

4 **ON NIL IDEALS AND JACOBSON RADICAL OF LEAVITT**
5 **PATH ALGEBRAS OVER COMMUTATIVE RINGS**

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15 **Abstract**

16 We show in this paper that for any graph E and for a commutative unital
17 ring R , the nil ideals of the Leavitt path algebra $L_R(E)$ depend solely on
18 the nil ideals of the ring R . A connection between the Jacobson radical of
19 $L_R(E)$ and the Jacobson radical of R is obtained. We also prove that for
20 a nil ideal I of a Leavitt path algebra $L_R(E)$ the ideal $M_2(I)$ is also nil,
21 thus obtaining that Leavitt path algebras over arbitrary graphs satisfy the
22 Köethe's conjecture.

23 **Keywords:** Leavitt path algebras, Nil ideals, Jacobson radical, arbitrary
24 graph.

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26 **1. INTRODUCTION**

27 Throughout this paper, R denotes a commutative ring with identity, $J(R)$ the
28 Jacobson radical of R , $N(R)$ the nilradical of R and $L_R(E)$ shall denote the
29 Leavitt path algebra of a directed graph E with coefficients from R . An important
30 note to make in these introductory lines about Leavitt path algebras is that they
31 are locally unital. We recall that a ring R is *locally unital* if for each finite set F of

elements of R , there is an idempotent u (i.e., $u^2 = u \in R$) such that $ua = au = a$ for all $a \in F$. The set of all such idempotents u is said to be a set of local units.

We begin this paper with some basic definitions concerning Leavitt path algebras. Some known properties of Leavitt path algebras which will be helpful to us later in establishing our main results are also included. We would like to refer to [1], [12] and [8] for the details of this section.

A quadruple $E = (E^0, E^1, r, s)$ consisting of a set of vertices E^0 , a set of edges E^1 and two maps $r, s : E^1 \rightarrow E^0$ (the range and source maps of E) is called a (*directed*) *graph*. A *sink* is a vertex that emits no edge. When a vertex emits a non-empty finite set of edges, it is called a *regular vertex*. We denote the set of regular vertices as E_{reg}^0 . A vertex which is a source of infinitely many edges is called an *infinite emitter*. For each $e \in E^1$, we call e^* a *ghost edge* such that the source and range of e^* is equal to the range and source of e respectively.

A *path* ρ of finite length $|\rho| = n \geq 0$ is a sequence of n edges $\rho = f_1 f_2 \cdots f_n$ with $r(f_i) = s(f_{i+1})$ for all $i = 1, \dots, n-1$. Accordingly $\rho^* = f_n^* \cdots f_2^* f_1^*$ will be considered as the corresponding ghost path of ρ . A vertex is a path of length 0. In this case, the vertex is considered as the ghost path of itself. The set of all vertices on the path μ is denoted by μ^0 . The set of all paths in E is denoted by $\text{Path}(E) := \cup_{n=0}^{\infty} E^n$, where E^n is the set of paths of length $n \geq 0$.

A path $\mu = e_1 e_2 \dots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be based at the vertex $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \dots, n$. A closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$, for every $i \neq j$.

Given an arbitrary graph E and a unital commutative ring R , the *Leavitt path algebra* $L_R(E)$ is defined to be the R -algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The CK-1 relations) For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.
4. (The CK-2 relations) For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

The first useful observation about $L_R(E)$ is that every element a can be written in the form $a = \sum_{i=1}^n k_i \alpha_i \beta_i^*$, where $k_i \in R$, α_i, β_i are paths in E and n is a suitable integer. Secondly, elements of the subset $\{v, e, e^* : v \in E^0, e \in E^1\}$

of $L_R(E)$ are all nonzero [Proposition 3.4 [11]]. Thirdly, a Leavitt path algebra is a \mathbb{Z} -graded algebra [11].

It may be recalled that a ring R is \mathbb{Z} -graded (or, simply, *graded*) if there exists a collection of additive subgroups $\{R_k\}_{k \in \mathbb{Z}}$ of R such that the following conditions hold:

1. $R = \bigoplus_{k \in \mathbb{Z}} R_k$
2. $R_j R_k \subseteq R_{j+k}$ for all $j, k \in \mathbb{Z}$.

