Discussiones Mathematicae
General Algebra and Applications xx (xxxx) 1–20

ALGEBRAS OF FULL TERMS CONSTRUCTED FROM TRANSFORMATIONS WITH FIXED SET

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Abstract

Based on th notion of full transformations with fixed set, in this paper, we present a novel concept of n-ary $Fix(I_n, Y)$-full terms. This term can be considered as a generalization of strongly full terms, permutational full terms and full terms. Together with the superposition operation, one can form a Menger algebra of rank n. The freeness of such algebra with respect to a variety of algebras of the same types is discussed. Furthermore, we apply hypersubstitution theory to define a $Fix(I_n, Y)$-full closed identity, a $Fix(I_n, Y)$-full closed variety and present some concrete examples.

Keywords: transformations with fixed set, full term, strongly full term, permutational full term, Menger algebra, hypersubstitution.

2020 Mathematics Subject Classification: 08B15, 08A62.

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1. Introduction

Term is one of the principal concept of the study in universal algebra, which can be considered as an appropriate language for describing classes of algebras. Let $\tau_n := (n_i)_{i\in I}$ be a type of algebras in which all operation symbols f_i are indexed by some set I and have arity $n_i = n$, for a fixed positive integer n. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-elements alphabet of variables. By $W_{\tau_n}(X_n)$ we denote the set of all *n-ary terms of type* τ_n. Recent contributions on terms can be found, for example, in [2, 3, 10, 12, 16]. Actually, in [5], K. Denecke and P. Jampachon defined *n-ary full terms of type* τ_n in the following way:

(i) Let $s : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and f_i be an operation symbol of type τ_n. Then $f_i(x_{s(1)}, \ldots, x_{s(n)})$ is an *n-ary full term of type* τ_n.

(ii) If $t_1, \ldots, t_n$ are n-ary full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an *n-ary full term of type* τ_n.

The set of all n-ary full terms of type τ_n is closed under finite application of (ii) and is denoted by $W^F_{\tau_n}(X_n)$. If s is an identity mapping, then $W^F_{\tau_n}(X_n)$ is denoted by $W^{SF}_{\tau_n}(X_n)$, and it is called the set of all *n-ary strongly full terms of type* τ_n [4]. If s is a permutation, then $W^F_{\tau_n}(X_n)$ is denoted by $W^{PF}_{\tau_n}(X_n)$, and it is called the set of all n-ary permutational full terms of type τ_n [13]. Obviously,

$$W^{SF}_{\tau_n}(X_n) \subseteq W^{PF}_{\tau_n}(X_n) \subseteq W^{F}_{\tau_n}(X_n) \subseteq W_{\tau_n}(X_n).$$

There are several possibilities to define other classes of terms by different mappings in a finite set. Recall that the semigroup of all mappings from a nonempty set X into itself under the usual composition is called the full transformation semigroup and denoted by $T(X)$. If $X = \{1, \ldots, n\}$, we may write T_n instead of $T(X)$.

Recently, Wattanatripop and Changphas introduced the notions of an $K^*(n,r)$-full terms [21] by considering a subsemigroup $K^*(n,r) := \{\alpha \in T_n \mid |im(\alpha)| \leq r\} \cup \{1_{id}\}$ of T_n in which each element is called a *restricted range transformation*. It is observed that $K^*(n,r) = K(n,r) = T_n$ if $r = n$. Thus, a clone denoted by $clone_{K^*(n,r)}(\tau_n)$ consisting of the set of all n-ary $K^*(n,r)$-full terms of type τ_n and a superposition S^n was constructed. On the other hand, the set $OD_n = \{\alpha \in T_n \mid \forall k \in \{1, \ldots, n\}, \alpha(k) \leq k\}$ of all *order-decreasing full transformations* on a finite chain which is a submonoid of T_n was applied to define an n-ary order-decreasing full term of type τ_n in [22]. An identity of a variety that determined by a pair of terms in $\mathcal{MA}_{OD_n}(\tau_n)$ and full closed variesties were examined. In [20], a semigroup $S(\bar{n}, Y) := \{\beta \in T_n \mid \beta(Y) \subseteq Y\}$ of transformations on a finite set $\bar{n}$ leaving $Y \subseteq \bar{n}$ invariant was applied to set a new term in such a way that each pair of these terms was extended to be $S(\bar{n}, Y)$-hyperidentity of a variety V. Similarly, in [18], the theorem which gave the freeness of an algebra consisting of the set of all terms generated by transformations with restricted range and $(n+1)$-ary operation satisfying certain equational laws was proved.

In [11], Honyam and Sanwong introduced a semigroup $Fix(X,Y)$ which is called a *transformation semigroup with fixed set*, which contains the identity mapping on X, denoted by 1_X. Actually, for a fixed subset Y of X,

$$Fix(X,Y) = \{\alpha \in T(X) \mid \alpha(a) = a \text{ for all } a \in Y\}.$$

It is clear that $Fix(X,Y) = T(X)$ if $Y = \emptyset$ and $Fix(X,Y)$ contains only the identity mapping 1_X if $|X| = 1$ or $X = Y$.

Our main goal of this paper is to generalize the concepts of strongly full terms, permutational full terms and full terms. In Section 2, applying the notion of transformations with fixed set, we introduce a special kind of n-ary terms of type τ_n, the so-called $Fix(I_n,Y)$-full terms. The combination between full terms and transformation with fixed set is established. This leads us to form a Menger algebra of $Fix(I_n,Y)$-full terms consisting the set of all $Fix(I_n,Y)$-full terms with $(n+1)$-ary superposition operation. The generating system and freeness of such algebra are studied. We continue the results in Section 3 by introducing the monoid of $Fix(I_n,Y)$-full hypersubstitutions and $Fix(I_n,Y)$-full substitutions. Particularly, the relation between these monoids is provided. The last section, we apply the former results for classifying the algebras of type τ_n.

2. THE MENGER ALGEBRA OF $Fix(I_n,Y)$-FULL TERMS

Let $I_n = \{1,\ldots,n\}$ where n is an arity of the operation symbol f_i. Throughout this paper, we consider $X = I_n$. This leads us to define

$$Fix(I_n,Y) = \{\alpha \in T(I_n) \mid \alpha(a) = a \text{ for all } a \in Y\}$$

where $T(I_n)$ is the semigroup of all mappings from I_n into itself under the usual composition of functions.

We then have the following example.

