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DISJUNCTIVE INCLUSION PROPERTY IN PSEUDO-COMPLEMENTED DISTRIBUTIVE LATTICES

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Abstract

Disjunctive inclusion property of several prime ideals and prime filters of pseudo-complemented lattices is studied. Algebraic structures like Boolean algebras and Stone lattices are characterized with the help of the disjunctive inclusion property of prime ideals and prime filters. A set of equivalent conditions is given for every Stone lattice to become a Boolean algebra.

Keywords: Disjunctive inclusion property; minimal prime ideal, minimal prime filter; kernel ideal, δ-ideal; Stone lattice; Boolean algebra.

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1. Introduction

The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by M.H. Stone [13], O. Frink [8], and George Gratzer [9]. Later many authors like R. Balbes [1], T.P. Speed [12], and O. Frink [8]etc., extended the study of pseudo-complements to characterize Stone lattices. In [4], I. Chajda, R. Halaš and J. Kühr extensively studied the structure of pseudo-complemented semilattices. In [6], W.H. Cornish investigated various significant properties of pseudo-complemented distributive lattices in terms of congruences. O. Frink in [8], generalized and extended most of the theory of pseudo-complements to semi-lattices without making use of the join operation. In [10], the concept of δ-ideals was introduced in pseudo-complemented distributive lattices and Stone lattices are characterized in terms of δ-ideals.

In this paper, the notion of disjunctive inclusion property is introduced in pseudo-complemented distributive lattices and observed that every maximal filter of a pseudo-complemented lattice satisfies this property. It is showed that every prime filter of a pseudo-complemented lattice satisfies the disjunctive inclusion property if and only if the pseudo-complemented lattice is a Boolean algebra. Similarly, it is showed that the disjunctive inclusion property of prime ideals of a pseudo-complemented lattice is equivalent to the lattice to become a Boolean algebra. Some equivalent conditions are given for every Stone lattice to become a Boolean algebra. A pseudo-complemented lattice is proved to be a Boolean algebra if and only if every minimal prime ideal satisfies the disjunctive inclusion property.

It is observed that every prime ideal of a pseudo-complemented lattice need not satisfy the disjunctive inclusion property and whenever every prime ideal satisfies the same than the lattice will become a Boolean algebra. It is proved that every prime $*$-ideal as well as every median prime ideal of a pseudo-complemented lattice satisfy the disjunctive inclusion property. Finally, the class of Stone lattice is characterized with the help of prime $*$-ideals, prime δ-ideals and median prime ideals of pseudo-complemented lattices.

2. Preliminaries

The reader is referred to [2], [3], [6], [10] and [12] for the elementary notions and notations of pseudo-complemented lattices. However some of the preliminary definitions and results are presented for the ready reference of the reader.

A non-empty subset A of a lattice L is called an ideal (filter) of L if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice. A proper ideal (resp. filter) P of a distributive lattice L is said to be *prime* if for any $x, y \in L$, $x \wedge y \in P$ (resp. $x \vee y \in P$) implies $x \in P$ or $y \in P$. A proper ideal (resp. proper filter) P of a lattice L is called *maximal* if there exists no proper ideal (resp. filter) Q of L such that $P \subset Q$. A proper ideal (resp. proper filter) P of a distributive lattice L is minimal if there exists no prime ideal (resp. prime filter) Q of L such that $Q \subset P$. Every maximal ideal (resp. maximal filter) of a distributive lattice is a prime ideal (resp. prime filter). A complemented distributive lattice is a *Boolean algebra*.

The *pseudo-complement* b^* of an element b is the element satisfying

$$a \wedge b = 0 \;\Leftrightarrow\; a \wedge b^* = a \;\Leftrightarrow\; a \leq b^*$$

where $\leq$ is the induced order of L.

A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice. For any two elements a, b of a pseudo-complemented semilattice [4], we have the following.

