow (\mathbf{Mod}_{\Bbbk_{\mathbf{C}}}; \otimes_{\Bbbk_C}; \Bbbk_C) \\ M &\longmapsto \mathrm{Hom}_C(M, I) \\ f : M \to N &\longmapsto F(f) \end{aligned}$$

where, $F(f)(h) = hf$, for every $h \in \mathrm{Hom}_C(N, I)$. F is then a contravariant semi-normal monoidal functor, with the following structures $F_0 = F_0^- = id_{\Bbbk_C}$, and

$$\begin{aligned} F_{2\ M,N} : \mathrm{Hom}_C(M, I) \otimes_{\Bbbk_C} \mathrm{Hom}_C(N, I) &\longrightarrow \mathrm{Hom}_C(M \otimes N, I) \\ f \otimes g &\longmapsto f \otimes g\ . \end{aligned}$$

∎

Remark 18. In the above example of the Proposition 17, this just reduces to the ordinary (linear) duality when considering C to be the category of finitely generated modules over a commutative noetherian ring R.

Proposition 19. *Let C be a monoidal Ab-category, equipped with a braiding c. Assume there exists a dualizable object A of C, with duality structures $(A^*; d_A; b_A)$, satisfying $d_A \circ c \circ b_A = id$. Then, C admits the following contravariant semi-normal monoidal functor to* $(\mathbf{Mod}_{\Bbbk_{\mathbf{C}}}; \otimes_{\Bbbk_C}; \Bbbk_C)$

$$\begin{aligned} G_A : C &\longrightarrow (\mathbf{Mod}_{\Bbbk_{\mathbf{C}}}; \otimes_{\Bbbk_C}; \Bbbk_C) \\ M &\longmapsto \mathrm{Hom}_C(M, A \otimes A^*) \\ f : M \to N &\longmapsto F(f)(g) = g \circ f \end{aligned}$$

for every $g \in \mathrm{Hom}_C(N, A \otimes A^*)$.

Proof. Define the following semi-normal monoidal structures on G_A

$$\begin{aligned} (G_A)_0 : \Bbbk_C &\longrightarrow \mathrm{Hom}_C(I, A \otimes A^*) \\ k &\longmapsto b_A \circ k \end{aligned}$$

$$\begin{aligned} (G_A)_0^- : \mathrm{Hom}_C(I, A \otimes A^*) &\longrightarrow \Bbbk_C \\ h &\longmapsto d_A \circ c \circ h \end{aligned}$$

$$\begin{aligned} (G_A)_{2\ M,N} : \mathrm{Hom}_C(M, A \otimes A^*) \otimes_{\Bbbk_C} \mathrm{Hom}_C(N, A \otimes A^*) &\longrightarrow \mathrm{Hom}_C(M \otimes N, A \otimes A^*) \\ f \otimes g &\longmapsto (id \otimes d_A \otimes id)(f \otimes g)\ . \end{aligned}$$

Clearly, we have $(G_A)_0^-(G_A)_0 = id$, and for every $k \in \Bbbk_C$ and $f \in \mathrm{Hom}_C(M, A \otimes A^*)$, we have

$$\begin{aligned}(G_A)_2\big((G_A)_0 \otimes id\big)(k \otimes f) &= (G_A)_2(b_A \circ k \otimes f)\\ &= (id \otimes d_A \otimes id)(b_A \circ k \otimes f)\\ &= (id \otimes d_A \otimes id)(b_A \otimes id \otimes id)(id \otimes f)(k \otimes id)\\ &= k \otimes f.\end{aligned}$$

Hence, the left unitality axiom holds, and similarly for the right unitality one. For the associativity axiom, we have

$$\begin{aligned}(G_A)_2\big((G_A)_2 \otimes id\big)(f \otimes g \otimes h) &= (id \otimes d_A \otimes id)\big((id \otimes d_A \otimes id)(f \otimes g) \otimes h\big)\\ &= (id \otimes d_A \otimes id)\big(f \otimes (id \otimes d_A \otimes id)(g \otimes h)\big)\\ &= (G_A)_2\big(id \otimes (G_A)_2\big)(f \otimes g \otimes h).\end{aligned}$$

Thus, $\big(G_A; ((G_A)_0, (G_A)_0^-); (G_A)_2\big)$ is a contravariant semi-normal monoidal functor. ■

In the next Proposition, we explicitly prove that a natural isomorphism between two functors transforms semi-normality structures from one to the other.

