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SOLVABILITY OF B-ALGEBRAS

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Abstract

In this paper, we introduce and characterize solvable B-algebras. We also establish some of the basic properties of solvable B-algebras.

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1. Introduction

A *B-algebra* [21] is an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms:
(I) $x * x = 0$,

(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$.
This algebra was introduced and established by Neggers and Kim (2002). From then on, several properties and characterizations as well as several notions relating to B-algebras were established, including the basic properties of B-algebras [2, 3, 7, 9, 11, 13, 29, 30], homomorphisms of B-algebras [14, 22, 28], B_p-subalgebras [8, 10, 12], cyclic B-algebras [15, 16], and fuzzy B-algebras [1, 4, 5, 6, 17, 18, 20, 23, 24, 25, 26, 27]. In this paper, we introduce and characterize solvable B-algebras. We also establish some of the basic properties of solvable B-algebras.

We recall first some concepts needed in this study. Throughout this paper, let X be a B-algebra $(X; *, 0)$. In [21], X is said to be *commutative* if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$.

Example 1. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table of operations:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra [22]. Since $2 * (0 * 3) = 5 \neq 4 = 3 * (0 * 2)$, X is not commutative.

In [22], a nonempty subset N of X is called a *subalgebra* of X if $x * y \in N$ for any $x, y \in N$. A subalgebra N of X is called *normal* in X if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. A map $\varphi : X \to Y$ is called a *B-homomorphism* if $\varphi(x * y) = \varphi(x) * \varphi(y)$ for any $x, y \in X$. The subset $\{x \in X : \varphi(x) = 0_Y\}$ of X is called the *kernel* of the B-homomorphism φ, denoted by $Ker\ \varphi$. If N is normal in X, then X/N is a B-algebra, called the *quotient B-algebra* of X by N, where binary operation in X/N is defined by $xN *' yN = (x * y)N$; $X/N = \{xN : x \in X\}$; $xN = \{y \in X : x \sim_N y\}$ the equivalence class containing x by xN; $x \sim_N y$ if and only if $x * y \in N$. In [7], for subalgebra H of X and $x \in X$, we have $xH = \{x * (0 * h) : h \in H\}$ and $Hx = \{h * (0 * x) : h \in H\}$, called the *left* and *right B-cosets* of H in X, respectively. In [14], if H, K are subalgebras of X, we define the subset HK of X to be the set $HK = \{x \in X : x = h * (0 * k)$ for some $h \in H, k \in K\}$. In [10], we say that a B-algebra is *B-simple* if it has no nontrivial normal subalgebras.

2. B-SERIES

This section presents the notions of subnormal, normal, composition, and solvable B-series of B-algebras.

Definition. Let $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n = \{0\}$ be a series of subalgebras of X. The series is called a *subnormal B-series* if each H_i is normal in H_{i-1}. The series is called a *normal B-series* if each H_i is normal in X. The series is called a *composition B-series* if each H_i is a maximal normal subalgebra of H_{i-1}. The number of proper inclusions $\supset$ in the series is called the *length* of the series. The quotient B-algebras H_{i-1}/H_i are called the *factors* of the series.

If $H_{i-1} = H_i$, then the quotient B-algebra H_{i-1}/H_i consists of a single element and is called a *trivial factor* of the series. Given a series of subalgebras $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n = \{0\}$ of X, then the length of the series is the number of nontrivial factors H_{i-1}/H_i of the series. Since $\{0\}$ is normal in X, every B-algebra has a normal B-series.