The subgroup R_k here is called the *homogeneous component of R of degree k* . For a Leavitt path algebra, the homogeneous components are given as $L_R(E)_k$

$$:= \left\{ \sum_{i=1}^N r_i \alpha_i \beta_i^* : \alpha_i, \beta_i \in \text{Path}(E), r_i \in R, \text{ and } |\alpha_i| - |\beta_i| = k, \forall i \right\}$$

In order to study the description of ideals in Leavitt path algebras, the following concepts concerning some subsets of $L_R(E)$ are needed.

A subset $H \subseteq E^0$ is *hereditary* if whenever a vertex $v \in H$, $r(\rho) \in H$ for any path $\rho \in \text{Path}(E)$ with $s(\rho) = v$. Also, a subset $S \subseteq E^0$ is *saturated* if whenever the set $\{r(e) | e \in E^1, s(e) = v\} \subseteq S$ for a regular vertex $v \in E^0$, $v \in S$. For a hereditary saturated subset H of E^0 , the set of *breaking vertices*, B_H of H is defined to be the collection of infinite emitters of $E^0 \setminus H$ emitting finitely many edges into itself, i.e.,

$$B_H := \{v \in E^0 \setminus H : |s^{-1}(v)| = \infty, 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

Also, for a vertex $v \in B_H$, we denote

$$v^H := v - \sum_{s(e)=v, r(e) \notin H} ee^*.$$

and $B_H^H := \{v^H \mid v \in B_H\}$.

Results about generators of ideals in Leavitt path algebras over a field have been studied extensively by [9], [2], [5], [3]. In [5] it has been proved that given a row-finite graph E , if K is a field and I is a two-sided ideal of $L_K(E)$, then I is generated by elements of the form $v + \sum_{i=1}^n \lambda_i g^i$ where $v \in E^0$, g is a cycle at v and $\lambda_i \in K$, for $1 \leq i \leq n$.

Larki [8] made a study for ideals with coefficients in a commutative ring. A recent paper by Rigby and van den Hove [10] about generators of ideals in Leavitt path algebras over a commutative ring R with identity, proves that a two-sided

85 ideal of a Leavitt path algebra $L_R(E)$ is generated by elements of the following
 86 three types:

87 **Type 1:** Scalar multiples of vertices.

88 **Type 2:** Scalar multiples of elements of the form

$$v^H = v - \sum_{e \in s^{-1}(v), r(e) \notin H} ee^*,$$

89 with v a breaking vertex for a hereditary saturated subset $H \subseteq E^0$.

90 **Type 3:** Laurent polynomials evaluated at cycles in the graph E .

91 Let us quickly recall here that, the formal expression of a Laurent polynomial
 92 f in a single indeterminate x with coefficients in a ring R is given by

$$f(x, x^{-1}) = a_{-n}x^{-n} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots + a_mx^m,$$

93 where $m, n \in \mathbb{Z}^+$, $a_i \in R$ (cf. [13]).

If c is any cycle in E , we write $f(c, c^*)$ to mean the evaluation of $f(x, x^{-1})$
 at the cycle c , that is,

$$f(c, c^*) = a_{-n}c^{*n} + \cdots + a_{-1}c^{*1} + a_0s(c) + a_1c + \cdots + a_mc^m.$$

94 Here $a_0s(c)$ is the degree 0 element of $f(c, c^*)$.

95 This recent discovery by Rigby and van den Hove prompted us to study the
 96 behaviour of nil ideals and Jacobson radical in Leavitt path algebras.

97 Recall that an ideal I of a ring R is a *nil ideal* if each of its elements is
 98 nilpotent. The well-known Köethe's conjecture asks whether the sum of two
 99 one-sided nil ideals is one-sided nil. In one of its equivalent forms the Köethe's
 100 conjecture asks whether the ring of 2×2 matrices over a nil ideal is nil. In this
 101 paper, we establish that for Leavitt path algebras (though they are in general
 102 non-commutative) over a commutative ring, the Köethe's conjecture is indeed
 103 true. We also prove here that the nil ideals of Leavitt path algebras are locally
 104 nilpotent.

105 Another algebraic object which is of interest to us is the Jacobson radical.
 106 In order to define Jacobson radical we first recall the definition of a right quasi-
 107 regular ideal. Following [6], an element a of an arbitrary ring R is called *right*
 108 *quasi-regular* if there exists an element $a' \in R$ (called the right quasi-inverse of
 109 a) such that $a + a' + aa' = 0$. A right ideal is right quasi-regular if all its elements
 110 are right quasi-regular. The Jacobson radical of a ring is the join of all right
 111 quasi-regular right ideals of the ring.

