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ON NIL IDEALS AND JACOBSON RADICAL OF LEAVITT PATH ALGEBRAS OVER COMMUTATIVE RINGS

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Abstract

We show in this paper that for any graph E and for a commutative unital ring R, the nil ideals of the Leavitt path algebra $L_R(E)$ depend solely on the nil ideals of the ring R. A connection between the Jacobson radical of $L_R(E)$ and the Jacobson radical of R is obtained. We also prove that for a nil ideal I of a Leavitt path algebra $L_R(E)$ the ideal $M_2(I)$ is also nil, thus obtaining that Leavitt path algebras over arbitrary graphs satisfy the Köethe's conjecture.

Keywords: Leavitt path algebras, Nil ideals, Jacobson radical, arbitrary graph.

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1. Introduction

Throughout this paper, R denotes a commutative ring with identity, $J(R)$ the Jacobson radical of R, $N(R)$ the nilradical of R and $L_R(E)$ shall denote the Leavitt path algebra of a directed graph E with coefficients from R. An important note to make in these introductory lines about Leavitt path algebras is that they

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are locally unital. We recall that a ring R is *locally unital* if for each finite set F of elements of R, there is an idempotent u (i.e., $u^2 = u \in R$) such that $ua = au = a$ for all $a \in F$. The set of all such idempotents u is said to be a set of local units.

We begin this paper with some basic definitions concerning Leavitt path algebras. Some known properties of Leavitt path algebras which will be helpful to us later in establishing our main results are also included. We would like to refer to [1, 12] and [8] for the details of this section.

A quadruple $E = (E^0, E^1, r, s)$ consisting of a set of vertices E^0, a set of edges E^1 and two maps $r, s : E^1 \longrightarrow E^0$ (the range and source maps of E) is called a (*directed*) *graph*. A *sink* is a vertex that emits no edge. When a vertex emits a non-empty finite set of edges, it is called a *regular vertex*. We denote the set of regular vertices as E^0_{reg}. A vertex which is a source of infinitely many edges is called an *infinite emitter*. For each $e \in E^1$, we call e^* a *ghost edge* such that the source and range of e^* is equal to the range and source of e respectively.

A *path* ρ of finite length $|\rho| = n \geq 0$ is a sequence of n edges $\rho = f_1 f_2 \cdots f_n$ with $r(f_i) = s(f_{i+1})$ for all $i = 1, \ldots, n-1$. Accordingly $\rho^* = f_n^* \cdots f_2^* f_1^*$ will be considered as the corresponding ghost path of ρ. A vertex is a path of length 0. In this case, the vertex is considered as the ghost path of itself. The set of all vertices on the path μ is denoted by μ^0. The set of all paths in E is denoted by $\mathrm{Path}(E) := \bigcup_{n=0}^{\infty} E^n$, where E^n is the set of paths of length $n \geqslant 0$.

A path $\mu = e_1 e_2 \cdots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be based at the vertex $s(e_1)$. A closed path μ as above is called *simple* provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. A closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$, for every $i \neq j$.

Given an arbitrary graph E and a unital commutative ring R, the *Leavitt path algebra* $L_R(E)$ is defined to be the R-algebra generated by a set $\{v : v \in E^0\}$ of pair-wise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$.
3. (The CK-1 relations) For all $e, f \in E^1$, $e^* e = r(e)$ and $e^* f = 0$ if $e \neq f$.
4. (The CK-2 relations) For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1,\ s(e)=v} ee^*.$$

The first useful observation about $L_R(E)$ is that every element a can be written in the form $a = \sum_{i=1}^{n} k_i \alpha_i \beta_i^*$, where $k_i \in R$, α_i, β_i are paths in E and n is a suitable integer. Secondly, elements of the subset $\{v, e, e^* : v \in E^0, e \in E^1\}$

of $L_R(E)$ are all nonzero (Proposition 3.4 [11]). Thirdly, a Leavitt path algebra is a $\mathbb{Z}-$*graded* algebra [11].

It may be recalled that a ring R is $\mathbb{Z}-$*graded* (or, simply, *graded*) if there exists a collection of additive subgroups $\{R_k\}_{k\in\mathbb{Z}}$ of R such that the following conditions hold:

1. $R = \bigoplus\limits_{k\in\mathbb{Z}} R_k$
2. $R_jR_k \subseteq R_{j+k}$ for all $j,k\in\mathbb{Z}$.

