

## ON $B^*$ -PURE ORDERED SEMIGROUP

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### Abstract

We introduce the concept of  $B^*$ -pure ordered semigroups, and give some properties of  $B^*$ -pure ordered semigroups.

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### 1. INTRODUCTION

A bi-ideal  $A$  of a semigroup  $S$  is said to be  $B$ -pure if  $A \cap xS = xA$  and  $A \cap Sx = Ax$  for all  $x \in S$ . A semigroup  $S$  is said to be  $B^*$ -pure if every bi-ideal of  $S$  is  $B$ -pure. The concept  $B^*$ -pure semigroups was studied by Kuroki [3]. In this paper, the concept of  $B^*$ -pure ordered semigroups is introduced. We shall give some properties of  $B^*$ -pure ordered semigroups, and characterize  $B^*$ -pure Archimedean ordered semigroups. We prove that any  $B^*$ -pure ordered semigroup is a semilattices of Archimedean semigroups. Let us recall some certain definitions and results used throughout the paper. A semigroup  $(S, \cdot)$  together with a partial order  $\leq$  that is *compatible* with the semigroup operation, meaning that, for any  $x, y, z$  in  $S$ ,

$$x \leq y \text{ implies } zx \leq zy \text{ and } xz \leq yz$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*) (see [2]). Under the trivial relation,  $x \leq y$  if and only if  $x = y$ , it is observed that every semigroup is an ordered semigroup. Let  $(S, \cdot, \leq)$  be an ordered semigroup. For  $A, B$  nonempty subsets of  $S$ , we write  $AB$  for the set of all elements  $xy$  in  $S$  where  $x \in A$  and  $y \in B$ , and write  $[A]$  for the set of all elements  $x$  in  $S$  such that  $x \leq a$  for some  $a$  in  $A$ , i.e.,

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

In particular, we write  $Ax$  for  $A\{x\}$ , and  $xA$  for  $\{x\}A$ . It was shown in [10] that the followings hold:

- (1)  $A \subseteq [A]$  and  $(([A]) = [A]$ ;
- (2)  $A \subseteq B \Rightarrow [A] \subseteq [B]$ ;
- (3)  $(([A])([B]) = ([A]B) = (A[B]) = (AB]$ ;
- (4)  $[A](B) \subseteq (AB]$ ;
- (5)  $[A]B \subseteq (AB]$  and  $A(B) \subseteq (AB]$ ;
- (6)  $(A \cup B) = [A] \cup [B]$ .

The concepts of left, right and two-sided ideals of an ordered semigroup can be found in [2]. Let  $(S, \cdot, \leq)$  be an ordered semigroup. A nonempty subset  $A$  of  $S$  is called a *left* (resp., *right*) *ideal* of  $S$  if it satisfies the following conditions:

- (i)  $SA \subseteq A$  (resp.,  $AS \subseteq A$ );
- (ii)  $A = [A]$ , that is, for any  $x$  in  $A$  and  $y$  in  $S$ ,  $y \leq x$  implies  $y \in A$ .

If  $A$  is both a left and a right ideal of  $S$ , then  $A$  is called a *two-sided ideal*, or simply an *ideal* of  $S$ . It is known that the union or intersection of two ideals of  $S$  is an ideal of  $S$ .

Let  $(S, \cdot, \leq)$  be an ordered semigroup. A left ideal  $A$  of  $S$  is said to be *proper* if  $A \subset S$ . The symbol  $\subset$  stands for proper subset of sets. A proper right and two-sided ideals are defined similarly.  $S$  is said to be *left* (resp., *right*) *simple* if  $S$  does not contain proper left (resp., right) ideals. If  $S$  does not contain proper *ideals* then we call  $S$  *simple*. A proper ideal  $A$  of  $S$  is said to be *maximal* if for any ideal  $B$  of  $S$ , if  $A \subset B \subseteq S$ , then  $B = S$ .