Example 1. Let $\tau_4 = (4)$ be a type. This means that we have $I_4 = \{1,2,3,4\}$. If we let $Y = \{2,4\} \subseteq I_4$, then

$$Fix(I_4,Y) =$$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 4 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 4 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 4 & 4 \end{pmatrix} \Bigg\}.$$

The following example shows that, if $Y = \emptyset$, then $Fix(I_n, Y) = T_n$.

Example 2. Consider a type $\tau_2 = (2)$. Then we have $I_2 = \{1, 2\}$. Let $Y = \emptyset$. Thus

$$Fix(I_2, Y) = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\} = T(I_2).$$

In the case that $I_n = Y$, we have the following example.

Example 3. Consider type $\tau_3 = (3)$. Then $I_3 = \{1, 2, 3\}$. If $Y = I_3$, then

$$Fix(I_3, Y) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}.$$

The next example shows that, if $|I_n| = 1$, then $Fix(I_n, Y) = 1_{I_n}$ where 1_{I_n} is the identity mapping on I_n.

Example 4. Consider a type $\tau_1 = (1)$. That is $I_1 = \{1\}$. For arbitrary subset Y of I_1. Then $Fix(I_1, Y) = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Now, we inductively define n-ary $Fix(I_n, Y)$-full terms of type τ_n as follows.

(i) If f_i is an n-ary operation symbol and $\alpha \in Fix(I_n, Y)$, then $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an *n-ary $Fix(I_n, Y)$-full term of type τ_n*.

(ii) If f_i is an n-ary operation symbol and $t_1, \ldots, t_n$ are n-ary $Fix(I_n, Y)$-full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an *n-ary $Fix(I_n, Y)$-full term of type τ_n*.

The set of all n-ary $Fix(I_n, Y)$-full terms of type τ_n which is closed under finite applications of (ii), is denoted by $W_{\tau_n}^{Fix(I_n,Y)}(X_n)$.

Example 5. Consider $\tau_4 = (4)$ with a 4-ary operation symbol f. Let $I_4 = \{1, 2, 3, 4\}$. If $Y = \emptyset$, then there are many elements in $W_{\tau_4}^{Fix(I_4,Y)}(X_4)$ such as
$f(x_1, x_1, x_2, x_3), f(x_2, x_3, x_4, x_1), f(x_2, x_4, x_1, x_1), f(x_1, x_4, x_4, x_4),$
$f(x_2, x_3, x_3, x_1), f(x_2, x_2, x_2, x_3), f(f(x_2, x_3, x_3, x_1), f(x_2, x_4, x_1, x_1),$
$f(x_1, x_2, x_3, x_4), f(x_1, x_1, x_2, x_3))$. If $Y = \{1, 2, 3\}$, then $W_{\tau_4}^{Fix(I_4,Y)}(X_4)$ consists many elements, for instance,
$f(x_1, x_2, x_3, x_1), f(x_1, x_2, x_3, x_2), f(x_1, x_2, x_3, x_3), f(x_1, x_2, x_3, x_4),$

$f(f(x_1, x_2, x_3, x_1), f(x_1, x_2, x_3, x_2), f(x_1, x_2, x_3, x_3), f(x_1, x_2, x_3, x_4))$.
If $Y = I_4$, then the followings are some elements in $W_{\tau_4}^{Fix(I_4,Y)}(X_4)$:
$f(x_1, x_2, x_3, x_4), f(f(x_1, x_2, x_3, x_4), f(x_1, x_2, x_3, x_4), f(x_1, x_2, x_3, x_4),$
$f(x_1, x_2, x_3, x_4))$.

Remark 6. Let $\tau_n = (n_i)$ where $n_i = n$ for every $i \in I$ be a type. Let $I_n = \{1, \ldots, n\}$ and $Y \subseteq I_n$. Then the following statements are valid.

(i) If $Y = \emptyset$, then $W_{\tau_n}^{SF}(X_n) \subset W_{\tau_n}^{PF}(X_n) \subset W_{\tau_n}^{Fix(I_n,Y)}(X_n) = W_{\tau_n}^{F}(X_n)$.

(ii) If $\emptyset \neq Y \subset I_n$, then $W_{\tau_n}^{SF}(X_n) \subset W_{\tau_n}^{Fix(I_n,Y)}(X_n) \subset W_{\tau_n}^{F}(X_n)$.

(iii) If $I_n = Y$, then $W_{\tau_n}^{SF}(X_n) = W_{\tau_n}^{Fix(I_n,Y)}(X_n) \subset W_{\tau_n}^{PF}(X_n) \subset W_{\tau_n}^{F}(X_n)$.

Example 7. By Example 4, we have $f(x_1), f(f(x_1)), f(f(f(x_1))) \in W_{\tau_1}^{Fix(I_1,Y)}(X_1)$.

Remark 8. If $|I_n| = 1$ and $Y \subseteq I_n$, then

$$W_{\tau_n}^{SF}(X_n) = W_{\tau_n}^{PF}(X_n) = W_{\tau_n}^{Fix(I_n,Y)}(X_n) = W_{\tau_n}^{F}(X_n).$$

For $W_{\tau_n}^{Fix(I_n,Y)}(X_n)$, the set of all n-ary $Fix(I_n, Y)$-full terms of type τ_n, the superposition operation

$$S^n : (W_{\tau_n}^{Fix(I_n,Y)}(X_n))^{n+1} \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$$

is defined by

(i) $S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n) := f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})$.

(ii) $S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$
where $\alpha \in Fix(I_n, Y)$.

Now we may consider the following algebra of type $(n + 1)$

$$MA_{Fix(I_n,Y)}(\tau_n) := (W_{\tau_n}^{Fix(I_n,Y)}(X_n), S^n).$$

An algebra (M, S^n) of type $\tau = (n + 1)$ is said to be a Menger algebra of rank n if (M, S^n) satisfies the superassociative law

$$\begin{aligned}&\widetilde{S}^n(\widetilde{S}^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n)\\ \approx\ &\widetilde{S}^n(X_0, \widetilde{S}^n(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n(Y_n, X_1, \ldots, X_n))\end{aligned}$$

where $\widetilde{S}^n$ is an $(n + 1)$-ary operation symbol and X_i, Y_j are variables. For more details, see [7, 8, 14, 17]. The following theorem shows that an algebra $MA_{Fix(I_n,Y)}(\tau_n)$ is a Menger algebra of rank n.