Proposition 20. *Let F, $G : C \longrightarrow D$ be functors between monoidal categories, and $\varphi : F \longrightarrow G$ a natural isomorphism. If F is a semi-normal monoidal* (*resp., comonoidal*) *functor, then G is as well a semi-normal monoidal* (*resp., comonoidal*) *functor.*

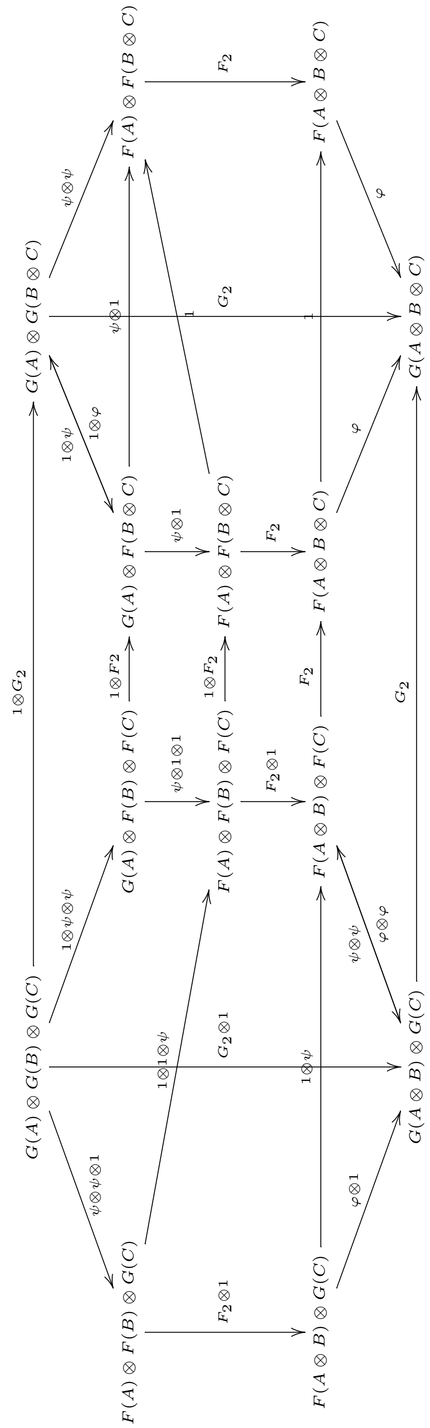
Proof. Let ψ denote the inverse of φ. Assume that F is a semi-normal monoidal functor. Then, G is a semi-normal monoidal functor via the maps given as follows. For any $A, B \in \mathbf{Ob}(C)$

$$G_{2A,B} = \varphi_{A\otimes B} \circ F_{2A,B} \circ (\psi_A \otimes \psi_B) G(A) \otimes G(B) \longrightarrow G(A \otimes B).$$

$$G_0 = \varphi_I \circ F_0 : I' \longrightarrow G(I).$$

$$G_0^- = F_0^- \circ \psi_I : G(I) \longrightarrow I'.$$

Consider the following diagram, expressing the associativity constraint of G (the larger square) :


G(A) ⊗ G(B) ⊗ G(C)
G(A) ⊗ G(B ⊗ C)
F(A) ⊗ F(B ⊗ C)
F(A ⊗ B ⊗ C)
G(A ⊗ B ⊗ C)
G(A) ⊗ F(B ⊗ C)
F(A) ⊗ F(B ⊗ C)
F(A ⊗ B ⊗ C)
G(A) ⊗ F(B) ⊗ F(C)
F(A) ⊗ F(B) ⊗ F(C)
F(A ⊗ B) ⊗ F(C)
F(A) ⊗ F(B) ⊗ G(C)
F(A ⊗ B) ⊗ G(C)
G(A ⊗ B) ⊗ G(C)
1⊗G2
ψ⊗ψ
F2
φ
G2
ψ⊗1
1
1
1⊗ψ
1⊗φ
φ
ψ⊗1
F2
1⊗F2
1⊗F2
F2
G2
ψ⊗1⊗1
F2⊗1
1⊗ψ⊗ψ
ψ⊗ψ
φ⊗φ
G2⊗1
1⊗1⊗ψ
1⊗ψ
ψ⊗ψ⊗1
φ⊗1
F2⊗1

This is a commutative diagram. In fact, the two (left and right) hexagonal diagrams are clearly commutative through the commutativity of the interior diagrams constituting a three diagrams decomposition of each one. The commutativity of the interior central square holds by using only the left invertibility $\psi \circ \varphi = id$ of φ (the double-headed arrows in the above diagram) and by the associativity constraint of F.