For any element  $a$  of an ordered semigroup  $(S, \cdot, \leq)$ , the *principal ideal generated* by  $a$  is of the form  $I(a) = (a \cup Sa \cup aS \cup SaS]$ .

A nonempty subset  $B$  is called a *bi-ideal* of  $S$  if

- (i)  $BSB \subseteq B$ ;
- (ii) for any  $x$  in  $B$  and  $y$  in  $S$ ,  $y \leq x$  implies  $y \in B$  (see [5]).

For any element  $a$  of an ordered semigroup  $(S, \cdot, \leq)$  the *bi-ideal generated* by  $a$  is of the form  $B(a) = (\{a\} \cup aSa]$ .

An equivalence relation  $\sigma$  on  $S$  is called *congruence* if  $(a, b) \in \sigma$  implies  $(ac, bc) \in \sigma$  and  $(ca, cb) \in \sigma$  for every  $c \in S$ . A congruence  $\sigma$  on  $S$  is called *semilattice congruence* if  $(a^2, a) \in \sigma$  and  $(ab, ba) \in \sigma$  for every  $a, b \in S$ . A semilattice congruence  $\sigma$  on  $S$  is called *complete* if  $a \leq b$  implies  $(a, ab) \in \sigma$ . An ordered semigroup  $S$  is called a *semilattice of Archimedean semigroups* (resp., *complete semilattice of Archimedean semigroups*) if there exists a semilattice congruence

(resp., complete semilattice congruence)  $\sigma$  on  $S$  such that the  $\sigma$ -class  $(x)_\sigma$  of  $S$  containing  $x$  is a Archimedean subsemigroup of  $S$  for every  $x \in S$ .

A subsemigroup  $F$  is called a *filter* of  $S$  if

- (i)  $a, b \in S$ ,  $ab \in F$  implies  $a \in F$  and  $b \in F$ ;
- (ii) if  $a \in F$  and  $b$  in  $S$ ,  $a \leq b$ , then  $b \in F$  (see [6]).

For an element  $x$  of  $S$ , we denote by  $N(x)$  the filter of  $S$  generated by  $x$  and  $\mathcal{N}$  the equivalence relation on  $S$  defined by  $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$ . The relation  $\mathcal{N}$  is the least complete semilattice congruence on  $S$ . An element  $e$  of an ordered semigroup  $(S, \cdot, \leq)$  is called an *ordered idempotent* if  $e \leq e^2$ . We call an ordered semigroup  $S$  *idempotent ordered semigroup* if every element of  $S$  is an ordered idempotent (see [1]). The set of all ordered idempotent of an ordered semigroup  $S$  denoted by  $E(S)$  and the set of all positive integers denoted by  $N$ .

An ordered semigroup  $(S, \cdot, \leq)$  is called *Archimedean* if for any  $a, b$  in  $S$  there exists a positive integer  $n$  such that  $a^n \in (SbS]$  (see [8]). An ordered semigroup  $S$  is called *regular* if for every  $a \in S$ , there exists  $x \in S$  such that  $a \leq axa$ . Equivalent definitions are as follows: (1)  $A \subseteq (ASA]$  for any  $A \subseteq S$  or (2)  $a \in (aSa]$  for any  $a \in S$  (see [7]). An ordered semigroup  $S$  is said to be *normal* if  $(xS] = (Sx]$  for all  $x \in S$ . An ordered semigroup  $S$  is said to be *weakly commutative* if for any  $a, b \in S$ , then there exists positive integer  $n$  such that  $(ab)^n \in (bSa]$  (see [4]). An ideal  $A$  of an ordered semigroup  $S$  is called *globally idempotent* if  $A = (A^2]$  (see [9]). An ideal  $A$  of an ordered semigroup  $S$  is called *complete* if  $A = (AS] = (SA]$  (see [9]).