The subgroup R_k here is called the *homogeneous component of R of degree k.* For a Leavitt path algebra, the homogeneous components are given as

$$L_R(E)_k := \left\{\sum_{i=1}^{N} r_i\alpha_i\beta_i^* : \alpha_i,\beta_i \in Path(E), r_i\in R, \text{ and } |\alpha_i|-|\beta_i| = k,\ \forall i\right\}.$$

In order to study the description of ideals in Leavitt path algebras, the following concepts concerning some subsets of $L_R(E)$ are needed.

A subset $H\subseteq E^0$ is *hereditary* if whenever a vertex $v\in H$, $r(\rho)\in H$ for any path $\rho\in Path(E)$ with $s(\rho)=v$. Also, a subset $S\subseteq E^0$ is *saturated* if whenever the set $\{r(e)|e\in E^1, s(e)=v\}\subseteq S$ for a regular vertex $v\in E^0$, $v\in S$. For a hereditary saturated subset H of E^0, the set of *breaking vertices*, B_H of H is defined to be the collection of infinite emitters of $E^0\setminus H$ emitting finitely many edges into itself, i.e.,

$$B_H := \left\{v\in E^0\setminus H : |s^{-1}(v)| = \infty,\ 0 < |s^{-1}(v)\cap r^{-1}(E^0\setminus H)| < \infty\right\}.$$

Also, for a vertex $v\in B_H$, we denote

$$v^H := v - \sum_{s(e)=v, r(e)\notin H} ee^*.$$

and $B_H^H := \{v^H \mid v\in B_H\}$.

Results about generators of ideals in Leavitt path algebras over a field have been studied extensively by [2, 3, 5, 9]. In [5] it has been proved that given a row-finite graph E, if K is a field and I is a two-sided ideal of $L_K(E)$, then I is generated by elements of the form $v+\sum_{i=1}^{n}\lambda_i g^i$ where $v\in E^0,\ g$ is a cycle at v and $\lambda_i\in K$, for $1\le i\le n$.

Larki [8] made a study for ideals with coefficients in a commutative ring. A recent paper by Rigby and van den Hove [10] about generators of ideals in Leavitt path algebras over a commutative ring R with identity, proves that a two-sided ideal of a Leavitt path algebra $L_R(E)$ is generated by elements of the following three types:

Type 1. Scalar multiples of vertices.

Type 2. Scalar multiples of elements of the form

$$v^H = v - \sum_{e \in s^{-1}(v), r(e) \notin H} ee^*,$$

with v a breaking vertex for a hereditary saturated subset $H \subseteq E^0$.

Type 3. Laurent polynomials evaluated at cycles in the graph E.

Let us quickly recall here that, the formal expression of a Laurent polynomial f in a single indeterminate x with coefficients in a ring R is given by

$$f(x, x^{-1}) = a_{-n}x^{-n} + \cdots + a_{-1}x^{-1} + a_0 + a_1 x + \cdots + a_m x^m,$$

where $m, n \in \mathbb{Z}^+,\ a_i \in R$ (cf. [13]).

If c is any cycle in E, we write $f(c, c^*)$ to mean the evaluation of $f(x, x^{-1})$ at the cycle c, that is,

$$f(c, c^*) = a_{-n}c^{*n} + \cdots + a_{-1}c^{*1} + a_0 s(c) + a_1 c + \cdots + a_m c^m.$$

Here $a_0 s(c)$ is the degree 0 element of $f(c, c^*)$.

This recent discovery by Rigby and van den Hove prompted us to study the behaviour of nil ideals and Jacobson radical in Leavitt path algebras.

Recall that an ideal I of a ring R is a *nil ideal* if each of its elements is nilpotent. The well-known Köethe's conjecture asks whether the sum of two one-sided nil ideals is one-sided nil. In one of its equivalent forms the Köethe's conjecture asks whether the ring of 2×2 matrices over a nil ideal is nil. In this paper, we establish that for Leavitt path algebras (though they are in general non-commutative) over a commutative ring, the Köethe's conjecture is indeed true. We also prove here that the nil ideals of Leavitt path algebras are locally nilpotent.

Another algebraic object which is of interest to us is the Jacobson radical. In order to define Jacobson radical we first recall the definition of a right quasi-regular ideal. Following [6], an element a of an arbitrary ring R is called *right quasi-regular* if there exists an element $a' \in R$ (called the right quasi-inverse of a) such that $a + a' + aa' = 0$. A right ideal is right quasi-regular if all its elements are right quasi-regular. The Jacobson radical of a ring is the join of all right quasi-regular right ideals of the ring.