On the other hand, the following unitality diagrams commute:

$$\begin{tikzcd}
I' \otimes G(A) \arrow[r, "1"] \arrow[d, "F_0 \otimes 1"'] \arrow[ddr, "F_0 \otimes \psi_A"] & G(A) \\
F(I) \otimes G(A) \arrow[d, "\varphi_I \otimes 1"'] \arrow[dr, "1 \otimes \psi_A"] & F(A) \arrow[u, "\varphi_A"'] \\
G(I) \otimes G(A) \arrow[r, "\psi_I \otimes \psi_A"] & F(I) \otimes F(A) \arrow[u, "F_2"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G(A) \otimes I' \arrow[r, "1"] \arrow[d, "1 \otimes F_0"'] \arrow[ddr, "\psi_A \otimes F_0"] & G(A) \\
G(A) \otimes F(I) \arrow[d, "1 \otimes \varphi_I"'] \arrow[dr, "\psi_A \otimes 1"] & F(A) \arrow[u, "\varphi_A"'] \\
G(A) \otimes G(I) \arrow[r, "\psi_A \otimes \psi_I"] & F(A) \otimes F(I) \arrow[u, "F_2"']
\end{tikzcd}$$

For the first diagram. The right upper triangle is commutative now due to the right invertibility $\varphi_A \circ \psi_A = id_{G(A)}$ of φ. Whilst, the left lower triangle is commutative due to the datum: $\psi_I \circ \varphi_I = id_{F(I)}$.

Similar arguments hold for the second diagram. Furthermore, we have

$$G_0^- \circ G_0 = F_0^- \circ \psi_I \circ \varphi_I \circ F_0 = id_{I'}.$$

Hence, G is a semi-normal monoidal functor.

Assume now that F is a semi-normal comonoidal functor. Then, G is as well a semi-normal comonoidal functor via the following maps: for all $A, B \in \mathbf{Ob}(C)$

$$G_{2A,B} = (\varphi_A \otimes \varphi_B) \circ F_{2A,B} \circ \psi_{A \otimes B} : G(A \otimes B) \longrightarrow G(A) \otimes G(B).$$

$$G_0 = \varphi_I \circ F_0 : I' \longrightarrow G(I).$$

$$G_0^- = F_0^- \circ \psi_I : G(I) \longrightarrow I'.$$

By reversing the arrows in the associativity and unitality diagrams, the proof in this case is done similarly. ∎

4. Functor category

Denote by $\mathbf{SNFun(C;D)}$, the category of semi-normal monoidal functors between monoidal categories C and D, with all the natural transformations between the functors.

Proposition 21. *If D is a (strict) braided monoidal category, then* $\mathbf{SNFun}(\mathbf{C};\mathbf{D})$ *is also a (strict) monoidal and braided category.*

Proof. The monoidal product in $\mathbf{SNFun}(\mathbf{C};\mathbf{D})$ is the pointwise monoidal product (we denote it also by " $\otimes$ "), which is obviously associative and unital. The unit object is the functor $\mathbb{I}$, sending any object of C to the unit object I' of D, and any morphism of C to the identity on I'. It is clear that id D is strict then $\mathbf{SNFun}(\mathbf{C};\mathbf{D})$ is as well. Now, for any semi-normal monoidal functors $F,\ G : C \longrightarrow D$, the following maps, where c will denote the braiding of D, define the semi-normality monoidal structures on $F \otimes G$.

(1) For any $A, B \in \mathbf{Ob}(C)$, $(F \otimes G)_{2A,B} = (F_{2A,B} \otimes G_{2A,B})(id \otimes c \otimes id)$

$$\begin{tikzcd}
F(A) \otimes' G(A) \otimes' F(B) \otimes' G(B) \arrow[r, "(F\otimes G)_{2A,B}"] \arrow[d, "1\otimes c\otimes 1"] & F(A \otimes B) \otimes' G(A \otimes B) \\
F(A) \otimes' F(B) \otimes' G(A) \otimes' G(B) \arrow[ur, "F_2\otimes G_2"'] &
\end{tikzcd}$$

(2) $(F \otimes G)_0 = F_0 \otimes G_0 : I' = I' \otimes I' \longrightarrow (F \otimes G)(I) = F(I) \otimes' G(I)$.

(3) $(F \otimes G)_0^- = F_0^- \otimes G_0^- : F(I) \otimes' G(I) \longrightarrow I'$.