Theorem 9. *The algebra $MA_{Fix(I_n,Y)}(\tau_n)$ satisfies the superassociative law.*

Proof. We prove the theorem by induction on the complexity of the $Fix(I_n,Y)$-full term which is substituted for X_0. Firstly, if $X_0 = f_i(x_{\alpha(1)},\dots,x_{\alpha(n)})$ where $\alpha \in Fix(I_n,Y)$, then

$$\begin{aligned}
&S^n(S^n(f_i(x_{\alpha(1)},\dots,x_{\alpha(n)}),s_1,\dots,s_n),t_1,\dots,t_n)\\
&= S^n(f_i(s_{\alpha(1)},\dots,s_{\alpha(n)}),t_1,\dots,t_n)\\
&= f_i(S^n(s_{\alpha(1)},t_1,\dots,t_n),\dots,S^n(s_{\alpha(n)},t_1,\dots,t_n))\\
&= S^n(f_i(x_{\alpha(1)},\dots,x_{\alpha(n)}),S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)).
\end{aligned}$$

Let $f_i(r_1,\dots,r_n) \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ be such that $r_1,\dots,r_n$ satisfy the superassociative law. Then

$$\begin{aligned}
&S^n(S^n(f_i(r_1,\dots,r_n),s_1,\dots,s_n),t_1,\dots,t_n)\\
&= S^n(f_i(S^n(r_1,s_1,\dots,s_n),\dots,S^n(r_n,s_1,\dots,s_n)),t_1,\dots,t_n)\\
&= f_i(S^n(S^n(r_1,s_1,\dots,s_n),t_1,\dots,t_n),\dots,S^n(S^n(r_n,s_1,\dots,s_n),t_1,\dots,t_n))\\
&= f_i(S^n(r_1,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)),\dots,\\
&\quad S^n(r_n,S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)))\\
&= S^n(f_i(r_1,\dots,r_n),S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n)).
\end{aligned}$$

This completes the proof. ■

According to Theorem 9, if $Y = I_n$ and $\alpha \in Fix(I_n,Y)$ is the identity mapping, then the following corollary is obtained.

Corollary 10 ([4], Proposition 2.1)**.** *Let $t,t_1,\dots,t_n,s_1,\dots,s_n$ be strongly full terms of type τ_n. Then*

$$S^n(S^n(t,t_1,\dots,t_n),s_1,\dots,s_n) = S^n(t,S^n(t_1,s_1,\dots,s_n),\dots,S^n(t_n,s_1,\dots,s_n)).$$

In addition, if a subset Y of I_n is empty, then by Theorem 9, we have the following.

Corollary 11 ([5], Proposition 1)**.** *Let $t,t_1,\dots,t_n,s_1,\dots,s_n \in W_{\tau_n}^F(X_n)$. Then*

$$S^n(S^n(t,t_1,\dots,t_n),s_1,\dots,s_n) = S^n(t,S^n(t_1,s_1,\dots,s_n),\dots,S^n(t_n,s_1,\dots,s_n)).$$

The next aim is to study the freeness of algebra $MA_{Fix(I_n,Y)}(\tau_n)$. First, the generating system of such algebra is constructed. We see that

$$F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} := \left\{ f_i(x_{\alpha(1)},\dots,x_{\alpha(n)}) \mid i \in I, \alpha \in Fix(I_n,Y) \right\}$$

generates $MA_{Fix(I_n,Y)}(\tau_n)$.

Let V_{Menger} be the variety of all Menger algebras of type $(n+1)$ satisfying (SASS), and let $\mathcal{F}_{V_{Menger}}(Z)$ be the free algebra with respect to V_{Menger}, freely generated by $Z := \{z_j \mid j \in J\}$ where Z is an alphabet of variables indexed by the set $J := \{(i,\alpha) \mid i \in I, \alpha \in Fix(I_n,Y)\}$. The operation of $\mathcal{F}_{V_{Menger}}(Z)$ will be denoted by $\widetilde{S}^n$. We have the following theorem.

Theorem 12. *The algebra $MA_{Fix(I_n,Y)}(\tau_n)$ is free with respect to the variety V_{Menger} of Menger algebras of rank n, freely generated by the set*

$$Z = \big\{z_{(i,\alpha)} \mid i \in I, \alpha \in Fix(I_n,Y)\big\}.$$

Proof. Claim that $MA_{Fix(I_n,Y)}(\tau_n)$ is isomorphic to $\mathcal{F}_{V_{Menger}}(Z)$ under the mapping

$$\varphi : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to \mathcal{F}_{V_{Menger}}(Z)$$

defined by

(i) $\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})) := z_{(i,\alpha)}$.

(ii) $\varphi(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n) := \widetilde{S}^n(z_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n))$.

We prove the theorem by induction on the complexity of the term t that

$$\varphi(S^n(t,t_1,\ldots,t_n)) = \widetilde{S}^n(\varphi(t),\varphi(t_1),\ldots,\varphi(t_n))$$

for all $t,t_1,\ldots,t_n \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$. If $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$, then

$$\begin{aligned}
\varphi(S^n(t,t_1,\ldots,t_n)) &= \varphi(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n)) \\
&= \widetilde{S}^n(z_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n)) \\
&= \widetilde{S}^n(\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})),\varphi(t_1),\ldots,\varphi(t_n)) \\
&= \widetilde{S}^n(\varphi(t),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

Let $t = f_i(r_1,\ldots,r_n)$ and assume that, for $1 \le k \le n$,

$$\varphi(S^n(r_k,t_1,\ldots,t_n)) = \widetilde{S}^n(\varphi(r_k),\varphi(t_1),\ldots,\varphi(t_n)).$$

By the fact,

$$\varphi(f_i(t'_1,\ldots,t'_n)) = \widetilde{S}^n(z_{(i,1_n)},\varphi(t'_1),\ldots,\varphi(t'_n))$$

for all $t'_1, \ldots, t'_n \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$, we then have

$$
\begin{aligned}
& \varphi(S^n(t, t_1, \ldots, t_n)) \\
= {} & \varphi(S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)) \\
= {} & \varphi(f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^n(r_n, t_1, \ldots, t_n)) \\
= {} & \widetilde{S}^n(z_{(i,1_n)}, \varphi(S^n(r_1, t_1, \ldots, t_n)), \ldots, \varphi(S^n(r_n, t_1, \ldots, t_n))) \\
= {} & \widetilde{S}^n(z_{(i,1_n)}, \widetilde{S}^n(\varphi(r_1), \varphi(t_1), \ldots, \varphi(t_n)), \ldots, \widetilde{S}^n(\varphi(r_n), \varphi(t_1), \ldots, \varphi(t_n))) \\
= {} & \widetilde{S}^n(\widetilde{S}^n(z_{(i,1_n)}, \varphi(r_1), \ldots, \varphi(r_n)), \varphi(t_1), \ldots, \varphi(t_n)) \\
= {} & \widetilde{S}^n(\varphi(t), \varphi(t_1), \ldots, \varphi(t_n)).
\end{aligned}
$$

Here, φ is a homomorphism.