(1) $a \leq b$ implies $b^* \leq a^*$,

(2) $a \leq a^{**}$,

(3) $a^{***} = a^*$,

(4) $(a \vee b)^* = a^* \wedge b^*$,

(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element a of a pseudo-complemented distributive lattice L is called a dense element if $a^* = 0$ and the set D of all dense elements of L forms a filter in L. A pseudo-complemented distributive lattice is a Boolean algebra if and only if every prime ideal is maximal if and only if $x \vee x^* = 1$ for all $x \in L$.

Definition. [2] A pseudo-complemented distributive lattice L is called a Stone lattice if $x^* \vee x^{**} = 1$ for all $x \in L$

Theorem 1. [12] *The following assertions are equivalent in a pseudo-complemented distributive lattice L:*

(1) *L is a Stone lattice;*

(2) *for $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$;*

(3) *for $x, y \in L$, $(x \vee y)^{**} = x^{**} \vee y^{**}$.*

Theorem 2. [6] *Let P be a prime ideal of a pseudo-complemented distributive lattice and $x \in L$. Then the following assertions are equivalent:*

(1) *P is minimal;*

(2) *$x \in P$ implies $x^* \notin P$;*

(3) *$x \in P$ if and only if $x^{**} \in P$.*

An ideal I of a pseudo-complemented lattice L is called a δ-*ideal* [10] if there exists a filter F such that $I = \delta(F) = \{x \in L \mid x^* \in F\}$. A prime ideal P of a pseudo-complemented lattice L is called *median prime* [11] if to each $x \in P$, there exists $y \notin P$ such that $x^* \vee y^* = 1$. A congruence θ of a pseudo-complemented lattice L is called $*$-*congruence* [3] if, for all $x, y \in L$, $(x, y) \in \theta$ implies $(x^*, y^*) \in \theta$. An ideal I of a pseudo-complemented lattice is called a *kernel ideal* [3] if there exists a $*$-congruence θ such that $I = ker\ \theta$. An ideal I of a pseudo-complemented lattice L is called a $*$-ideal if for all $x, y \in L$, $x^* = y^*$ and $x \in I$ imply that $y \in I$. Throughout this note, all lattices are bounded pseudo-complemented distributive lattices unless otherwise mentioned.

3. Disjunctive Inclusion Property in Lattices

In this section, the notion of disjunctive inclusion property is introduced in pseudo-complemented lattices. The algebraic structures like Boolean algebras and Stone lattices are characterized with the help of disjunctive inclusion property. The disjunctive inclusion properties of certain classes of prime ideals and prime filters are derived.

Lemma 3. *The following properties hold in a pseudo-complemented lattice L:*

(1) *every prime ideal contains either x or x^* for all $x \in L$,*

(2) *every maximal ideal contains either x or x^* for all $x \in L$,*

(3) *every maximal filter contains exactly one of x and x^* for all $x \in L$.*

Proof. (1) Let P be a prime ideal of L and $x \in L$. Clearly $x \wedge x^* = 0 \in P$. Since P is prime, we get either $x \in P$ or $x^* \in P$.

(2) Since every maximal ideal is prime, it is clear.

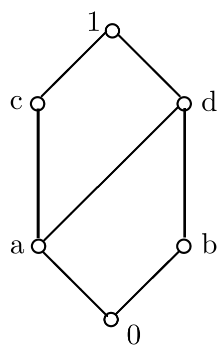
(3) Let M be a maximal filter of L. Let $x \in D$. Suppose $x \notin M$. Since M is maximal, there exists $0 \neq y \in M$ such that $x \wedge y = 0$. Hence $y \leq x^* = 0$, which is a contradiction. Thus $x \in M$, which gives that $D \subseteq M$. Hence $x \vee x^* \in D \subseteq M$. Since M is prime, we get $x \in M$ or $x^* \in M$. Suppose M contains both x and x^*. Then $0 = x \wedge x^* \in M$, which is a contradiction. Therefore M contains exactly one of x and x^*. ∎

Definition. A subset A of a pseudo-complemented lattice L is said to satisfy *disjunctive inclusion property* if A contains exactly one of x and x^* for all $x \in L$.