112 It may be noted here that the Jacobson radical of an arbitrary ring is a (right)
 113 quasi-regular two sided ideal [Theorem 1 [6]].

114 To summarize, in this paper, we show how the nil ideals and the Jacobson
 115 radical of the Leavitt path algebra $L_R(E)$ depend on the ring R . Indeed, the nil

ideals of the Leavitt path algebra over R are defined by the nil ideals of the ring R .

2. RESULTS

We begin this section with a lemma citing an example of a particular class of idempotent elements in $L_R(E)$.

Lemma 1. *For any $u \in E^0$ and edges e_1, e_2, \dots, e_n with $s(e_i) = u$, the element $u - \sum_{i=1}^n e_i e_i^*$ is idempotent.*

Proof. We first observe that for a single edge, say e_1 .

$$(u - e_1 e_1^*)(u - e_1 e_1^*) = (u - e_1 e_1^* - e_1 e_1^* + e_1 e_1^*) = (u - e_1 e_1^*)$$

Assuming that the result is true for $(n - 1)$ edges, we get that

$$\begin{aligned} & (u - e_1 e_1^* - e_2 e_2^* - \dots - e_{n-1} e_{n-1}^* - e_n e_n^*)^2 \\ &= (u - e_1 e_1^* - \dots - e_{n-1} e_{n-1}^*)^2 - e_n e_n^* (u - e_1 e_1^* - \dots - e_{n-1} e_{n-1}^*) + e_n e_n^* - \\ & \quad (u - e_1 e_1^* - \dots - e_{n-1} e_{n-1}^*) e_n e_n^* \\ &= (u - e_1 e_1^* - \dots - e_{n-1} e_{n-1}^*) - e_n e_n^* + e_n e_n^* - e_n e_n^* \\ &= (u - e_1 e_1^* - e_{n-1} e_{n-1}^* - e_n e_n^*) \end{aligned}$$

Hence, $u - \sum_{i=1}^n e_i e_i^*$ is idempotent. ■

Remark 2. For a breaking vertex u of a hereditary saturated subset H of E^0 , the set $\{e \in E^1 \mid e \in s^{-1}(u), r(e) \notin H\}$ is finite and hence u^H is an idempotent element of $L_R(E)$.

Lemma 3. *All coefficients of a nilpotent Laurent polynomial f evaluated at a cycle x of E are nilpotent in R .*

Proof. Let $af(x, x^*)$ be a nilpotent Laurent monomial evaluated at a cycle x of E with index of nilpotency k , where $a \in R$ is the coefficient of the monomial. Then it follows easily that $a^k = 0$, i.e., a is nilpotent in R . Hence, the lemma is true for nilpotent Laurent monomials. We assume this to be true for nilpotent Laurent polynomials with less than n monomials. Let $f'(x, x^*) = bh(x, x^*) + g(x, x^*)$ be a nilpotent Laurent polynomial with n monomials and $bh(x, x^*)$ be its highest degree monomial with coefficient b . Suppose m is the

index of nilpotency of $f'(x, x^*)$, then $0 = \{f'(x, x^*)\}^m = g'(x, x^*) + \{bh(x, x^*)\}^m$, where $g'(x, x^*) = \{g(x, x^*)\}^m + bH(x, x^*)$ for some polynomial $H(x, x^*)$ with $\deg(g'(x, x^*)) < m\{\deg(bh(x, x^*))\}$. This implies that $b^m = 0$, i.e., b is nilpotent and $g'(x, x^*) = 0$. So, $\{g(x, x^*)\}^m + bH(x, x^*) = 0$. i.e., $\{g(x, x^*)\}^m = -bH(x, x^*)$, yielding $\{g(x, x^*)\}^{m^2} = (-b)^m \{H(x, x^*)\}^m = 0$. Thus g is nilpotent. But g is a Laurent polynomial with less than n monomials. Hence according to our assumption, each coefficient of g is nilpotent. This implies that each coefficient of $f'(x, x^*)$ is nilpotent. Thus the lemma is true for any nilpotent Laurent polynomial. ■

For each ideal I in $L_R(E)$, we define $I(R)$ to be the ideal of R generated by the coefficients of a system of generators of I .