For a bijection of mapping φ, it can be proved by the following

$$z_{(i,\alpha)} = z_{(j,\beta)} \Rightarrow (i, \alpha) = (j, \beta) \Rightarrow f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = f_j(x_{\beta(1)}, \ldots, x_{\beta(n)})$$

and

$$z_{(i,\alpha)} \in Z \Rightarrow \varphi(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})) = z_{(i,\alpha)}.$$

Thus φ is an isomorphism. ■

As we have seen in Theorem 12, if a mapping $\alpha \in Fix(I_n, Y)$ is identity and a subset Y of I_n is empty, then we have Theorem 2.2 in [4] and Theorem 1 in [5], respectively.

3. EMBEDDING THEOREM OF $Fix(I_n, Y)$-FULL HYPERSUBSTITUTIONS AND $Fix(I_n, Y)$-FULL SUBSTITUTIONS

The concept of hypersubstitutions was introduced by Graczyńska and Schweigert [9]. For more details about hypersubstitution theory, see [6]. In this section, the concept of a mapping which maps from the set of all operation symbols of type τ_n to the set of all n-ary $Fix(I_n, Y)$-full terms of type τ_n is defined as follows.

A $Fix(I_n, Y)$*-full hypersubstitution of type* τ_n is a mapping

$$\sigma : \{f_i \mid i \in I\} \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$$

taking every n-ary operation symbol of type τ_n to an n-ary $Fix(I_n, Y)$-full term of the same type. The set of all $Fix(I_n, Y)$-full hypersubstitutions of type τ_n is denoted by $Hyp^{Fix(I_n,Y)}(\tau_n)$.

For $t \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ and $\alpha, \beta \in Fix(I_n, Y)$, we define a $Fix(I_n, Y)$-full term arising from a mapping β as follows:

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then $t_\beta := f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))})$.

(ii) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta := f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

It is observed that if t is a $Fix(I_n, Y)$-full term of type τ_n, then t_β is a $Fix(I_n, Y)$-full term of type τ_n for all $\beta \in Fix(I_n, Y)$.

Any $Fix(I_n, Y)$-full hypersubstitution $\sigma : \{f_i \mid i \in I\} \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ of type τ_n can be extended to a mapping

$$\hat{\sigma} : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$$

defined by the following steps:

(i) $\hat{\sigma}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] := (\sigma(f_i))_\alpha$ where $\alpha \in Fix(I_n, Y)$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

Now, we define a binary operation $\circ_h$ as follows

$$(\sigma_1 \circ_h \sigma_2) := \hat{\sigma}_1 \circ \sigma_2$$

where $\sigma_1, \sigma_2 \in Hyp^{Fix(I_n,Y)}(\tau_n)$ and $\circ$ is the usual composition of functions.

Now, we present connections between the superposition operation and $\hat{\sigma}$.

Lemma 13. *Let $t, t_1, \ldots, t_n \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$. Then*

$$S^n(t, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) = S^n(t_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $\alpha \in Fix(I_n, Y)$.

Proof. Let $t = f_i(x_{\beta(1)}, \ldots, x_{\beta(n)})$ where $\beta \in Fix(I_n, Y)$. For $\alpha \in Fix(I_n, Y)$, we then have

$$\begin{aligned} S^n(t, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) &= S^n(f_i(x_{\beta(1)}, \ldots, x_{\beta(n)}), \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\ &= f_i(\hat{\sigma}[t_{\alpha(\beta(1))}], \ldots, \hat{\sigma}[t_{\alpha(\beta(n))}]) \\ &= S^n(f_i(x_{\alpha(\beta(1))}, \ldots, x_{\alpha(\beta(n))}), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\ &= S^n(t_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \end{aligned}$$

Let $t = f_i(s_1, \ldots, s_n)$ and assume that

$$S^n(s_k, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) = S^n((s_k)_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $1 \leq k \leq n$ and for all $\alpha \in Fix(I_n, Y)$. Then, for $\alpha \in Fix(I_n, Y)$, we have

$$\begin{aligned} & S^n(t, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\ = \; & S^n(f_i(s_1, \ldots, s_n), \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\ = \; & f_i(S^n(s_1, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]), \ldots, S^n(s_n, \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}])) \\ = \; & f_i(S^n((s_1)_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]), \ldots, S^n((s_n)_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) \\ = \; & S^n(f_i((s_1)_\alpha, \ldots, (s_n)_\alpha), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\ = \; & S^n(t_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]). \end{aligned}$$

■

Applying Theorem 9 and Lemma 13, we can prove the following theorem.

Theorem 14. *For $\sigma \in Hyp^{Fix(I_n,Y)}(\tau_n)$, an extension*

$$\hat{\sigma} : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$$

is an endomorphism on the algebra $MA_{Fix(I_n,Y)}(\tau_n)$.

Proof. We prove the theorem by induction on the complexity of t_0 that for any $t_0, t_1, \ldots, t_n \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$, $\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

Firstly, if we substitute for t_0 a term $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in Fix(I_n, Y)$, then

$$\begin{aligned}
\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] &= \hat{\sigma}[S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})] \\
&= S^n(\sigma(f_i), \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\
&= S^n((\sigma(f_i))_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

Assume $t_0 = f_i(s_1, \ldots, s_n)$ such that

$$\hat{\sigma}[S^n(s_k, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[s_k], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $1 \leq k \leq n$. Then

$$\begin{aligned}
&\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] \\
&= \hat{\sigma}[S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)] \\
&= S^n(\sigma(f_i), \hat{\sigma}[S^n(s_1, t_1, \ldots, t_n)], \ldots, \hat{\sigma}[S^n(s_n, t_1, \ldots, t_n)]) \\
&= S^n(\sigma(f_i), S^n(\hat{\sigma}[s_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]), \ldots, S^n(\hat{\sigma}[s_n], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) \\
&= S^n(S^n(\sigma(f_i), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_n]), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[f_i(s_1, \ldots, s_n)], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

■

The following proposition shows a property of term that arise from a mapping and the extension of each element in $Hyp^{Fix(I_n,Y)}(\tau_n)$.