Lemma 2. *H is a maximal normal in X if and only if X/H is B-simple.*

Proof. This follows from [8, Corollary 16]. ■

Theorem 3. *Every finite B-algebra has a composition B-series.*

Proof. Let X be a finite B-algebra. Since X is finite, there exists a maximal normal subalgebra H_1 of X. Thus, by Lemma 2, X/H_1 is B-simple. If $H_1 \neq \{0\}$, then since H_1 is finite, there exists a maximal normal subalgebra H_2 of H_1. Hence, H_1/H_2 is B-simple. If $H_2 \neq \{0\}$, then continuing the process, we obtain the following series $X = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_n \supset \cdots$ such that H_i/H_{i+1} is B-simple for all i. Since X is finite, there exists $n \geq 0$ such that $H_n = \{0\}$. Thus, $X = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_n = \{0\}$ is a composition B-series for X. ■

Example 4. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be a set with the following table of operations:

$*$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	11	10	9	8	7	6	5	4	3	2	1
1	1	0	11	10	9	8	7	6	5	4	3	2
2	2	1	0	11	10	9	8	7	6	5	4	3
3	3	2	1	0	11	10	9	8	7	6	5	4
4	4	3	2	1	0	11	10	9	8	7	6	5
5	5	4	3	2	1	0	11	10	9	8	7	6
6	6	5	4	3	2	1	0	11	10	9	8	7
7	7	6	5	4	3	2	1	0	11	10	9	8
8	8	7	6	5	4	3	2	1	0	11	10	9
9	9	8	7	6	5	4	3	2	1	0	11	10
10	10	9	8	7	6	5	4	3	2	1	0	11
11	11	10	9	8	7	6	5	4	3	2	1	0

Then $(X; *, 0)$ is a B-algebra [10]. Moreover, X is commutative. Thus, by [30, Corollary 2.3], the subalgebras $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, $\{0, 2, 4, 6, 8, 10\}$ are normal in X. The following series are normal B-series for X:

$$X \supset \{0,6\} \supset \{0\},$$
$$X \supset \{0,3,6,9\} \supset \{0,6\} \supset \{0\},$$
$$X \supset \{0,2,4,6,8,10\} \supset \{0,6\} \supset \{0\},$$
$$X \supset \{0,2,4,6,8,10\} \supset \{0,4,8\} \supset \{0\}.$$

The first normal B-series is not a composition B-series for X. The remaining three normal B-series are composition B-series for X.

Definition. Let

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}\,, \tag{1}$$

be a subnormal B-series in X. A *one-step refinement* of this series is any series of the form

$$X = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{i-1} \supseteq H \supseteq H_i \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}\,,$$

where H is a normal subalgebra of H_{i-1} and H_i is a normal subalgebra of H, $i = 1, 2, \ldots, n$. A *refinement* of (1) is a subnormal B-series which is obtained from (1) by a finite sequence of one-step refinements. A refinement

$$X = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{m-1} \supseteq K_m = \{0\}\,, \tag{2}$$

of (1) is called a *proper refinement* if there exists a subalgebra K_j in (2) which is different from each H_i of (1). Thus, a series of subalgebras

$$X = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{m-1} \supseteq K_m = \{0\}$$

of X is called a *refinement* of a series of subalgebras

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = 0,$$

of X if

$$\{H_0, H_1, H_2, \dots, H_n\} \subseteq \{K_0, K_1, K_2, \dots, K_m\}$$

and is called a *proper refinement* if

$$\{H_0, H_1, H_2, \dots, H_n\} \subset \{K_0, K_1, K_2, \dots, K_m\}.$$

Example 5. In Example 4, $X \supset \{0,3,6,9\} \supset \{0,6\} \supset \{0\}$ is a refinement of $X \supset \{0,6\} \supset \{0\}$ while $X \supset \{0,2,4,6,8,10\} \supset \{0,4,8\} \supset \{0\}$ is not.