It may be noted here that the Jacobson radical of an arbitrary ring is a (right) quasi-regular two sided ideal (Theorem 1 [6]).

To summarize, in this paper, we show how the nil ideals and the Jacobson radical of the Leavitt path algebra $L_R(E)$ depend on the ring R. Indeed, the nil ideals of the Leavitt path algebra over R are defined by the nil ideals of the ring R.

2. Results

We begin this section with a lemma citing an example of a particular class of idempotent elements in $L_R(E)$.

Lemma 1. *For any $u \in E^0$ and edges $e_1, e_2, \ldots, e_n$ with $s(e_i) = u$, the element $u - \sum_{i=1}^{n} e_i e_i^*$ is idempotent.*

Proof. We first observe that for a single edge, say e_1.

$$(u - e_1e_1^*)(u - e_1e_1^*) = (u - e_1e_1^* - e_1e_1^* + e_1e_1^*) = (u - e_1e_1^*).$$

Assuming that the result is true for $(n-1)$ edges, we get that

$$\begin{aligned}
&\left(u - e_1e_1^* - e_2e_2^* - \cdots - e_{n-1}e_{n-1}^* - e_ne_n^*\right)^2 \\
&= \left(u - e_1e_1^* - \cdots - e_{n-1}e_{n-1}^*\right)^2 - e_ne_n^*\left(u - e_1e_1^* - \cdots - e_{n-1}e_{n-1}^*\right) + e_ne_n^* \\
&- \left(u - e_1e_1^* - \cdots - e_{n-1}e_{n-1}^*\right)e_ne_n^* \\
&= \left(u - e_1e_1^* - \cdots - e_{n-1}e_{n-1}^*\right) - e_ne_n^* + e_ne_n^* - e_ne_n^* \\
&= \left(u - e_1e_1^* - e_{n-1}e_{n-1}^* - e_ne_n^*\right).
\end{aligned}$$

Hence, $u - \sum_{i=1}^{n} e_ie_i^*$ is idempotent. ■

Remark 2. For a breaking vertex u of a hereditary saturated subset H of E^0, the set $\{e \in E^1 | e \in s^{-1}(u), r(e) \notin H\}$ is finite and hence u^H is an idempotent element of $L_R(E)$.

Lemma 3. *All coefficients of a nilpotent Laurent polynomial f evaluated at a cycle x of E are nilpotent in R.*

Proof. Let $af(x, x^*)$ be a nilpotent Laurent monomial evaluated at a cycle x of E with index of nilpotency k, where $a \in R$ is the coefficient of the monomial. Then it follows easily that $a^k = 0$, i.e., a is nilpotent in R. Hence, the lemma is true for nilpotent Laurent monomials. We assume this to be true for nilpotent Laurent polynomials with less than n monomials. Let $f'(x, x^*) = bh(x, x^*) + g(x, x^*)$ be a nilpotent Laurent polynomial with n monomials and $bh(x, x^*)$ be its highest degree monomial with coefficient b. Suppose m is the index of nilpotency of $f'(x, x^*)$, then $0 = \{f'(x, x^*)\}^m = g'(x, x^*) + \{bh(x, x^*)\}^m$, where $g'(x, x^*) = \{g(x, x^*)\}^m + bH(x, x^*)$ for some polynomial $H(x, x^*)$ with $deg(g'(x, x^*)) < m\{deg(bh(x, x^*)\}$. This implies that $b^m = 0$, i.e., b is nilpotent and $g'(x, x^*) = 0$. So, $\{g(x, x^*)\}^m + bH(x, x^*) = 0$. i.e., $\{g(x, x^*)\}^m = -bH(x, x^*)$, yielding $\{g(x, x^*)\}^{m^2} = (-b)^m\{H(x, x^*)\}^m = 0$. Thus g is nilpotent. But g is a Laurent polynomial with less than n monomials. Hence according to our assumption, each coefficient of g is nilpotent. This implies that each coefficient of $f'(x, x^*)$ is nilpotent. Thus the lemma is true for any nilpotent Laurent polynomial. ■

For each ideal I in $L_R(E)$, we define $I(R)$ to be the ideal of R generated by the coefficients of a system of generators of I.