Proposition 4. *Every minimal prime ideal of a pseudo-complemented lattice satisfies disjunctive inclusion property.*

Proof. Let P be a minimal prime ideal of a pseudo-complemented lattice L. Then $L-P$ is a maximal filter. By Lemma 3(3), we get $L-P$ satisfies disjunctive inclusion property. Therefore P satisfies disjunctive inclusion property. ∎

Example 5. Consider the following bounded and finite distributive lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given by:



Clearly L is a pseudo-complemented lattice. Observed that $a^* = b$, $b^* = c$, $c^* = b$ and $d^* = 0$. This lattice contains only two maximal filters $F_1 = \{1, b, d\}$ and $F_2 = \{1, a, c, d\}$. Clearly F_1 and F_2 are both satisfying disjunctive inclusion property. Observe that L is not a Boolean algebra because of $a \vee a^* = d \neq 1$. Further, the lattice contains only two maximal ideals, precisely $M_1 = \{0, a, c\}$ and $M_2 = \{0, a, b, d\}$. Clearly neither of them are satisfying the property.

From Lemma 3(3), every maximal filter of a pseudo-complemented lattice satisfies the disjunctive inclusion property. In general, every prime filter of a pseudo-complemented lattice need not satisfy the disjunctive inclusion property. In deed, consider the finite distributive lattice $0 < a, bc < d < 1$ where $a^* = b, b^* = a, c^* = d^* = 0$. Clearly the prime filter $P = \{1, d\}$ neither contains a nor a^*. However, we have the following result:

Theorem 6. *The following assertions are equivalent in a pseudo-complemented lattice L:*

(1) *L is a Boolean algebra;*
(2) *every prime filter satisfies disjunctive inclusion property;*
(3) *every prime filter contains D;*
(2) *every minimal prime filter contains D.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a Boolean algebra. Let $x \in L$ and P be a prime filter of L. Since L is Boolean, we get $x \vee x^* = 1 \in P$. Since P is prime, we get either $x \in P$ or $x^* \in P$. Suppose P contains both of x or x^*. Then $0 = x \wedge x^* \in P$, which is a contradiction. Therefore P contains exactly one of x and x^* for all $x \in L$.
(2) $\Rightarrow$ (3): Assume condition (2). Let P be a prime filter of L. Let $x \in D$. Then $x^* = 0 \notin P$. By the assumption, we must have $x \in P$. Therefore P contains D.
(3) $\Rightarrow$ (4): It is obvious.
(4) $\Rightarrow$ (1): Assume condition (4). Let $x \in L$. Clearly $x \vee x^* \in D$. Since $x \wedge x^* = 0$, it is enough to show that $x \vee x^* = 1$. Suppose $x \vee x^* \neq 1$. Then there exists a maximal ideal M such that $x \vee x^* \in M$. Then $L - M$ is a minimal prime filter of L such that $x \vee x^* \notin L - M$. Hence $D \nsubseteq L - M$, which is contradicting the hypothesis. ■

Theorem 7. *The following assertions are equivalent in a pseudo-complemented lattice L:*

(1) *L is a Boolean algebra;*
(2) *every prime ideal satisfies disjunctive inclusion property;*

(3) *every maximal ideal satisfies disjunctive inclusion property;*

(2) *no maximal ideal contains a dense element.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a Boolean algebra. Let P be a prime ideal of L and $x \in L$. Since L is Boolean, we get $x \vee x^* = 1$. By Lemma 3(1), we get either $x \in P$ or $x^* \in P$. Suppose P contains both x and x^*. Then $1 = x \vee x^* \in P$, which is a contradiction. Therefore P contains exactly one of x and x^*.