Theorem 6. *A subnormal B-series in X is a composition B-series if and only if it has no proper refinement.*

Proof. Let

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\} \tag{3}$$

be a composition B-series of X. Suppose that

$$X = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{i-1} \supseteq H \supseteq H_i \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$$

is a one-step refinement of (3). Since (3) is a composition B-series, H_i is a normal subalgebra of H_{i-1}. Thus, either $H = H_{i-1}$ or $H = H_i$. Hence, it follows that (3) has no proper refinement. Conversely, suppose that

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\} \tag{4}$$

is a subnormal B-series which has no proper refinement. Suppose that (4) is not a composition B-series. Then there exists a subalgebra H_i in (4) such that H_i is not a maximal normal subalgebra in H_{i-1}. Thus, there exists a subalgebra H such that $H_{i-1} \neq H \neq H_i$, H is normal in H_{i-1}, and H_i is normal in H. This produces a proper refinement of (4), a contradiction. Therefore, (4) is a composition B-series. ■

Definition. Two subnormal B-series

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\} \tag{5}$$

and

$$X = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{m-1} \supseteq K_m = \{0\} \tag{6}$$

for a B-algebra X are called *equivalent* if there is a one-one correspondence between the nontrivial factors of (5) and (6) such that correponding factors are B-isomorphic.

Lemma 7. *Let H', H, K', and K be subalgebras of X such that H' is a normal subalgebra of H and K' is a normal subalgebra of K. Then $H'(H \cap K')$ is a normal subalgebra of $H'(H \cap K)$ and $K'(H' \cap K)$ is a normal subalgebra of $K'(H \cap K)$. Furthermore,*

$$H'(H \cap K)/H'(H \cap K') \cong K'(H \cap K)/K'(H' \cap K).$$

Proof. Since H' is normal in H and K' is normal in K, $H \cap K'$ and $H' \cap K$ are normal subalgebras of $H \cap K$ by [14, Lemma 2.10]. Also $(H \cap K')(H' \cap K)$ is normal in $H \cap K$ by [14, Lemma 2.12]. For simplicity, let $J = (H \cap K')(H' \cap K)$. Define $f : H'(H \cap K) \to (H \cap K)/J$ as follows: if $x \in H'(H \cap K)$, then $x = h' * (0 * y)$, where $h' \in H'$ and $y \in H \cap K$. Set $f(x) = Jy$.

Let $a_1, a_2 \in H'(H \cap K)$. Then $a_1 = h'_1 * (0 * b_1)$ and $a_2 = h'_2 * (0 * b_2)$ for some $h'_1, h'_2 \in H'$ and $b_1, b_2 \in H \cap K$.

Claim 1: f is well-defined.

Suppose that $a_1 = a_2$. Then by (III), (I), and [21, Lemma 2.6], we have

$$\begin{aligned} h'_1 * (0 * b_1) &= h'_2 * (0 * b_2) \\ b_2 * (h'_1 * (0 * b_1)) &= b_2 * (h'_2 * (0 * b_2)) \\ (b_2 * b_1) * h'_1 &= (b_2 * b_2) * h'_2 \\ (b_2 * b_1) * h'_1 &= 0 * h'_2 \\ ((b_2 * b_1) * h'_1) * (0 * h'_1) &= (0 * h'_2) * (0 * h'_1) \\ b_2 * b_1 &= (0 * h'_2) * (0 * h'_1) \end{aligned}$$

Thus, $(0 * h'_2) * (0 * h'_1) = b_2 * b_1 \in H \cap K$. Hence, $(0 * h'_2) * (0 * h'_1) \in H'(H \cap K) \subseteq H' \cap K \subseteq J$. It follows that $b_2 * b_1 \in J$. By [7, Theorem 3.3(ii)], $f(a_1) = Jb_1 = Jb_2 = f(a_2)$. This proves Claim 1.

Claim 2: f is a B-homomorphism.

First, take note that $H'(H \cap K) = (H \cap K)H'$. Since H' and $H \cap K$ are subalgebras of H with H' normal in H, by [14, Lemma 2.11], $H'(H \cap K)$ is a subalgebra of H. And by [14, Theorem 2.8], $H'(H \cap K) = (H \cap K)H'$.