Theorem 4. *An ideal I of $L_R(E)$ is nil iff $I(R)$ is nil in R.*

Proof. By Corollary 5.6 of [10], each ideal I of $L_R(E)$ is generated by generators of the form $k_1v_1, k_2v_2, \ldots, l_1u_1^{H_1}, l_2u_2^{H_2}, \ldots, f_1, f_2, \ldots$ for some $k_i, l_j \in R$ and $v_i \in E^0$, $u_j \in B_{H_j}$, and Laurent polynomials f_h's evaluated at cycles of E over R. Let $I(R)$ be the ideal of R generated by the coefficients k_i, l_j and the coefficients of the monomials of the Laurent polynomials f_h.

We first assume that $I(R)$ is nil in R. If $\alpha \in I$, then α is a finite sum of monomials with coefficients in $I(R)$. If J is the ideal of R generated by the coefficients of the monomials occurring in α, then J is a subideal of $I(R)$. Again since J is finitely generated, J is nilpotent of index (say) k. Now, the coefficient of each monomial in α^k belongs to J^k. This yields $\alpha^k = 0$

Conversely, if I is a nil ideal in $L_R(E)$, then the generators are nilpotent. So, for k_iv_i, there exists a non negative integer d_i such that $0 = (k_iv_i)^{d_i} = k_i^{d_i}v_i$ (as v_i is idempotent). Since each v_i is a vertex, we get $k_i^{d_i} = 0$ and so each k_i is nilpotent. Similarly each l_j is nilpotent. Further as each f_h is nilpotent, Lemma 3 suggests that the coefficients of f_h are nilpotent for all $h = 1, 2, \ldots$. ■

Recall that a ring R is a reduced ring if it has no non zero nilpotent elements [7].

Corollary 5. *Over a reduced ring R, $L_R(E)$ has no non-trivial nil ideal.*

Corollary 6. *Sum of two nil ideals is again nil in $L_R(E)$.*

Theorem 7. *For a Leavitt path algebra $L_R(E)$, and a nil ideal I of $L_R(E)$, $M_2(I)$ is a nil ideal of $M_2(L_R(E))$.*

Proof. Let A be a matrix with coefficients in a nil ideal I of $L_R(E)$, i.e., let $A \in M_2(I)$ and $I(R)$ be its corresponding ideal in R generated by the coefficients of a system of generators of I. If J is the ideal of R generated by the coefficients of the monomials of the entries of A, then we may observe that J is a finitely generated subideal of $I(R)$. Thus by Theorem 4, J is also nil in R. Being finitely generated, J is nilpotent. But the coefficients of the monomials of the entries of A^k belong to J^k, therefore the matrix A is also nilpotent. ■

The above theorem shows that the Leavitt path algebra of an arbitrary graph over a unital commutative ring satisfies the Köethe's conjecture.

Below, we record another result about nil ideals in $L_R(E)$. Recall that a subset S of a ring T is locally nilpotent (see [7]) if for any finite subset $\{s_1, \ldots, s_n\}$ of S, there exists a positive integer k, such that any product of k elements from $\{s_1, \ldots, s_n\}$ is zero.

Theorem 8. *Every nil ideal of $L_R(E)$ is locally nilpotent.*

Proof. As in Theorem 4, if I is a nil ideal of $L_R(E)$, the ideal $I(R)$ is a nil ideal of R and so is locally nilpotent. Therefore the ideal I is also locally nilpotent. ■

We now turn our attention towards Jacobson radicals of Leavitt path algebras over a unital commutative ring. We first record the following two lemmas:

Lemma 9. *For any vertex $v \in E^0 \setminus E^0_{reg}$ and a non-zero element $a \in R$, if $av \in J(L_R(E))$ then $a \in J(R)$.*

Proof. Let $a(\neq 0) \in R$ and $v \in E^0 \setminus E^0_{\text{reg}}$ such that $av \in J(L_R(E))$. That implies $rv.av = rav \in J(L_R(E))$, $\forall r \in R$. Since any element in $J(L_R(E))$ is right quasi regular, for each $r \in R$ there exists $b_r \in L_R(E)$ such that $rav + b_r + (rav)b_r = 0$.

Without loss of generality, we may assume

$$b_r = vb_rv = s'v + \sum_{i=1}^{n} k_i\alpha_i\beta_i^*$$

with $\alpha_i\beta_i^* \neq v$ where $r(\alpha_i) = r(\beta_i)$ and $s', k_i \in R$, for $1 \leq i \leq n$.