Proposition 15. *Let $t \in W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ and $\beta \in Fix(I_n, Y)$. Then*

$$\hat{\sigma}[t]_\beta = \hat{\sigma}[t_\beta].$$

Proof. It can be proved by induction on the complexity of the term t. ■

By using Theorem 14 and Proposition 15, we get the following result showing the relationship between the operation $\circ_h$ and the extension of σ.

Lemma 16. *Let* $\hat{\sigma}_1, \hat{\sigma}_2 \in Hyp^{Fix(I_n,Y)}(\tau_n)$. *Then*

$$(\sigma_1 \circ_h \sigma_2)^{\hat{}} = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Proof. We prove the lemma by induction on the complexity of the $Fix(I_n, Y)$-full term which is substituted for t. If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in Fix(I_n, Y)$, then

$$\begin{aligned}
(\sigma_1 \circ_h \sigma_2)^{\hat{}}[t] &= (\sigma_1 \circ_h \sigma_2)^{\hat{}}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] \\
&= (\hat{\sigma}_1 \circ \sigma_2)^{\hat{}}[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})] \\
&= ((\hat{\sigma}_1 \circ \sigma_2)(f_i))_\alpha \\
&= (\hat{\sigma}_1[\sigma_2(f_i)])_\alpha \\
&= \hat{\sigma}_1[\sigma_2(f_i)]_\alpha \\
&= \hat{\sigma}_1[\hat{\sigma}_2[f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})]] \\
&= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t].
\end{aligned}$$

Let $t = f_i(s_1, \ldots, s_n)$. Assume that $(\sigma_1 \circ_h \sigma_2)^{\hat{}}[s_k] = \hat{\sigma}_1 \circ \hat{\sigma}_2[s_k]$ for all $1 \leq k \leq n$. Then

$$\begin{aligned}
(\sigma_1 \circ_h \sigma_2)^{\hat{}}[t] &= (\sigma_1 \circ_h \sigma_2)^{\hat{}}[f_i(s_1, \ldots, s_n)] \\
&= S^n((\hat{\sigma}_1 \circ \sigma_2)(f_i), (\sigma_1 \circ_h \sigma_2)^{\hat{}}[s_1], \ldots, (\sigma_1 \circ_h \sigma_2)^{\hat{}}[s_n]) \\
&= S^n((\hat{\sigma}_1 \circ \sigma_2)(f_i), (\hat{\sigma}_1 \circ \hat{\sigma}_2)[s_1], \ldots, (\hat{\sigma}_1 \circ \hat{\sigma}_2)[s_n]) \\
&= S^n(\hat{\sigma}_1[\sigma_2(f_i)], \hat{\sigma}_1[\hat{\sigma}_2[s_1]], \ldots, \hat{\sigma}_1[\hat{\sigma}_2[s_n]]) \\
&= \hat{\sigma}_1[S^n(\sigma_2(f_i), \hat{\sigma}_2[s_1], \ldots, \hat{\sigma}_2[s_n])] \\
&= \hat{\sigma}_1[\hat{\sigma}_2[f_i(s_1, \ldots, s_n)]] \\
&= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t].
\end{aligned}$$

Therefore $(\sigma_1 \circ_h \sigma_2)^{\hat{}} = \hat{\sigma}_1 \circ \hat{\sigma}_2$. ■

Now, we have the important result.

Theorem 17. $(Hyp^{Fix(I_n,Y)}(\tau_n); \circ_h, \sigma_{id})$ *is a monoid where* σ_{id} *is the identity hypersubstitution which is defined by* $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_n)$.

Proof. The associativity of a binary operation $\circ_h$ on $Hyp^{Fix(I_n,Y)}(\tau_n)$ follows directly from Lemma 16. Furthermore, the proof of the identity element $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_n)$ with respect to $\circ_h$ is clearly straightforward. ■

If a subset Y of I_n is empty, then by Theorem 17 we have

Corollary 18 [5]**.** *$(Hyp^{F}(\tau_n); \circ_h, \sigma_{id})$ forms a monoid.*

If $Y = I_n$ and $\alpha \in Fix(I_n, Y)$ is the identity mapping, then by Theorem 17 we obtain

Corollary 19 [4]**.** *$(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$ forms a monoid.*

Applying an operation $+$ on the set of all hypersubstitutions mentioned in [1], our next purpose is to define a second binary operation on $Hyp^{Fix(I_n,Y)}(\tau_n)$. Let σ_1 and σ_2 be elements in $Hyp^{Fix(I_n,Y)}(\tau_n)$. Define a binary operation on $Hyp^{Fix(I_n,Y)}(\tau_n)$ by

$$(\sigma_1 + \sigma_2)(f_i) := S^n(\sigma_2(f_i), \sigma_1(f_i), \ldots, \sigma_1(f_i)).$$

It is observed that $\sigma_1 + \sigma_2$ is an element in $Hyp^{Fix(I_n,Y)}(\tau_n)$.

The following theorem shows that the set $Hyp^{Fix(I_n,Y)}(\tau_n)$ together with two binary operations $\circ_h$ and $+$ forms a left-seminearring.

Theorem 20. *$(Hyp^{Fix(I_n,Y)}(\tau_n), \circ_h, +)$ forms a left-seminearring.*

Proof. We first prove that the operation $+$ is associative. For this, let $\sigma_1, \sigma_2, \sigma_3$ be elements in $Hyp^{Fix(I_n,Y)}(\tau_n)$. By Theorem 9, we have

$$\begin{aligned}
&((\sigma_1 + \sigma_2) + \sigma_3)(f_i) \\
= {}& S^n(\sigma_3(f_i), (\sigma_1 + \sigma_2)(f_i), \ldots, (\sigma_1 + \sigma_2)(f_i)) \\
= {}& S^n(\sigma_3(f_i), S^n(\sigma_2(f_i), \sigma_1(f_i), \ldots, \sigma_1(f_i)), \ldots, S^n(\sigma_2(f_i), \sigma_1(f_i), \ldots, \sigma_1(f_i))) \\
= {}& S^n(S^n(\sigma_3(f_i), \sigma_2(f_i), \ldots, \sigma_2(f_i)), \sigma_1(f_i), \ldots, \sigma_1(f_i)) \\
= {}& S^n((\sigma_2 + \sigma_3)(f_i), \sigma_1(f_i), \ldots, \sigma_1(f_i)) \\
= {}& (\sigma_1 + (\sigma_2 + \sigma_3))(f_i).
\end{aligned}$$