(2) $\Rightarrow$ (3): Since every maximal ideal is prime, it is clear.

(3) $\Rightarrow$ (4): Assume condition (3). Let M be a maximal ideal of L. Let $x \in D$. Clearly $x^* = 0 \in M$. Since M satisfies disjunctive inclusion property, we must have $x \notin M$. Therefore M contains no dense element.

(4) $\Rightarrow$ (1): Assume condition (4). Let $a \in L$. Clearly $a \wedge a^* = 0$. It is enough to show that $a \vee a^* = 1$. Suppose $a \vee a^* \neq 1$. Then there exists a maximal ideal M such that $a \vee a^* \in M$. Since $a \vee a^* \in D$, it is contradicting the hypothesis. ∎

Corollary 8. *A pseudo-complemented lattice L is a Boolean algebra if and only if every minimal prime filter satisfies disjunctive inclusion property.*

Proof. Assume that L is a Boolean algebra. Let P be a minimal prime filter of L. Then $L - P$ is a maximal ideal of L. By Theorem 7, we get that $L - P$ satisfies disjunctive inclusion property. Therefore P satisfies disjunctive inclusion property. Conversely, assume that every minimal prime filter satisfies disjunctive inclusion property. Then every maximal ideal satisfies disjunctive inclusion property. By Theorem 7, it concludes that L is a Boolean algebra. ∎

Proposition 9. *Every Boolean algebra is a Stone lattice.*

Proof. Let L be a Boolean algebra. By Theorem 7, every maximal ideal satisfies disjunctive inclusion property. Let $x \in L$. Suppose $x^* \vee x^{**} \neq 1$. Then there exists a maximal ideal M such that $x^* \vee x^{**} \subseteq M$. Hence $x \vee x^* \in M$. Thus $x \in M$ and $x^* \in M$, which is a contradiction. Therefore L is a Stone lattice. ∎

The converse of Proposition 9 is not true. For, consider any infinite chain $L = \{0, a_1, a_2, \ldots, 1\}$. Clearly $a_i^* = 1^* = 0$ and $0^* = 1$. It can be easily seen that L is a Stone lattice. Clearly $M = \{x \in L \mid x \neq 1\}$ is the unique maximal ideal of the chain L. Then $a_i \in M$ and $a_i^* = 0 \in M$. Hence L is not a Boolean algebra. Though, every Stone lattice is not a Boolean algebra, in the following result, a set of equivalent conditions is given for every Stone lattice to Boolean.

Theorem 10. *Let L be a pseudo-complemented lattice. Suppose L is a Stone lattice and $x, y \in L$. Then the following assertions are equivalent in L:*

(1) *L is a Boolean algebra;*

(2) *for any maximal ideal M, $x \in M$ if and only if $x^{**} \in M$;*

(3) *for any maximal ideal M, $x^* = y^*$ and $x \in M$ imply that $y \in M$.*

Proof. (1) $\Rightarrow$ (2): Assume that L is Boolean. Let M be a maximal ideal of L and $x \in M$. By Theorem 7, we get $x^* \notin M$. Hence $x^{**} \in M$. Converse is clear.
(2) $\Rightarrow$ (3): Assume that $x^* = y^*$. Let M be a maximal ideal of L. Suppose $x \in M$. By (2), we get $y^{**} = x^{**} \in M$. Since $y \leq y^{**}$, we get $y \in M$.
(3) $\Rightarrow$ (1): Assume condition (3). Let M be a maximal ideal of L and $x \in L$. Clearly $x \wedge x^* = 0 \in M$. Since M is prime, we get either $x \in M$ or $x^* \in M$. Suppose M contains both x and x^*. Since $x^* = x^{***}$ and $x \in M$, by (3), we get $x^{**} \in M$. Hence $1 = x^* \vee x^{**} \in M$, which is a contradiction. Hence M contains exactly one of x and x^*. Thus M satisfies disjunctive inclusion property. By Theorem 7, L is Boolean. ■