Thus the expression $rav + b_r + (rav)b_r = 0$ becomes

$$(ra + s' + ras')v + (1 + ra)\sum_{i=1}^{n} k_i\alpha_i\beta_i^* = 0.$$

Since $\alpha_i\beta_i^* \neq v$ for any $1 \leq i \leq n$, we get $ra + s' + ras' = 0$ and hence ra is a right quasi regular element of R. Thus $(1 + ra)$ is an unit in R for all $r \in R$ and by Proposition 1.9 [14], $a \in J(R)$. ■

Lemma 10. *For a breaking vertex v of hereditary saturated subset H of E^0 and for a non-zero element $a \in R$, if $av^H \in J(L_R(E))$, $a \in J(R)$.*

Proof. Let $a(\neq 0) \in R$ and $v \in B_H$ for a hereditary saturated subset H of E^0 such that $av^H \in J(L_R(E))$. This implies that $rv.av^H = rav^H \in J(L_R(E))$. Hence, there exists an element $b \in L_R(E)$ such that $rav^H + b + (rav^H)b = 0$.

Since it is clear by the definition of breaking vertices that the set $A = \{e \mid e \in s^{-1}(v), r(e) \notin H\}$ is finite, let $A = \{e_1, \dots, e_n\}$. Then $v^H = v - \sum_{e\in A} ee^*$ and we may assume

$$b = vbv = s'v + \sum_{e\in A} s_e ee^* + \sum_{i=1}^{m} k_i\alpha_i\beta_i^*,$$

with $ee^* \neq \alpha_i\beta_i^* \neq v$ for each $e \in A$, and $s', k_i, s_e \in R$ for $1 \leq i \leq m$ and $e \in A$. The expression $rav^H + b + rav^Hb = 0$ becomes

$$(ra + s' + ras')v - (ra + ras')\sum_{e\in A} ee^* + \sum_{e\in A} s_e ee^* + (1 + ra)\sum_{i=1}^{m} k_i\alpha_i\beta_i^*$$

$$-ra\sum_{e\in A} ee^*\sum_{i=1}^{m} k_i\alpha_i\beta_i^* = 0.$$

Since $ee^*, \alpha_i\beta_i^*$ and $ee^*\alpha_i\beta_i^*$ are not equal to v for any $e \in A$ and $1 \leq i \leq m$, we get $ra + s' + ras' = 0$. Thus $(1+ra)$ is a unit for all $r \in R$ and hence $a \in J(R)$ (Proposition 1.9 [14]). ■

Remark 11. Since the Jacobson radical of a $\mathbb{Z}$-graded ring is a homogeneous ideal (Corollary 2 [4]), a Laurent polynomial evaluated at a cycle of E over R can be a generator of $J(L_R(E))$ if the polynomial is homogeneous, i.e., it is a monomial of a Laurent polynomial.

The fact that the Jacobson radical is a two-sided ideal, the monomials of a Laurent polynomial can be substituted by an R-multiple of their source vertex. Therefore they can be reduced to Type 1 generators of the Jacobson radical and hence for a graph E, with no regular vertex, we have the following result.

Theorem 12.

$$J(L_R(E)) \subseteq J(R)\left(E^0 \cup \bigcup_{H\in\mathcal{H}} B_H^H\right)$$

where $\mathcal{H}$ is the set of the hereditary and saturated subsets of E^0 and B_H is the set of the breaking vertices of H.

Proof. The proof of this theorem follows directly from the proofs of Lemmas 9 and 10 and also Remark 11. ■

Question. *For any vertex $v \in E^0_{reg}$ with $av \in J(L_R(E))$, is it necessary that $a \in J(R)$?*

It may be remarked that the converse of the above question is false in general. For if we take the power series $R := \mathbb{Q}[[Y]]$ in one indeterminate Y and if E is the graph having one vertex v and a single loop c, then $a = Y$ is in $J(R)$ but av is not in $J(L_R(E))$: indeed $1 + A(Y)Y$ is invertible in R for each $A(Y) \in R$, while $v + Yc = 1_{L_R(E)} + Yc$ is not invertible in $L_R(E)$. However, if the element a is in $N(R)$, then we get the following result.

Lemma 13. *For all $v \in E^0$, if $a \in N(R)$ then $av \in J(L_R(E))$.*

Proof. Let $a \in N(R)$ and $n \in \mathbb{Z}_{>0}$ be its index of nilpotency. We claim that $avL_R(E)$ is a right quasi regular ideal of $L_R(E)$.