Next, the left distributive law $\sigma_1 \circ_h (\sigma_2 + \sigma_3) = (\sigma_1 \circ_h \sigma_2) + (\sigma_1 \circ_h \sigma_3)$ is satisfied by using Theorem 14. In fact, we have

$$\begin{aligned}
(\sigma_1 \circ_h (\sigma_2 + \sigma_3))(f_i) &= \hat{\sigma}_1[(\sigma_2 + \sigma_3)(f_i)] \\
&= \hat{\sigma}_1[S^n(\sigma_3(f_i), \sigma_2(f_i), \ldots, \sigma_2(f_i))] \\
&= S^n(\hat{\sigma}_1[\sigma_3(f_i)], \hat{\sigma}_1[\sigma_2(f_i)], \ldots, \hat{\sigma}_1[\sigma_2(f_i)]) \\
&= S^n((\sigma_1 \circ_h \sigma_3)(f_i), (\sigma_1 \circ_h \sigma_2), \ldots, (\sigma_1 \circ_h \sigma_2)) \\
&= (\sigma_1 \circ_h \sigma_2) + (\sigma_1 \circ_h \sigma_3).
\end{aligned}$$

■

It is not difficult to see that the right distributivity, $(\sigma_1 + \sigma_2) \circ_h \sigma_3 = (\sigma_1 \circ_h \sigma_3) + (\sigma_2 \circ_h \sigma_3)$, is not true for arbitrary $Fix(I_n, Y)$-full hypersubstitutions $\sigma_1, \sigma_2, \sigma_3$. The following counter example shows such statement.

Example 21. Let I be a singleton indexed set and $\tau_2 = (2)$ with a binary operation symbol f. Assume that $\sigma_1, \sigma_2, \sigma_3$ are elements in $Hyp^{Fix(I_2,\{2\})}(2)$ which are defined by

$$\sigma_1(f) = f(x_{\alpha_1(1)}, x_{\alpha_1(2)}) \text{ where } \alpha_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\sigma_2(f) = f(x_{\alpha_2(1)}, x_{\alpha_2(2)}) \text{ where } \alpha_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\sigma_3(f) = f(f(x_{\beta_1(1)}, x_{\beta_1(2)}), f(x_{\beta_2(1)}, x_{\beta_2(2)})) \text{ where } \beta_1 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \text{ and}$$

$$\beta_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Consider

$$\begin{aligned}
&((\sigma_1 + \sigma_2) \circ_h \sigma_3)(f) \\
&= (\sigma_1 + \sigma_2)^{\hat{\text{ffl}}}\, [f(f(x_{\beta_1(1)}, x_{\beta_1(2)}), f(x_{\beta_2(1)}, x_{\beta_2(2)}))] \\
&= S^2((\sigma_1 + \sigma_2)(f), (\sigma_1 + \sigma_2)^{\hat{\text{ffl}}}\, [f(x_{\beta_1(1)}, x_{\beta_1(2)})], (\sigma_1 + \sigma_2)^{\hat{\text{ffl}}}\, [f(x_{\beta_2(1)}, x_{\beta_2(2)})]) \\
&= S^2(f(f(x_1, x_2), f(x_1, x_2)), f(f(x_1, x_2), f(x_1, x_2))_{\beta_1}, f(f(x_1, x_2), f(x_1, x_2))_{\beta_2}) \\
&= f(f(f(f(x_2, x_2), f(x_2, x_2)), f(f(x_1, x_2), f(x_1, x_2))), \\
&\quad\; f(f(f(x_2, x_2), f(x_2, x_2)), f(f(x_1, x_2), f(x_1, x_2))))
\end{aligned}$$

and

$$((\sigma_1 \circ_h \sigma_3) + (\sigma_2 \circ_h \sigma_3))(f) = S^2((\sigma_2 \circ_h \sigma_3)(f), (\sigma_1 \circ_h \sigma_3)(f), (\sigma_1 \circ_h \sigma_3)(f)).$$

Since $(\sigma_1 \circ_h \sigma_3)(f) = f(f(x_2, x_2), f(x_1, x_2))$ and $(\sigma_2 \circ_h \sigma_3)(f) = f(f(x_2, x_2), f(x_2, x_2))$, a term $((\sigma_1 \circ_h \sigma_3) + (\sigma_2 \circ_h \sigma_3))(f)$ differs from $((\sigma_1 + \sigma_2) \circ_h \sigma_3)(f)$. Hence, $(\sigma_1 + \sigma_2) \circ_h \sigma_3 \neq (\sigma_1 \circ_h \sigma_3) + (\sigma_2 \circ_h \sigma_3)$. This means that the right distributivity does not hold.

To close this section, we discuss an embedding theorem for $Fix(I_n, Y)$-full hypersubstitution. Since the algebra $MA_{Fix(I_n,Y)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} := \left\{ f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) \mid i \in I, \alpha \in Fix(I_n, Y) \right\},$$

then any mapping

$$\eta : F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} \to W_{\tau_n}^{Fix(I_n,Y)}(X_n),$$

called a $Fix(I_n, Y)$-*full substitution*, can be uniquely extended to an endomorphism

$$\bar{\eta} : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to W_{\tau_n}^{Fix(I_n,Y)}(X_n).$$

Let $Subst_{Fix(I_n,Y)}(\tau_n)$ be the set of all $Fix(I_n, Y)$-full substitutions.

For $\eta_1, \eta_2 \in Subst_{Fix(I_n,Y)}(\tau_n)$, define

$$\eta_1 \odot \eta_2 := \bar{\eta}_1 \circ \eta_2$$

where $\circ$ is the usual composition. Let $id_{F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}}$ be the identity mapping on $F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}$. The equation $\overline{\overline{\eta_1} \circ \eta_2} = \overline{\eta_1} \circ \eta_2$ holds due to an application of Theorem 12. In fact, every clone substitution η in $Subst_{Fix(I_n,Y)}(\tau_n)$ that maps from a generating set $F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}$ to $W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ can be uniquely extended to an endomorphism $\overline{\eta}$ from the algebra $MA_{Fix(I_n,Y)}(\tau_n)$ to itself because $MA_{Fix(I_n,Y)}(\tau_n)$ is free. Thus we have that $(Subst_{Fix(I_n,Y)}(\tau_n); \odot, id_{F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}})$ is a monoid.