So, for $h'_2 * (0 * (b_2 * b_1) \in H'(H \cap K)$, $h'_2 * (0 * (b_2 * b_1) \in (H \cap K)H'$. That is, $h'_2 * (0 * (b_2 * b_1) = (b_2 * b_1) * (0 * h'_3)$, for some h'_3 in H'.

Now, by (III), [29, Lemma 2.3(v)], and [21, Proposition 2.8], we have,

$$\begin{aligned} a_1 * a_2 &= (h'_1 * (0 * b_1)) * (h'_2 * (0 * b_2)) \\ &= h'_1 * ((h'_2 * (0 * b_2)) * b_1) \\ &= h'_1 * (h'_2 * (b_1 * b_2)) \\ &= h'_1 * (h'_2 * (0 * (b_2 * b_1))) \\ &= h'_1 * ((b_2 * b_1) * (0 * h'_3)) \end{aligned}$$

$$\begin{aligned} &= (h_1' * h_3') * (b_2 * b_1) \\ &= h_4' * (0 * (b_1 * b_2)) \end{aligned}$$

for $h_4' \in H'$.
Then,

$$\begin{aligned} f(a_1 * a_2) &= f(h_4' * (0 * (b_1 * b_2))) \\ &= J(b_1 * b_2) \\ &= Jb_1 * Jb_2 \\ &= f(a_1) * f(a_2) \end{aligned}$$

This proves Claim 2.
Claim 3: f is onto.
Let $Jy \in (H \cap K)/J$. Then $y = 0 * (0 * y) \in H'(H \cap K)$ and $f(y) = Jy$. This proves Claim 3.
Therefore, by [22, Theorem 3.11], $H'(H \cap K)/Kerf \cong (H \cap K)/J$.
Claim 4: $Kerf = H'(H \cap K')$.
Let $(h_1' * (0 * b_1)) \in Kerf$, for $h_1' \in H'$ and $b_1 \in H \cap K$. Then $J = f(h_1' * (0 * b_1)) = Jb_1$. By [7, Theorem 3.3(ii)], $(0 * b_1) \in J$. If $(0 * b_1) \in J = (H \cap K')(H' \cap K)$, then $(0 * b_1) = h_2' * (0 * b_2)$ for $h_2' \in H \cap K'$ and $b_2 \in H' \cap K$.
Hence, $(h_1' * (0 * b_1)) \in Kerf$ if and only if $h_1' * (0 * b_1) = h_1' * (h_2' * (0 * b_2)) = (h_1' * b_2) * h_2'$. Note that $h_1' * b_2 = h_1' * (0 * (0 * b_2)) \in H'(H' \cap K)$ implies that, by [14, Lemma 2.7], $h_1' * b_2 \in H'$. Hence, $(h_1' * (0 * b_1)) \in H'(H \cap K'$. Therefore, $Kerf = H'(H \cap K')$.
Therefore, $H'(H \cap K)/H'(H \cap K') \cong (H \cap K)/(H \cap K)(H' \cap K)$.
Similar argument applies for $K'(H \cap K)/K'(H' \cap K) \cong H \cap K/(H \cap K')(H' \cap K)$.
Therefore, $H'(H \cap K)/H'(H \cap K') \cong K'(H \cap K)/K'(H' \cap K)$. ∎

Theorem 8. *Any two subnormal B-series*

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$$

and

$$X = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{m-1} \supseteq K_m = \{0\}$$

of X have refinements which are equivalent.