Indeed, we have

$$(F \otimes G)_0^-(F \otimes G)_0 = (F_0^- \otimes G_0^-)(F_0 \otimes G_0) = F_0^- F_0 \otimes G_0^- G_0 = id \otimes id = id.$$

On the other hand, it is not so difficult to see that $(F \otimes G)_2$ and $(F \otimes G)_0$ satisfy the commutativity of the associativity, left and right unitality diagrams.

The braiding is given by $c_{F;G} : F \otimes G \longrightarrow G \otimes F$, subject to

$$(c_{F;G})_{A\in\mathbf{Ob}(C)} = c_{F(A);G(A)} : F(A) \otimes' G(A) \longrightarrow G(A) \otimes' F(A).$$

∎

Corollary 22. *Let C, D and D' be three monoidal categories and $G : D \longrightarrow D'$ a semi-normal monoidal functor. Then, the category* $\mathbf{SNFun}(\mathbf{C};\mathbf{D})$ *of semi-normal monoidal functors from C to D admits a semi-normal monoidal functor to the category* $\mathbf{SNFun}(\mathbf{C};\mathbf{D'})$ *via the following functor*

$$\begin{array}{cccc}
\varphi : & \mathbf{SNFun}(\mathbf{C};\mathbf{D}) & \longrightarrow & \mathbf{SNFun}(\mathbf{C};\mathbf{D}') \\
& F & \longmapsto & \varphi(F) = G \circ F \\
& \alpha = \left\{(\alpha)_A : F(A) \to F'(A)\right\}_A & \longmapsto & \varphi(\alpha) = \left\{(\varphi(\alpha))_A = G((\alpha)_A)\right\}_A
\end{array}$$

for every $A \in \mathbf{Ob}(C)$.

Proof. The functor φ is clearly well defined by Proposition 11. Thus, we have to show that φ is semi-normal and monoidal. The monoidal structures on φ are given by

(1) $\varphi_0 : \mathbb{I}' \longrightarrow \varphi(\mathbb{I}) = G \circ \mathbb{I}$, given by

$$\begin{aligned}\varphi_0 &= \Big\{(\varphi_0)_A : \mathbb{I}'(A) \longrightarrow G \circ \mathbb{I}(A)\Big\}_{A \in \mathbf{Ob}(C)}\\ &= \Big\{(\varphi_0)_A : I_{D'} \longrightarrow G(I_D)\Big\}_{A \in \mathbf{Ob}(C)}\end{aligned}$$

where $(\varphi_0)_A = G_0$, for every $A \in \mathbf{Ob}(C)$.

(2) $\varphi_0^- : \varphi(\mathbb{I}) \longrightarrow \mathbb{I}'$, given by

$$\begin{aligned}\varphi_0^- &= \Big\{(\varphi_0^-)_A : G \circ \mathbb{I}(A) \longrightarrow \mathbb{I}'(A)\Big\}_{A \in \mathbf{Ob}(C)}\\ &= \Big\{(\varphi_0^-)_A : G(I_D) \longrightarrow I_{D'}\Big\}_{A \in \mathbf{Ob}(C)}\end{aligned}$$

where $(\varphi_0^-)_A = G_0^-$, for every $A \in \mathbf{Ob}(C)$. Here, $\mathbb{I}'$ is the unit object of $\mathbf{SNFun}(\mathbf{C};\mathbf{D}')$, which is the functor associating to each object of C, the unit object of D, and I_D and $I_{D'}$ are the unit objects of D and D' respectively.

(3) $\varphi_{2F,F'}$ is defined as follows

$$\begin{aligned}\varphi_{2F,F'} &= \Big\{(\varphi_{2F,F'})_A : \big(\varphi(F) \otimes \varphi(F')\big)(A) \longrightarrow \varphi(F \otimes F')(A)\Big\}_{A \in \mathbf{Ob}(C)}\\ &= \Big\{(\varphi_{2F,F'})_A : G\big(F(A)\big) \otimes G\big(F'(A)\big) \longrightarrow G\big(F(A) \otimes F'(A)\big)\Big\}_{A \in \mathbf{Ob}(C)}\end{aligned}$$

where $(\varphi_{2F,F'})_A = G_{2F(A),F'(A)}$. Thus defined, it is not difficult to see that $\big(\varphi; (\varphi_0, \varphi_0^-); \varphi_2\big)$ is a semi-normal monoidal functor. ■

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