**Definition.** Let  $(S, \cdot, \leq)$  be an ordered semigroup. A bi-ideal  $A$  of  $S$  is said to be *B-pure* if  $A \cap (xS] = (xA]$  and  $A \cap (Sx] = (Ax]$  for all  $x \in S$ . An ordered semigroup  $S$  is said to be  *$B^*$ -pure* if every bi-ideal of  $S$  is *B-pure*.

**Example 1.** Let  $S = \{a, b\}$ ,  $xy = b$  for all  $x, y \in S$ ,  $\leq = \{(a, a), (b, b), (a, b)\}$ . It is clear that  $S$  is an ordered semigroup. We show that  $S$  is  $B^*$ -pure. We determine all bi-ideals in  $S$ . We have two candidates:  $\{a\}$  and  $S$ . Of course,  $S$  is a bi-ideal, but  $\{a\}$  is not a bi-ideal, because  $\{a\}S\{a\} = \{b\}$ . So there exists only one bi-ideal in  $S$ , namely  $S$ . Bi-ideal  $S$  is *B-pure*, because  $S \cap (xS] = (Sx]$  and  $S \cap (xS] = (xS]$  for all  $x \in S$ .

## 2. MAIN RESULTS

First, we have the following lemma.

**Lemma 2.** *Any normal ordered semigroups are weakly commutative.*

**Proof.** Let  $S$  be a normal ordered semigroup and  $a, b \in S$ . We have

$$(ab)^3 = ababab \in (SbSaS] \subseteq ((Sb]S(aS]) \subseteq ((bS]S(Sa]) \subseteq (bSa].$$

Hence  $S$  is weakly commutative. ■

**Lemma 3.** *Let  $S$  be a  $B^*$ -pure ordered semigroup. Then  $S$  has the following properties:*

- (1)  $(aS] = (a^2S]$  and  $(Sa] = (Sa^2]$  for all  $a \in S$ ;
- (2)  $S$  is normal;
- (3)  $S$  is weakly commutative;
- (4) for each  $x \in S$ ,  $N(x) = \{y \in S \mid x^n \in (ySy] \text{ for some } n \in \mathbb{N}\}$ ;
- (5)  $a^2$  is regular for all  $a \in S$ .

**Proof.** (1) Let  $a \in S$ . Since  $S$  is  $B^*$ -pure, the bi-ideal  $(aS]$  is  $B$ -pure. Thus  $(aS] = (aS] \cap (aS] = (a(aS]) \subseteq (a^2S]$ . The converse is obvious. Hence  $(aS] = (a^2S]$ . Similarly, we have  $(Sa] = (Sa^2]$ .

(2) Let  $a \in S$ . By (1), we have

$$(aS] = (a^2S] \subseteq (SaS] \subseteq ((Sa]S] = (Sa] \cap (SS] \subseteq (Sa].$$

Similarly, we have  $(Sa] \subseteq (aS]$ . It follows that  $(aS] = (Sa]$ . Hence  $S$  is normal.

(3) This follows by (2) and Lemma 2.

(4) This follows by (3) and lemma in [4].

(5) Let  $a$  be any element of  $S$ . By (1) and (2) we have

$$a^2 \in (aS] = (a^2S] = (a^4S] \subseteq (a^2(a^2S]) = (a^2(Sa^2]) \subseteq (a^2Sa^2].$$

Thus  $a^2$  is regular. ■

The following Corollary 4 can be obtained from Lemma 2 and theorem in [4].

**Corollary 4.** *Any normal ordered semigroups are semilattices of Archimedean semigroups.*

The following Theorem 5 can be obtained from Lemma 3 and theorem in [4].

**Theorem 5.** *Any  $B^*$ -pure ordered semigroups are semilattices of Archimedean semigroups.*

**Theorem 6.** *Let  $(S, \cdot, \leq)$  be an ordered semigroup such that  $(aS] = (a^2S]$  and  $(Sa] = (Sa^2]$  for all  $a$  in  $S$ . The following statements are equivalent:*

- (1)  $(Se] = (eS]$  for all  $e$  in  $E(S)$ ;
- (2)  $S$  is normal;
- (3)  $S$  is weakly commutative;

(4) for each  $x \in S$ ,  $N(x) = \{y \in S \mid x^n \in (ySy] \text{ for some } n \in \mathbb{N}\}$ .