Consider $\sigma \in Hyp^{Fix(I_n,Y)}(\tau_n)$. By Theorem 14,

$$\hat{\sigma} : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$$

is an endomorphism. Since $F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}$ generates $MA_{Fix(I_n,Y)}(\tau_n)$, we have $\hat{\sigma}/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}$ is a $Fix(I_n, Y)$-full substitution with

$$\overline{\hat{\sigma}/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}} = \hat{\sigma}.$$

Define a mapping $\psi : Hyp^{Fix(I_n,Y)}(\tau_n) \to Subst_{Fix(I_n,Y)}(\tau_n)$ by

$$\psi(\sigma) = \hat{\sigma}/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}.$$

Let $\sigma_1, \sigma_2 \in Hyp^{Fix(I_n,Y)}(\tau_n)$. By Lemma 16, we then have

$$\begin{aligned}
\psi(\sigma_1 \circ_h \sigma_2) &= \widehat{(\sigma_1 \circ_h \sigma_2)}/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} \\
&= (\hat{\sigma}_1 \circ \hat{\sigma}_2)/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} \\
&= \overline{\hat{\sigma}_1/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}} \circ \hat{\sigma}_2/F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)} \\
&= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\
&= \psi(\sigma_1) \odot \psi(\sigma_2).
\end{aligned}$$

Here, ψ is a homomorphism. Clearly, ψ is an injection. Hence we have the following theorem.

Theorem 22. *$(Hyp^{Fix(I_n,Y)}(\tau_n); \circ_h, \sigma_{id})$ can be embedded into $(Subst_{Fix(I_n,Y)}(\tau_n); \odot, id_{F_{W_{\tau_n}^{Fix(I_n,Y)}(X_n)}})$.*

Using the definition of a $Fix(I_n, Y)$-full substitution, if we take $Y = \emptyset$ or $|I_n| = 1$, then $W^{Fix(I_n,Y)}_{\tau_n}(X_n) = W^F_{\tau_n}(X_n)$. According to Theorem 22, we have the following corollary.

Corollary 23 ([5], Proposition 3)**.** *The monoid* $(Hyp^F(\tau_n); \circ_h, \sigma_{id})$ *can be embedded into the monoid* $(Subst_{FC}; \odot, id_{F_{s\tau_n}})$*.*

Moreover, if $Y = I_n$ and $\alpha \in Fix(I_n, Y)$ is an identity mapping, then the following result decribing a close connection between the monoid of strongly full hypersubstitutions and the monoid of strongly full substitutions is obtained directly.

Corollary 24 ([4], Proposition 3.3)**.** *The monoid* $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$ *can be embedded into the monoid* $(Subst_{SF}; \odot, id_{F_{s\tau_n}})$*.*

4. Algebraic applications of $Fix(I_n, Y)$-full terms

Let V be a variety of algebras of type τ_n, and let IdV be the set of all identities of V. Let $Id^{Fix(I_n,Y)}V$ be the set of all identities $s \approx t$ of V such that s and t are both $Fix(I_n, Y)$-full terms of type τ_n; that is

$$Id^{Fix(I_n,Y)}V := (W^{Fix(I_n,Y)}_{\tau_n}(X_n))^2 \cap IdV.$$

It is well-known that IdV is a congruence on the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$. However, in general, this is not true for $Id^{Fix(I_n,Y)}V$.

Recall from [6] that a congruence relation θ on an algebra $\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I})$ of type τ_n is said to be *fully invariant* if for all endomorphisms $\varphi : \mathcal{A} \to \mathcal{A}$, $(a, b) \in \theta \Rightarrow (\varphi(a), \varphi(b)) \in \theta$ for all $a, b \in A$.

The following theorem shows that $Id^{Fix(I_n,Y)}V$ is a congruence relation on $MA_{Fix(I_n,Y)}(\tau_n)$.

Theorem 25. *Let* V *be a variety of type* τ_n*. Then* $Id^{Fix(I_n,Y)}V$ *is a congruence relation on* $MA_{Fix(I_n,Y)}(\tau_n)$*.*

Proof. Assume that

$$r \approx t, r_1 \approx t_1, \ldots, r_n \approx t_n \in Id^{Fix(I_n,Y)}V.$$

We will prove that

$$S^n(r, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

Firstly, we prove by induction on the complexity of the $Fix(I_n, Y)$-full term r that

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

Assume that $r = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ for some $\alpha \in Fix(I_n, Y)$. By the fact that IdV is compatible with the operations $\bar{f}_i$ of the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$ and the definition of $Fix(I_n, Y)$-full terms, we have

$$f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}) \approx f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)}) \in Id^{Fix(I_n,Y)}V.$$

That is

$$\begin{aligned} &S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), r_1, \ldots, r_n) \\ \approx\ &S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V. \end{aligned}$$

And,

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

Assume that $r = f_i(s_1, \ldots, s_n)$ such that for all $1 \leq k \leq n$,

$$S^n(s_k, r_1, \ldots, r_n) \approx S^n(s_k, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

Thus

$$\begin{aligned} &f_i(S^n(s_1, r_1, \ldots, r_n), \ldots, S^n(s_n, r_1, \ldots, r_n)) \\ \approx\ &f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)) \in Id^{Fix(I_n,Y)}V \end{aligned}$$

and

$$S^n(f_i(s_1, \ldots, s_n), r_1, \ldots, r_n) \approx S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

This means

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

So we have the claim.

Since IdV is a fully invariant congruence relation on the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$, $r \approx t \in Id^{Fix(I_n,Y)}V$ implies

$$S^n(r, t_1, \ldots, t_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V.$$

Finally, assume that $r \approx t, r_i \approx t_i \in Id^{Fix(I_n,Y)}V$ for all $i = 1, \ldots, n$. Then

$$S^n(r, r_1, \ldots, r_n) \approx S^n(t, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{Fix(I_n,Y)}V,$$

which shows that $Id^{Fix(I_n,Y)}V$ is a congruence relation on the algebra $MA_{Fix(I_n,Y)}(\tau_n)$. ■

By using the concept of $Fix(I_n, Y)$-full hypersubstitutions, a $Fix(I_n, Y)$-full closed identity and a $Fix(I_n, Y)$-full closed variety are introduced.