Theorem 4. *An ideal I of $L_R(E)$ is nil iff $I(R)$ is nil in R .*

Proof. By Corollary 5.6 of [10], each ideal I of $L_R(E)$ is generated by generators of the form $k_1v_1, k_2v_2, \dots, l_1u_1^{H_1}, l_2u_2^{H_2}, \dots, f_1, f_2, \dots$ for some $k_i, l_j \in R$ and $v_i \in E^0$, $u_j \in B_{H_j}$, and Laurent polynomials f_h 's evaluated at cycles of E over R . Let $I(R)$ be the ideal of R generated by the coefficients k_i, l_j and the coefficients of the monomials of the Laurent polynomials f_h .

We first assume that $I(R)$ is nil in R . If $\alpha \in I$, then α is a finite sum of monomials with coefficients in $I(R)$. If J is the ideal of R generated by the coefficients of the monomials occurring in α , then J is a subideal of $I(R)$. Again since J is finitely generated, J is nilpotent of index (say) k . Now, the coefficient of each monomial in α^k belongs to J^k . This yields $\alpha^k = 0$.

Conversely, if I is a nil ideal in $L_R(E)$, then the generators are nilpotent. So, for k_iv_i , there exists a non negative integer d_i such that $0 = (k_iv_i)^{d_i} = k_i^{d_i}v_i$ (as v_i is idempotent). Since each v_i is a vertex, we get $k_i^{d_i} = 0$ and so each k_i is nilpotent. Similarly each l_j is nilpotent. Further as each f_h is nilpotent, Lemma 3 suggests that the coefficients of f_h are nilpotent for all $h = 1, 2, \dots$. ■

Recall that a ring R is a reduced ring if it has no non zero nilpotent elements [7].

Corollary 5. *Over a reduced ring R , $L_R(E)$ has no non-trivial nil ideal.*

Corollary 6. *Sum of two nil ideals is again nil in $L_R(E)$.*

Theorem 7. *For a Leavitt path algebra $L_R(E)$, and a nil ideal I of $L_R(E)$, $M_2(I)$ is a nil ideal of $M_2(L_R(E))$.*

Proof. Let A be a matrix with coefficients in a nil ideal I of $L_R(E)$, i.e., let $A \in M_2(I)$ and $I(R)$ be its corresponding ideal in R generated by the coefficients of a system of generators of I . If J is the ideal of R generated by the coefficients

173 of the monomials of the entries of A , then we may observe that J is a finitely
 174 generated subideal of $I(R)$. Thus by Theorem 4, J is also nil in R . Being finitely
 175 generated, J is nilpotent. But the coefficients of the monomials of the entries of
 176 A^k belong to J^k , therefore the matrix A is also nilpotent. ■

177 The above theorem shows that the Leavitt path algebra of an arbitrary graph
 178 over a unital commutative ring satisfies the Köethe's conjecture.

179 Below, we record another result about nil ideals in $L_R(E)$. Recall that a
 180 subset S of a ring T is locally nilpotent (see [7]) if for any finite subset $\{s_1, \dots, s_n\}$
 181 of S , there exists a positive integer k , such that any product of k elements from
 182 $\{s_1, \dots, s_n\}$ is zero.

183 **Theorem 8.** *Every nil ideal of $L_R(E)$ is locally nilpotent.*

184 **Proof.** As in Theorem 4, if I is a nil ideal of $L_R(E)$, the ideal $I(R)$ is a nil ideal
 185 of R and so is locally nilpotent. Therefore the ideal I is also locally nilpotent. ■

186 We now turn our attention towards Jacobson radicals of Leavitt path algebras
 187 over a unital commutative ring. We first record the following two lemmas:

188 **Lemma 9.** *For any vertex $v \in E^0 \setminus E_{\text{reg}}^0$ and a non-zero element $a \in R$, if*
 189 *$av \in J(L_R(E))$ then $a \in J(R)$.*

190 **Proof.** Let $a(\neq 0) \in R$ and $v \in E^0 \setminus E_{\text{reg}}^0$ such that $av \in J(L_R(E))$. That implies
 191 $rv.av = rav \in J(L_R(E))$, $\forall r \in R$. Since any element in $J(L_R(E))$ is right quasi
 192 regular, for each $r \in R$ there exists $b_r \in L_R(E)$ such that $rav + b_r + (rav)b_r = 0$.

Without loss of generality, we may assume

$$b_r = vb_rv = s'v + \sum_{i=1}^n k_i \alpha_i \beta_i^*$$

193 with $\alpha_i \beta_i^* \neq v$ where $r(\alpha_i) = r(\beta_i)$ and $s', k_i \in R$, for $1 \leq i \leq n$.