**Proof.** By Lemma 2, (2) implies (3). We have that (3) and (4) are equivalent by lemma in [4].

(1) $\Rightarrow$ (2). Let  $a \in S$ . We have  $a^2 \in (aS] = (a^2S] = (a^4S]$  and  $a^2 \in (Sa] = (Sa^2] = (Sa^4]$ . Thus  $a^2 \leq a^4x$  and  $a^2 \leq ya^4$  for some  $x, y$  in  $S$ . This implies that  $a^4 \leq a^4xya^4$ . Hence  $xya^4 \in E(S)$ . Let  $b \in (aS] = (a^2S] = (a^4S]$ . Then  $b \leq a^4z$  for some  $z$  in  $S$ . We have

$$\begin{aligned} b \leq a^4z \leq a^4xya^4z &\in (a^4xya^4S] \subseteq (a^4(xya^4S]) \\ &= (a^4(Sxya^4]) \\ &\subseteq (a^4Sxya^4] \\ &\subseteq (Sa^4] \\ &\subseteq (Sa]. \end{aligned}$$

Similarly, we have  $(Sa] \subseteq (aS]$ . Hence  $S$  is normal.

(3) $\Rightarrow$ (1). Let  $e \in E(S)$  and  $x \in (eS]$ . Then  $x \leq ea$  for some  $a \in S$ . Since  $S$  is weakly commutative, then there exists a positive integer  $n$  such that  $(ea)^n \in (aSe]$ . It follows that

$$x \leq ea \leq eea \in (Sea] \subseteq (S(ea)^n] \subseteq (S(aSe]) \subseteq (SaSe] \subseteq (SSSe] \subseteq (Se].$$

Similarly, we have  $(Se] \subseteq (eS]$ . Hence  $(Se] = (eS]$ . This complete the proof. ■

Now we have shown that if an ordered semigroup  $S$  is  $B^*$ -pure, then the converse of Lemma 2 holds.

The following Theorem 7 can be obtained from Lemma 3 and Theorem 6.

**Theorem 7.** For a  $B^*$ -pure ordered semigroup  $S$ . The following statements are equivalent:

- (1)  $(Se] = (eS]$  for all  $e$  in  $E(S)$ ;
- (2)  $S$  is normal;
- (3)  $S$  is weakly commutative;
- (4) for each  $x \in S$ ,  $N(x) = \{y \in S \mid x^n \in (ySy] \text{ for some } n \in \mathbb{N}\}$ .

**Theorem 8.** For a  $B^*$ -pure ordered semigroup  $S$ . The following statements are equivalent:

- (1) every ideal of  $S$  is globally idempotent;
- (2) every ideal of  $S$  is complete.

**Proof.** By Theorem 2.3 in [9], (1) implies (2).

(2) $\Rightarrow$ (1). Let  $A$  be any ideal of  $S$  and  $b \in A$ . Since  $A$  is complete,  $A = (AS]$ . We have  $b \in (aS]$  for some  $a \in A$ . Since  $S$  is  $B^*$ -pure and every ideal is a bi-ideal,  $A \cap (aS] = (aA]$ . We have

$$b \in A \cap (aS] = (aA] \subseteq (A^2].$$

Thus  $A \subseteq (A^2]$ . As is easily seen,  $(A^2] \subseteq A$ . Hence  $A = (A^2]$ . ■

**Theorem 9.** *For an idempotent ordered semigroup  $S$ . The following statements are equivalent:*

- (1)  $S$  is  $B^*$ -pure;
- (2)  $S$  is normal and  $(Sa] = (Sa^2]$  for all  $a \in S$ .