Lemma 11. *No minimal prime ideal of a pseudo-complemented lattice contains a dense element.*

Proof. Let P be a minimal prime ideal of a pseudo-complemented lattice L. Suppose $P \cap D \neq \emptyset$. Choose $x \in P \cap D$. Then $x \in P$ and $x^* = 0$. Since P is minimal, there exists $y \notin P$ such that $x \wedge y = 0$. Hence $y \leq x^*$. Since $y \notin P$, we get $0 = x^* \notin P$, which is a contradiction. Thus P contains no dense elements. ■

Theorem 12. *The following assertions are equivalent in a pseudo-complemented lattice L:*

(1) *L is a Boolean algebra;*
(2) *every prime ideal is minimal;*
(3) *every prime ideal satisfies disjunctive inclusion property;*
(4) *every prime filter satisfies disjunctive inclusion property;*
(5) *for any $x, y \in L$, $x^* = y^*$ implies $x = y$;*
(6) *L has a unique dense element;*
(7) *every prime ideal is maximal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is Boolean. Let $x \in L$ and P be a prime ideal of L. Suppose $x \in P$. Since L is Boolean, we get $x \vee x^* = 1$. Suppose $x^* \in P$. Then $1 = x \vee x^* \in P$ which is a contradiction. Hence $x^* \notin P$. Thus P is minimal.
(2) $\Rightarrow$ (3): Assume that every prime ideal is minimal. Let $x \in L$ and P be a prime ideal of L. By Lemma 3(1), $x \in P$ or $x^* \in P$. Suppose P contains both x and x^*. Then, we get $x \vee x^* \in P \cap D$. Since P is minimal, by Lemma 11, we get that $x \vee x^* \notin P$. Thus we have arrived at a contradiction. Therefore P contains exactly one of x and x^*.
(3) $\Rightarrow$ (4): Assume condition (3). Let P be a prime filter of L. Then $L - P$ is a prime ideal of L. Let $x \in L$. By (3), $L - P$ contains exactly one of x and x^*. Hence P must contain exactly one of x and x^*. Therefore P satisfies disjunctive inclusion property.

(4) $\Rightarrow$ (5): Assume condition (4). Let $x, y \in L$ be such that $x^* = y^*$. Suppose $x \neq y$. Then there exists a prime filter P such that $x \in P$ and $y \notin P$. By (4), we must have $x^* \notin P$ and $x^* = y^* \in P$ which is a contradiction. Therefore $x = y$.
(5) $\Rightarrow$ (6): Assume condition (5). Let x and y be two dense elements of L. Then $x^* = 0 = y^*$. By (5), we get $x = y$. Therefore L contains a unique dense element.
(6) $\Rightarrow$ (7): Assume that L has a unique dense element, precisely 1. Let P be a prime ideal of L. Suppose Q is a proper ideal of L such that $P \subset Q$. Choose $x \in Q - P$. Clearly $x \vee x^* \in D = \{1\}$. Since $x \notin P$, we must have $x^* \in P \subset Q$. Hence $1 = x \vee x^* \in Q$, which is a contradiction. Therefore P is a maximal ideal.
(7) $\Rightarrow$ (1): Let $x \in L$. Clearly $x \wedge x^* = 0$. It is enough to show that $x \vee x^* = 1$. Suppose $x \vee x^* \neq 1$. Then there exists a prime ideal P such that $x \vee x^* \in P$. Suppose Q is a prime ideal of L such that $Q \subseteq P$. By (7), Q is maximal and hence $Q = P$. Therefore P is minimal. Since $x \vee x^* \in D$, we get $P \cap D \neq \emptyset$ that contradicts Theorem 6. Therefore x^* is the complement of x. ■

In [3], T.S. Blyth studied the properties of kernel ideals and $*$-ideals of pseudo-complemented distributive lattices. In [10], the author introduced the notion of δ-ideals of pseudo-complemented distributive lattices. In [11], the author introduced the notion of median prime ideals and investigated certain properties of these classes of ideals and then characterized Stone lattices and Boolean algebras with the help of these ideals. In the following, we present the disjunctive inclusion properties of these class of ideals.