Proof. Between each H_i and H_{i+1}, insert the subalgebra

$$H_{i+1}\left(H_i \cap K_j\right), j = 0, 1, 2, \ldots, m.$$

Between each K_j and K_{j+1}, insert the subalgebra

$$K_{j+1}\left(K_j \cap H_i\right), i = 0, 1, 2, \ldots, n.$$

These refinements are subnormal B-series with mn inclusions. The final refinements are

$$\cdots \supseteq H_{i+1}(H_i \cap K_j) \supseteq H_{i+1}(H_i \cap K_{j+1}) \supseteq \cdots$$

and

$$\cdots \supseteq K_{j+1}(K_j \cap H_i) \supseteq K_{j+1}(K_j \cap H_{i+1}) \supseteq \cdots.$$

By Lemma 7,

$$H_{i+1}(H_i \cap K_j)/H_{i+1}(H_i \cap K_{j+1}) \cong K_{j+1}(K_j \cap H_i)/K_{j+1}(K_j \cap H_{i+1}).$$

The result follows. ■

Theorem 9. *Any two composition B-series of X are equivalent.*

Proof. Any two composition B-series of X have equivalent refinements and by Theorem 6, a composition B-series has no proper refinements. Thus, a composition B-series is equivalent to every refinement of itself. Therefore, any two composition B-series of X are equivalent. ■

If X has a subnormal B-series $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$ such that H_i/H_{i+1} is commutative, $i = 0, 1, \ldots, n-1$, then we say that X is *solvable*. Such a subnormal B-series is called a *solvable B-series* for X.

Remark 10. Every commutative B-algebra is solvable.

Example 11. The noncommutative B-algebra X in Example 1 is solvable since $X \supset \{0, 1, 2\} \supset \{0\}$ is a solvable B-series for X.

3. Properties of Solvable B-algebra

We now present some of the basic properties of solvable B-algebras.

Theorem 12. *Every subalgebra of a solvable B-algebra is solvable.*

Proof. Let $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$ be a solvable B-series of X. Let K be any subalgebra of X. Set $K_i = K \cap H_i$, $i = 0, 1, \ldots, n$. Since H_{i+1} is a normal subalgebra of H_i, $H_{i+1} \cap K$ is a normal subalgebra of $H_i \cap K$. Thus, K_{i+1} is a normal subalgebra of K_i. Now, $K_{i+1} = K \cap H_{i+1} = K \cap H_i \cap H_{i+1} = K_i \cap H_{i+1}$. Hence, $K_i/K_{i+1} = K_i/(K_i \cap H_{i+1})$. By [14, Theorem 3.4], $K_i/K_{i+1} \cong K_i H_{i+1}/H_{i+1}$. Since $K_i H_{i+1}/H_{i+1}$ is a subalgebra of H_i/H_{i+1} and H_i/H_{i+1} is commutative, $K_i H_{i+1}/H_{i+1}$ is commutative. Therefore, K_i/K_{i+1} is commutative and so the series

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{n-1} \supseteq K_n = \{0\}$$

is a solvable B-series for K. Consequently, K is a solvable. ■

Theorem 13. *Every homomorphic image of a solvable B-algebra is solvable.*

Proof. Let $f : X \to Y$ be a B-epimorphism. Suppose that X is solvable. Let $X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$ be a solvable B-series of X. Set $K_i = f(H_i)$, $i = 0, 1, \ldots, n$. Since f is a B-epimorphism, $f(H_{i+1})$ is a normal subalgebra of $f(H_i)$. Since $H_i \supseteq H_{i+1}$, $f(H_i) \supseteq f(H_{i+1})$. Hence, $Y = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{n-1} \supseteq K_n = \{0\}$ is a subnormal B-series of Y. Define $g : H_i \to K_i/K_{i+1}$ by $g(h_i) = f(h_i)K_{i+1}$. Since f is a B-epimorphism, g is a B-epimorphism of H_i onto K_i/K_{i+1}. Note that for any $h_{i+1} \in K_{i+1} \subseteq K_i$, $g(h_{i+1}) = f(h_{i+1})K_{i+1} = f(h_{i+1})f(H_{i+1}) = f(H_{i+1})$. Hence, $H_{i+1} \subseteq Ker g$. Thus, g induces a B-epimorphism of H_i/H_{i+1} onto K_i/K_{i+1}. Since H_i/H_{i+1} is commutative, K_i/K_{i+1} is commutative. Therefore, the subnormal B-series $Y = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_{n-1} \supseteq K_n = \{0\}$ is a solvable B-series for Y and so Y is solvable. ■