**Proof.** By Lemma 3, (1) implies (2).

(2)  $\Rightarrow$  (1). Let  $A$  be any bi-ideal of  $S$ ,  $x \in S$ . Let  $a \in A \cap (Sx] = A \cap (Sx^2]$ . Then  $a \leq yx^2$  for some  $y \in S$ . Since  $ay \in (aS] = (Sa] = (Sa^2]$ ,  $ay \leq za^2$  for some  $z \in S$ . We have

$$\begin{aligned} a \leq a^2 \leq ayx^2 \leq za^2x^2 &\in (SaaSx] \\ &\subseteq ((Sa](aS)x] \\ &= ((aS](Sa)x] \\ &\subseteq (aSSax] \\ &\subseteq (ASSAx] \\ &\subseteq (Ax]. \end{aligned}$$

Thus  $A \cap (Sx] \subseteq (Ax]$ . Let  $b \in (Ax]$ . Then  $b \leq ax$  for some  $a$  in  $A$ . We have

$$b \leq ax \in (aS] = (Sa] = (Sa^2] \subseteq (aSa] \subseteq (ASA] \subseteq A,$$

and so  $(Ax] \subseteq A$ . Since  $(Ax] \subseteq (Sx]$ , then  $(Ax] \subseteq A \cap (Sx]$ . Thus  $A \cap (Sx] = (Ax]$ . Similarly, we have  $A \cap (xS] = (xA]$ . Hence  $A$  is  $B$ -pure. ■

**Theorem 10.** *Any normal regular ordered semigroups are  $B^*$ -pure.*

**Proof.** Let  $S$  be a normal regular ordered semigroup,  $A$  be a bi-ideal of  $S$  and  $x \in S$ . Let  $b \in (xA]$ . Then  $b \leq xa$  for some  $a$  in  $A$ . Since  $S$  is regular, then  $a \leq aya$  for some  $y$  in  $S$ . We have

$$\begin{aligned} b \leq xa \leq xaya &\in (SaSa] \subseteq ((Sa]Sa] = ((aS]Sa] \\ &\subseteq (aSSa] \\ &\subseteq (aSa] \\ &\subseteq (ASA] \subseteq A. \end{aligned}$$

Thus  $(xA] \subseteq A$ . Since  $(xA] \subseteq (xS]$ , then  $(xA] \subseteq A \cap (xS]$ . Let  $a \in A \cap (xS]$ . Then  $a \leq xb$  for some  $b$  in  $S$ . Since  $S$  is regular, then  $a \leq aya$  for some  $y$  in  $S$ . We have

$$\begin{aligned}
a \leq aya \leq ayaya \leq xbyaya = x(by)aya &\in (xSaya] \\
&\subseteq (x(Sa]ya] \\
&\subseteq (x(aS]SA] \\
&\subseteq (xaSSA] \\
&\subseteq (xASSA] \\
&\subseteq (xA].
\end{aligned}$$

Thus  $A \cap (xS] = (xA]$ . Similarly, we have  $A \cap (Sx] = (Ax]$ . Hence  $A$  is a  $B$ -pure.  $\blacksquare$

The following Corollary 11 can be obtained from Lemma 3 and Theorem 10.

**Corollary 11.** *For a regular ordered semigroup  $S$ . The following statements are equivalent:*

- (1)  $S$  is  $B^*$ -pure;
- (2)  $S$  is normal.

**Theorem 12.** *For a  $B^*$ -pure ordered semigroup  $S$ . The following statements are equivalent:*

- (1)  $S$  is Archimedean;
- (2)  $(SaS] = (SbS]$  for all  $a, b \in S$ ;
- (3)  $(aS] = (bS]$  for all  $a, b \in S$ ;
- (4)  $(aSa] = (bSb]$  for all  $a, b \in S$ ;
- (5) for any  $e, f \in E(S)$ ,  $(e, f) \in \mathcal{N}$ ;
- (6) every bi-ideal of  $S$  is Archimedean.