Theorem 13. *Every prime $*$-ideal of a pseudo-complemented lattice satisfies disjunctive inclusion property and hence a prime kernel ideal too.*

Proof. Let P be a prime $*$-ideal of a pseudo-complemented lattice L. Since P is proper, it contains no dense element. Otherwise, if $d \in D \cap P$. Then $1 = d^{**} \in P$, which is a contradiction. Let $x \in L$. Since P is prime, we get that P contains either x or x^*. Suppose P contains both of x and x^*. Then $x \vee x^* \in D \cap P$, which is a contradiction. Therefore P contains exactly one of x and x^* for all $x \in L$. ■

Corollary 14. *Every prime δ-ideal of a pseudo-complemented lattice satisfies disjunctive inclusion property.*

Proof. Let P be a prime δ-ideal of a pseudo-complemented lattice L. Then $P = \delta(F)$ for some filter F of L. Let $x, y \in L$ be such that $x^* = y^*$. Suppose $x \in P = \delta(F)$. Then $y^* = x^* \in F$, which gives $y \in \delta(F) = P$. Hence P is a prime $*$-ideal of L. By Theorem 13, P satisfies disjunctive inclusion property. ■

Theorem 15. *Every median prime ideal of a pseudo-complemented lattice satisfies disjunctive inclusion property.*

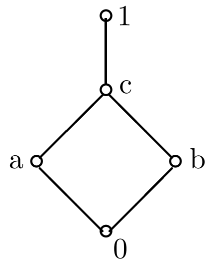
Proof. Let P be a median prime ideals of a pseudo-complemented lattice L. Let $x \in L$. Since P is prime, we get that P contains either x or x^*. Suppose $x \in P$. Since P is median, there exists $y \notin P$ such that $x^* \vee y^* = 1$. Then $x \wedge y \leq x^{**} \wedge y^{**} = (x^* \vee y^*)^* = 1^* = 0$. Hence $x \wedge y = 0$ and thus $y \leq x^*$. If $x^* \in P$, then $y \in P$ which is a contradiction. Hence $x^* \notin P$. Suppose $x^* \in P$. Similarly, we get $x \notin P$. Hence P contains exactly one of x and x^*. Thus P satisfies disjunctive inclusion property. ∎

Corollary 16. *Every median prime ideal of a pseudo-complemented lattice is a $*$-ideal as well as a kernel ideal.*

Proof. Let P be a median prime ideal of a pseudo-complemented lattice L. By the main theorem, P satisfies disjunctive inclusion property. Suppose $x, y \in L$ such that $x^* = y^*$ and $x \in P$. Since P satisfies disjunctive inclusion property, we must have $y^* = x^* \notin P$. Since P is prime and $y \wedge y^* = 0 \in P$, one must have $y \in P$. Hence P is a $*$-ideal of L. Since every $*$-ideal is a kernel ideal, the remaining part is clear. ∎

The converse of Corollary 16 is not true. In fact, every prime $*$-ideal need not to be a median prime ideal. Further, in [11], it is proved that every median prime ideal is a minimal prime ideal but not the converse. For consider the following example:

Example 17. Consider the following bounded and finite distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



Consider the prime ideal $P = \{0, a\}$ of the lattice L. It can be routinely verified that P is prime $*$-ideal of L. Choose $a \in P$. Observe that there exists no $x \notin P$ such that $a^* \vee x^* = 1$. Therefore P is not a median prime ideal of L. Further, it can be easily observed that P is a minimal prime ideal of L which is not a median prime ideal.