Thus the expression $rav + b_r + (rav)b_r = 0$ becomes

$$(ra + s' + ras')v + (1 + ra) \sum_{i=1}^n k_i \alpha_i \beta_i^* = 0.$$

194 Since $\alpha_i \beta_i^* \neq v$ for any $1 \leq i \leq n$, we get $ra + s' + ras' = 0$ and hence ra is a
 195 right quasi regular element of R . Thus $(1 + ra)$ is an unit in R for all $r \in R$ and
 196 by Proposition 1.9 [14], $a \in J(R)$.
 197 ■

198 **Lemma 10.** *For a breaking vertex v of hereditary saturated subset H of E^0 and*
 199 *for a non-zero element $a \in R$, if $av^H \in J(L_R(E))$, $a \in J(R)$.*

Proof. Let $a(\neq 0) \in R$ and $v \in B_H$ for a hereditary saturated subset H of E^0 such that $av^H \in J(L_R(E))$. This implies that $rv.av^H = rav^H \in J(L_R(E))$. Hence, there exists an element $b \in L_R(E)$ such that $rav^H + b + (rav^H)b = 0$.

Since it is clear by the definition of breaking vertices that the set $A = \{e \mid e \in s^{-1}(v), r(e) \notin H\}$ is finite, let $A = \{e_1, \dots, e_n\}$. Then $v^H = v - \sum_{e \in A} ee^*$ and we may assume

$$b = vbv = s'v + \sum_{e \in A} s_e ee^* + \sum_{i=1}^m k_i \alpha_i \beta_i^*,$$

with $ee^* \neq \alpha_i \beta_i^* \neq v$ for each $e \in A$, and $s', k_i, s_e \in R$ for $1 \leq i \leq m$ and $e \in A$. The expression $rav^H + b + rav^H b = 0$ becomes

$$\begin{aligned} (ra + s' + ras')v - (ra + ras') \sum_{e \in A} ee^* + \sum_{e \in A} s_e ee^* + (1 + ra) \sum_{i=1}^m k_i \alpha_i \beta_i^* \\ - ra \sum_{e \in A} ee^* \sum_{i=1}^m k_i \alpha_i \beta_i^* = 0. \end{aligned}$$

Since $ee^*, \alpha_i \beta_i^*$ and $ee^* \alpha_i \beta_i^*$ are not equal to v for any $e \in A$ and $1 \leq i \leq m$, we get $ra + s' + ras' = 0$. Thus $(1 + ra)$ is a unit for all $r \in R$ and hence $a \in J(R)$ [Proposition 1.9 [14]]. ■

Remark 11. Since the Jacobson radical of a \mathbb{Z} -graded ring is a homogeneous ideal [Corollary 2 [4]], a Laurent polynomial evaluated at a cycle of E over R can be a generator of $J(L_R(E))$ if the polynomial is homogeneous, i.e., it is a monomial of a Laurent polynomial.

The fact that the Jacobson radical is a two-sided ideal, the monomials of a Laurent polynomial can be substituted by an R -multiple of their source vertex. Therefore they can be reduced to Type 1 generators of the Jacobson radical and hence for a graph E , with no regular vertex, we have the following result.

Theorem 12.

$$J(L_R(E)) \subseteq J(R) \left(E^0 \cup \bigcup_{H \in \mathcal{H}} B_H^H \right)$$

where \mathcal{H} is the set of the hereditary and saturated subsets of E^0 and B_H is the set of the breaking vertices of H .

Proof. The proof of this theorem follows directly from the proofs of Lemmas 9 and 10 and also Remark 11. ■

218 Question : For any vertex $v \in E_{reg}^0$ with $av \in J(L_R(E))$, is it necessary that
 219 $a \in J(R)$?

220 It may be remarked that the converse of the above question is false in general.
 221 For if we take the power series $R := \mathbb{Q}[[Y]]$ in one indeterminate Y and if E is
 222 the graph having one vertex v and a single loop c , then $a = Y$ is in $J(R)$ but
 223 av is not in $J(L_R(E))$: indeed $1 + A(Y)Y$ is invertible in R for each $A(Y) \in R$,
 224 while $v + Yc = 1_{L_R(E)} + Yc$ is not invertible in $L_R(E)$. However, if the element
 225 a is in $N(R)$, then we get the following result.

226 **Lemma 13.** *For all $v \in E^0$, if $a \in N(R)$ then $av \in J(L_R(E))$.*

Proof. Let $a \in N(R)$ and $n \in \mathbb{Z}_{>0}$ be its index of nilpotency. We claim that $avL_R(E)$ is a right quasi regular ideal of $L_R(E)$.