Corollary 14. *If X is solvable and H is normal in X, then H and X/H are solvable.*

Theorem 15. *Let H be normal in X. If both H and X/H are solvable, then X is solvable.*

Proof. Suppose that H and X/H are solvable. Let

$$X/H = K'_0 \supseteq K'_1 \supseteq K'_2 \supseteq \cdots \supseteq K'_{m-1} \supseteq K'_m = \{0H\} = \{H\}$$

be a solvable B-series for X/H. By [8, Corollary 16], there are subalgebras K_i of X, $i = 0, 1, \ldots, m$, such that K_{i+1} is a normal subalgebra of K_i, $K'_i = K_i/H$, $i = 0, 1, \ldots, m-1$, $X = K_0$, and $H = K_m$. By [22, Theorem 3.14],

$$K_i/K_{i+1} \cong K'_i/K'_{i+1}.$$

Since H is solvable, H has a solvable B-series

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}.$$

Thus,

$$X = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_{m-1} \supseteq H \supseteq H_1 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\}$$

is a solvable B-series for X. Therefore, X is solvable. ■

Corollary 16. *Let H and K be subalgebras of X and H be normal in X. If both H and K are solvable, then HK is solvable.*

Proof. Suppose that H and K are solvable. By [14, Lemma 2.11], HK is a subalgebra of X. By [14, Theorem 3.4], $HK/H \cong K/H \cap K$. By [14, Lemma 2.1], $H \cap K$ is a subalgebra of K. Thus, by Theorem 12, $H \cap K$ is solvable. Hence, $K/H \cap K$ is solvable by Corollary 14. Therefore, HK/H is solvable. Therefore, by Theorem 15, HK is solvable. ■

Corollary 17. *Let H and K be normal subalgebras of X such that X/H and X/K are solvable. Then X is solvable if and only if $H \cap K$ is solvable.*

Proof. Suppose that X is solvable. By Theorem 12, $H \cap K$ is solvable. Conversely, suppose that $H \cap K$ is solvable. By [14, Theorem 3.4], $HK/H \cong K/H \cap K$. Since HK/H is a subalgebra of a solvable B-algebra X/H, HK/H is solvable by Theorem 12. Thus, $K/H \cap K$ is solvable. By Theorem 15, K is solvable. Therefore, by Theorem 15, X is solvable. ■

Theorem 18. *Any refinement of a solvable B-series of X is a solvable B-series.*

Proof. Let

$$X = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\} \tag{7}$$

be a solvable B-series for X and let

$$X = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{i-1} \supseteq H \supseteq H_i \supseteq \cdots \supseteq H_{n-1} \supseteq H_n = \{0\} \tag{8}$$

be a one-step refinement of (7). From (7), H_{i-1}/H_i is commutative. Since H/H_i is a subalgebra of H_{i-1}/H_i, H/H_i is commutative. By [22, Theorem 3.14], $(H_{i-1}/H_i)/(H/H_i) \cong H_{i-1}/H$ and so H_{i-1}/H is commutative. Thus, (8) is a solvable B-series. Hence, any one-step refinement of (7) is a solvable B-series. By induction, any refinement of (7) is a solvable B-series. ■

Recall from [2] that the center of X is given by

$$Z(X) = \{x \in X : x * (0 * y) = y * (0 * x) \text{ for all } y \in X\}.$$

Note that $Z(X)$ is a subalgebra of X [2]. Moreover, it is normal in X [30].

Theorem 19. *X is solvable if and only if $X/Z(X)$ is solvable.*

Proof. If X is solvable, then $X/Z(X)$ is solvable by Corollary 14. Conversely, if $X/Z(X)$ is solvable, then X is solvable by Remark 10 and Theorem 15. ■

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