**Proof.** It is clear that (6) implies (1).

(1) $\Rightarrow$ (2). Let  $a, b \in S$ . Since  $S$  is Archimedean, then there exists positive integer  $n$  such that  $a^n \in (SbS]$ . By Lemma 3, we have

$$(SaS] \subseteq (Sa^nS] \subseteq (S(SbS]S] \subseteq (SSbSS] \subseteq (SbS].$$

Similarly, we have  $(SbS] \subseteq (SaS]$ . Hence  $(SaS] = (SbS]$ . It follows from Lemma 3 (1) and (3) that (2) implies (3) and (3) implies (4).

(4) $\Rightarrow$ (5). Let  $e, f \in E(S)$ . Then  $(eSe] = (fSf]$ . This implies that  $N(e) = N(f)$ . Hence  $(e, f) \in \mathcal{N}$ .

(5) $\Rightarrow$ (6). Let  $A$  be a bi-ideal of  $S$  and  $a, b \in A$ . Since  $S$  is  $B^*$ -pure,  $a^2$  and  $b^2$  are regular by Lemma 3. Then  $a^2 \leq a^2xa^2$  and  $b^2 \leq b^2yb^2$  for some  $x, y \in S$ . This implies that  $a^2x, b^2y \in E(S)$ . We have  $b^2y \in N(a^2x)$ . Then  $(a^2x)^n \in (b^2ySb^2y]$  for some positive integer  $n$ . Thus  $(a^2x)^n \leq b^2yzb^2y$  for some  $z \in S$ . We have

$$\begin{aligned}
a^3 &\leq aa^2xa^2 \leq aa^2xa^2xa^2 = a(a^2x)a^2xa^2 \\
&\leq a(a^2x)^na^2 \\
&\leq a(b^2yzb^2y)a^2 \\
&= ab(b(yzb^2ya)a) \\
&\in (Ab(ASA)) \\
&\subseteq (AbA].
\end{aligned}$$

Hence  $A$  is Archimedean. This completes the proof of the theorem. ■

**Theorem 13.** *Any  $B^*$ -pure Archimedean regular ordered semigroup  $S$  does not contain proper bi-ideals.*

**Proof.** Let  $A$  be any bi-ideal of  $S$ . Let  $a \in A$  and  $b \in S$ . Since  $S$  is Archimedean, then there exists positive integer  $n$  such that  $b^n \in (SaS]$ . Since  $S$  is  $B^*$ -pure,  $(aSa]$  is  $B$ -pure. Then by the regularity of  $S$  and Lemma 3, we have

$$\begin{aligned}
b \in (bSb] &\subseteq (b^nSb^n] \subseteq ((SaS]S(SaS)] \\
&\subseteq (SaSSSaS] \\
&\subseteq (SaSSS(aS)] \\
&\subseteq (SaSSS(Sa)] \\
&\subseteq (SaSSSSa] \\
&\subseteq (S(aSa)] \\
&= (SS] \cap (aSa] \\
&\subseteq (ASA] \\
&\subseteq A.
\end{aligned}$$

Thus  $S \subseteq A$ . Hence  $S = A$ . ■

The following Theorem 14 can be obtained from Theorem 13 .

**Theorem 14.** *Any  $B^*$ -pure Archimedean regular ordered semigroups are left and right simple.*

**Theorem 15.** *For a  $B^*$ -pure Archimedean regular ordered semigroup  $S$ . The following statements are equivalent:*

- (1)  $S$  is regular;
- (2)  $S$  does not contain proper bi-ideals;
- (3)  $S$  are left and right simple.

**Proof.** By Theorem 13, (1) implies (2). It is clear that (2) implies (3).

(3) $\Rightarrow$ (1). Let  $a \in S$ . As is easily seen,  $(Sa]$  is a left ideal and  $(aS]$  is a right ideal. Since  $S$  are left and right simple, then  $S = (Sa]$  and  $S = (aS]$ . We have  $a \in (aS] = (a(Sa)] \subseteq (aSa]$ . This completes the proof of the theorem. ■



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