Let $av \sum_{i=1}^m \alpha_i \beta_i^* = a \sum_{i=1}^m \alpha_i \beta_i^*$ be any arbitrary element of $avL_R(E)$, where $s(\alpha_i) = v$ and $r(\alpha_i) = r(\beta_i)$ for all $1 \leq i \leq m$. We now choose

$$b = -a \sum_{i=1}^m \alpha_i \beta_i^* + a^2 \left(\sum_{i=1}^m \alpha_i \beta_i^* \right)^2 - \cdots + (-1)^{n-1} a^{n-1} \left(\sum_{i=1}^m \alpha_i \beta_i^* \right)^{n-1}.$$

It is now easy to see that

$$av \sum_{i=1}^m \alpha_i \beta_i^* + b + (av \sum_{i=1}^m \alpha_i \beta_i^*)b = 0.$$

227 Hence every element of $avL_R(E)$ is right quasi regular. Thus it is a right quasi
 228 regular ideal and is contained in $J(L_R(E))$. Therefore, $av \in J(L_R(E))$. ■

229 **Lemma 14.** *For a breaking vertex v of a hereditary saturated subset H of E^0 , if
 230 $a \in N(R)$ then $av^H \in J(L_R(E))$.*

231 It may be noted that the reverse implication of Lemma 13 may not always
 232 hold. Taking R to be the power series ring $\mathbb{Q}[[Y]]$ and E to be the oriented 2-line
 233 graph with two vertices and a single edge, we see that as $L_R(E) \cong M_2(R)$ of 2×2
 234 matrices over R , $Y \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is an element of $J(L_R(E))$ but Y is not nilpotent in
 235 R .

236 However, if we let $I^0 = \{v \in E^0 \setminus E_{reg}^0 \mid \text{there exists a closed path } \gamma \text{ such that}$
 237 $v \geq s(\gamma)\}$, where $v \geq s(\gamma)$ denotes that there is a path from v to $s(\gamma)$, then we
 238 have the following result.

239 **Theorem 15.** *For any $v \in I^0$ and a non-zero element $a \in R$, $av \in J(L_R(E))$ iff
 240 $a \in N(R)$.*

Proof. The proof of (\Leftarrow) follows from Lemma 13.

(\Rightarrow) . Let $av \in J(L_R(E))$, where $v \in I^0$. Since $v \in I^0$, there exists a closed path γ and a path β in E such that $0 \neq av\beta\gamma \in J(L_R(E))$, implying that $a\gamma \in J(L_R(E))$. Let $\omega \in L_R(E)$ such that

$$a\gamma + \omega + a\gamma\omega = 0.$$

Without loss of generality, let us assume that

$$\omega = (s'v' + \sum_{i=1}^n a_i\gamma^i + \sum_{j=1}^m b_j\alpha_j\beta_j^*)$$

where $v' = s(\gamma)$, $r(\alpha_j) = r(\beta_j)$, $\alpha_j\beta_j^* \neq \gamma^i \neq v'$ and $s', a_i, b_j \in R$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Putting the value of ω in the equation $a\gamma + \omega + a\gamma\omega = 0$ we get

$$s'v' + (as' + a + a_1)\gamma + \sum_{i=1}^n (a_i + aa_{i-1})\gamma^i + aa_n\gamma^{n+1} + \sum_{j=1}^m b_j\alpha_j\beta_j^* + \sum_{j=1}^m ab_j\gamma\alpha_j\beta_j^* = 0$$

Since $\alpha_j\beta_j^*$ and $\gamma\alpha_j\beta_j^*$ are not equal to v' or γ^i for any i, j , by comparing the coefficients of v' and γ^i for each i of both the sides we have the following equations,

$$s' = 0$$

$$as' + a + a_1 = 0 \implies a_1 = -a$$

$$a_2 + aa_1 = 0 \implies a_2 = a^2$$

$$\vdots$$

$$a_n + aa_{n-1} = 0 \implies a_n = (-1)^n a^n$$

$$aa_n = 0 \implies a(-1)^n a^n = 0 \implies a^{n+1} = 0.$$

241 Thus $a \in N(R)$.

242

■

243 We end this paper with the following question:

244

245 Question : If $v \in E^0 \setminus I^0$ and $a \in J(R) \setminus N(R)$, is it necessary that av should
246 be an element of $J(L_R(E))$?

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