In [7], W.H. Cornish introduced the notion of σ-ideals of distributive lattice. In [5], S.A. Celani investigated the properties of σ-ideals of distributive pseudo-complemented residuated lattices. In the following, we generalize these ideals in pseudo-complemented lattices and characterize Stone lattices with the help of σ-ideals of lattices.

Definition. For any ideal I of a pseudo-complemented lattice L, defined $\sigma(I) = \{x \in L \mid (x^*] \vee I = L\}$. Then clearly $\sigma(I) \subseteq I$. An ideal I of a pseudo-complemented lattice is called a σ-ideal if $I = \sigma(I)$.

In the following theorem, a set of equivalent conditions is given for every minimal prime ideal is a median prime ideal as well as every prime $*$-ideal is a median prime ideal which together leads to a characterization of Stone lattices.

Theorem 18. *Let L be a pseudo-complemented lattice. Then the following assertions are equivalent in L:*

(1) *L is a Stone lattice;*
(2) *every prime $*$-ideal is median;*
(3) *every prime δ-ideal is median;*
(4) *every minimal prime ideal is median;*
(5) *every minimal prime ideal is a σ-ideal.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a Stone lattice. Let $x \in L$ and P be a prime $*$-ideal of L. Suppose $x \in P$. Since P is a $*$-ideal, we get $x^{**} \in P$. Suppose $x^* \in P$. Then $1 = x^* \vee x^{**} \in P$, which is a contradiction. Hence $x^* \notin P$. Thus, for each $x \in P$, there exists $x^* \notin P$ such that $x^* \vee x^{**} = 1$. Therefore P is a median prime ideal of L.

(2) $\Rightarrow$ (3): Since every δ-ideal is a $*$-ideal, it is clear.

(3) $\Rightarrow$ (4): Since every minimal prime ideal is a prime δ-ideal [10], it is clear.

(4) $\Rightarrow$ (5): Assume condition (4). It is enough to show that every median prime ideal is a σ-ideal. Let P be a median prime ideal of L. Clearly $\sigma(P) \subseteq P$. Conversely, let $x \in P$. Since P is median, there exists $y \notin P$ such that $x^* \vee y^* = 1$. Hence $(x^*] \vee (y^*] = L$. Since $y \notin P$, we get $y^* \in P$ and thus $(y^*] \subseteq P$. Hence $L = (x^*] \vee (y^*] \subseteq (x^*] \vee P$. Thus $x \in \sigma(P)$, which gives $P \subseteq \sigma(P)$. Therefore P is a σ-ideal.

(5) $\Rightarrow$ (1): Assume that every minimal prime ideal is a σ-ideal. Let $x \in L$. Suppose $x^* \vee x^{**} \neq 1$. Then there exists a prime filter P such that $x^* \vee x^{**} \notin P$. Since every prime filter is contained in a maximal filter, there exists a maximal filter M such that $P \subseteq M$. Then $L - M$ is a minimal prime ideal of L. By (4), $L - M$ is a σ-ideal of L and thus $L - M = \sigma(L - M)$. Suppose $x \in M$. Since M is maximal, there exists $y \notin M$ such that $x \vee y = 1 \in P$. Since $y \notin M$ and $P \subseteq M$, one must have $y \notin P$. Since P is prime, we get $x \in P$. Clearly $x \leq x^{**} \leq x^* \vee x^{**}$. Since $x^* \vee x^{**} \notin P$, we must have $x \notin P$ which is a contradiction. Hence $x \notin M$. Thus $x \in L - M = \sigma(L - M)$. Hence $(x^*] \vee (L - M) = L$, which gives that $1 = x^* \vee a$ for some $a \in L - M$. Hence $a \notin M$. Since $P \subseteq M$, we get $a \notin P$. Since $x^* \vee a = 1 \in P$, we get $x^* \in P$. Hence $x^* \vee x^{**} \in P$, which is a contradiction. Therefore $x^* \vee x^{**} = 1$ for all $x \in L$. ■

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