Definition. Let V be a variety of type τ_n.

(i) An identity $s \approx t \in Id^{Fix(I_n,Y)}V$ is called a *$Fix(I_n, Y)$-full closed identity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{Fix(I_n,Y)}V$ for all $\sigma \in Hyp^{Fix(I_n,Y)}(\tau_n)$.

(ii) A variety V is called *$Fix(I_n, Y)$-full closed* if every identity in $Id^{Fix(I_n,Y)}V$ is a $Fix(I_n, Y)$-full closed identity.

Then we have the following theorem.

Theorem 26. *Let V be a variety of type τ_n. If $Id^{Fix(I_n,Y)}V$ is a fully invariant congruence on $MA_{Fix(I_n,Y)}(\tau_n)$, then V is $Fix(I_n, Y)$-full closed.*

Proof. Assume that $Id^{Fix(I_n,Y)}V$ is a fully invariant congruence on $MA_{Fix(I_n,Y)}(\tau_n)$. Let $s \approx t \in Id^{Fix(I_n,Y)}V$ and $\sigma \in Hyp^{Fix(I_n,Y)}(\tau_n)$. By Theorem 14, $\hat{\sigma}$ is an endomorphism of $MA_{Fix(I_n,Y)}(\tau_n)$. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{Fix(I_n,Y)}V$, that is V is $Fix(I_n, Y)$-full closed. ■

One of the concrete application of Theorem 26 can be shown by the following example.

Example 27. Let CA be a variety of commutative algebras type $\tau_2 = (2)$. This means that $CA = Mod\{f(x_1, x_2) \approx f(x_2, x_1)\}$. We easily see that $Id^{Fix(I_2,Y)}CA$ is a fully invariant congruence on $MA_{Fix(I_2,Y)}(\tau_2)$. By Theorem 26, CA is a $Fix(I_2, Y)$-full closed variety.

For a variety V of type τ_n, $Id^{Fix(I_n,Y)}V$ is a congruence on $MA_{Fix(I_n,Y)}(\tau_n)$ by Theorem 25. We then form the quotient algebra

$$MA_{Fix(I_n,Y)}(V) := MA_{Fix(I_n,Y)}(\tau_n)/Id^{Fix(I_n,Y)}V.$$

The quotient algebra obtained belongs to V_{Menger}. Note that we have a natural homomorphism

$$nat_{Id^{Fix(I_n,Y)}V} : MA_{Fix(I_n,Y)}(\tau_n) \to MA_{Fix(I_n,Y)}(V)$$

such that

$$nat_{Id^{Fix(I_n,Y)}V}(t) = [t]_{Id^{Fix(I_n,Y)}V}.$$

Finally, we prove the following theorem.

Theorem 28. *Let V be a variety of type τ_n. If $s \approx t \in Id^{Fix(I_n,Y)}V$, then $s \approx t$ is a $Fix(I_n, Y)$-full closed identity of V.*

Proof. Assume that $s \approx t \in Id^{Fix(I_n,Y)}V$ and $\sigma \in Hyp^{Fix(I_n,Y)}(\tau_n)$. By Theorem 14, we have that $\hat{\sigma} : W_{\tau_n}^{Fix(I_n,Y)}(X_n) \to W_{\tau_n}^{Fix(I_n,Y)}(X_n)$ is an endomorphism on the algebra $MA_{Fix(I_n,Y)}(\tau_n)$. Thus

$$nat_{Id^{Fix(I_n,Y)}V} \circ \hat{\sigma} : MA_{Fix(I_n,Y)}(\tau_n) \to MA_{Fix(I_n,Y)}(V)$$

is a homomorphism. By the assumption,

$$nat_{Id^{Fix(I_n,Y)}V} \circ \hat{\sigma}(s) = nat_{Id^{Fix(I_n,Y)}V} \circ \hat{\sigma}(t).$$

That is

$$nat_{Id^{Fix(I_n,Y)}V}(\hat{\sigma}[s]) = nat_{Id^{Fix(I_n,Y)}V}(\hat{\sigma}[t]).$$

Thus

$$[\hat{\sigma}[s]]_{Id^{Fix(I_n,Y)}V} = [\hat{\sigma}[t]]_{Id^{Fix(I_n,Y)}V}.$$

That is

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{Fix(I_n,Y)}V.$$

Hence $s \approx t$ is a $Fix(I_n, Y)$-full closed identity of V. ■

Example 29. Let CA be a variety of commutative algebras of type $\tau_2 = (2)$. We see that $f(x_1, x_2) \approx f(x_2, x_1)$ is an identity in $MA_{Fix(I_2,Y)}(CA)$ where $f(x_1, x_2)$ and $f(x_2, x_1)$ are binary $Fix(I_2, Y)$-full terms of type $\tau_2 = (2)$. By Theorem 28, we obtain that $f(x_1, x_2) \approx f(x_2, x_1)$ is a $Fix(I_2, Y)$-full closed identity of a variety of commutative algebras of type $\tau_2 = (2)$.

5. CONCLUSIONS

In this paper, we define n-ary $Fix(I_n, Y)$-full terms of type τ_n by using the concept of transformations with fixed set and n-ary full terms of type τ_n. The relationship between $Fix(I_n, Y)$-full terms, strongly full terms, permutational full terms and full terms of type τ_n are given. After that, the superposition operation for n-ary $Fix(\mathbb{N}, Y)$-full terms is established. It turns out that the set of all such terms together with the superposition forms a Menger algebra. In Section 3, the concept of mapping from the collection of all operation symbols to the set of all n-ary $Fix(I_n, Y)$-full terms of type τ_n is studied. This leads us to construct three algebraic structures. Finally, we studied some properties of identities of $Fix(I_n, Y)$-full terms. Our results play essentially significant roles for the study of classical algebras in various directions, for examples, the algebraic construction of new algebras and the classification of algebras via $Fix(I_n, Y)$-full closed identity. The extending from a $Fix(I_n, Y)$-full terms to a generalized $Fix(I_n, Y)$-full terms (see the papers [15, 19] for this research direction) and the characterization of some special elements in the monoid $(Hyp^{Fix(I_n,Y)}(\tau_n); \circ_h, \sigma_{id})$ based on theory of semigroups still remain open problem.

Acknowledgment

This research was supported by Rajamangala University of Technology Rattanakosin, Thailand. The authors are deeply grateful to the referee for comments and valuable suggestions.

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