Let $av\sum_{i=1}^{m}\alpha_i\beta_i^* = a\sum_{i=1}^{m}\alpha_i\beta_i^*$ be any arbitrary element of $avL_R(E)$, where $s(\alpha_i) = v$ and $r(\alpha_i) = r(\beta_i)$ for all $1 \leq i \leq m$. We now choose

$$b = -a\sum_{i=1}^{m}\alpha_i\beta_i^* + a^2\left(\sum_{i=1}^{m}\alpha_i\beta_i^*\right)^2 - \cdots + (-1)^{n-1}a^{n-1}\left(\sum_{i=1}^{m}\alpha_i\beta_i^*\right)^{n-1}.$$

It is now easy to see that

$$av\sum_{i=1}^{m}\alpha_i\beta_i^* + b + \left(av\sum_{i=1}^{m}\alpha_i\beta_i^*\right)b = 0.$$

Hence every element of $avL_R(E)$ is right quasi regular. Thus it is a right quasi regular ideal and is contained in $J(L_R(E))$. Therefore, $av \in J(L_R(E))$. ∎

Lemma 14. *For a breaking vertex v of a hereditary saturated subset H of E^0, if $a \in N(R)$ then $av^H \in J(L_R(E))$.*

It may be noted that the reverse implication of Lemma 13 may not always hold. Taking R to be the power series ring $\mathbb{Q}[[Y]]$ and E to be the oriented 2-line graph with two vertices and a single edge, we see that as $L_R(E) \cong M_2(R)$ of 2×2 matrices over R, $Y\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ is an element of $J(L_R(E))$ but Y is not nilpotent in R.

However, if we let $I^0 = \{v \in E^0 \setminus E^0_{\text{reg}} \mid$ there exists a closed path γ such that $v \geq s(\gamma)$ $\}$, where $v \geq s(\gamma)$ denotes that there is a path from v to $s(\gamma)$, then we have the following result.

Theorem 15. *For any $v \in I^0$ and a non-zero element $a \in R$, $av \in J(L_R(E))$ iff $a \in N(R)$.*

Proof. The proof of ($\Leftarrow$) follows from Lemma 13.

($\Rightarrow$) Let $av \in J(L_R(E))$, where $v \in I^0$. Since $v \in I^0$, there exists a closed path γ and a path β in E such that $0 \neq av\beta\gamma \in J(L_R(E))$, implying that $a\gamma \in J(L_R(E))$. Let $\omega \in L_R(E)$ such that

$$a\gamma + \omega + a\gamma\omega = 0.$$

Without loss of generality, let us assume that

$$\omega = \left(s'v' + \sum_{i=1}^{n}a_i\gamma^i + \sum_{j=1}^{m}b_j\alpha_j\beta_j^*\right)$$

where $v' = s(\gamma)$, $r(\alpha_j) = r(\beta_j)$, $\alpha_j\beta_j^* \neq \gamma^i \neq v'$ and $s', a_i, b_j \in R$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Putting the value of ω in the equation $a\gamma + \omega + a\gamma\omega = 0$ we get

$$s'v' + (as' + a + a_1)\gamma + \sum_{i=1}^{n}(a_i + aa_{i-1})\gamma^i + aa_n\gamma^{n+1} + \sum_{j=1}^{m}b_j\alpha_j\beta_j^* + \sum_{j=1}^{m}ab_j\gamma\alpha_j\beta_j^* = 0.$$

Since $\alpha_j\beta_j^*$ and $\gamma\alpha_j\beta_j^*$ are not equal to v' or γ^i for any i, j, by comparing the coefficients of v' and γ^i for each i of both the sides we have the following equations,

$$\begin{gathered} s' = 0 \\ as' + a + a_1 = 0 \implies a_1 = -a \\ a_2 + aa_1 = 0 \implies a_2 = a^2 \\ \vdots \\ a_n + aa_{n-1} = 0 \implies a_n = (-1)^n a^n \\ aa_n = 0 \implies a(-1)^n a^n = 0 \implies a^{n+1} = 0. \end{gathered}$$

Thus $a \in N(R)$. ■

We end this paper with the following question.

Question. *If $v \in E^0 \backslash I^0$ and $a \in J(R)\backslash N(R)$, is it necessary that av should be an element of $J(L_R(E))$?*

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