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ON B*-PURE ORDERED SEMIGROUP

PISAN SUMMAPRAB

Department of Mathematics Rajamangala University of Technology Isan Khon Kaen Campus, Khon Kaen 40000, Thailand **e-mail:** pisansu9999@gmail.com

Abstract

We introduce the concept of B^* -pure ordered semigroups, and give some properties of B^* -pure ordered semigroups. **Keywords:** semigroup, ordered semigroup, B^* -pure, normal, weakly commutative, Archimedean, semilattice, bi-ideal. **2020** Mathematics Subject Classification: 06F05.

1. INTRODUCTION

A bi-ideal A of a semigroup S is said to be B-pure if $A \cap xS = xA$ and $A \cap Sx = Ax$ for all $x \in S$. A semigroup S is said to be B^* -pure if every bi-ideal of S is B-pure. The concept B^* -pure semigroups was studied by Kuroki [3]. In this paper, the concept of B^* -pure ordered semigroups is introduced. We shall give some properties of B^* -pure ordered semigroups, and characterize B^* -pure Archimedean ordered semigroups. We prove that any B^* -pure ordered semigroup is a semilattices of Archimedean semigroups. Let us recall some certain definitions and results used throughout the paper. A semigroup (S, \cdot) together with a partial order \leq that is compatible with the semigroup operation, meaning that, for any x, y, z in S,

 $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*)(see [2]). Under the trivial relation, $x \leq y$ if and only if x = y, it is observed that every semigroup is an ordered semigroup. Let (S, \cdot, \leq) be an ordered semigroup. For A, B nonempty subsets of S, we write AB for the set of all elements xy in S where $x \in A$ and $y \in B$, and write (A] for the set of all elements x in S such that $x \leq a$ for some a in A, i.e.,

$$(A] = \{ x \in S \mid x \le a \text{ for some } a \in A \}.$$

In particular, we write Ax for $A\{x\}$, and xA for $\{x\}A$. It was shown in [10] that the followings hold:

- (1) $A \subseteq (A]$ and ((A]] = (A];
- (2) $A \subseteq B \Rightarrow (A] \subseteq (B];$
- (3) ((A](B]] = ((A]B] = (A(B)] = (AB];
- $(4) \ (A](B] \subseteq (AB];$
- (5) $(A|B \subseteq (AB) \text{ and } A(B) \subseteq (AB);$
- (6) $(A \cup B] = (A] \cup (B].$

The concepts of left, right and two-sided ideals of an ordered semigroup can be found in [2]. Let (S, \cdot, \leq) be an ordered semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of S if it satisfies the following conditions:

- (i) $SA \subseteq A$ (resp., $AS \subseteq A$);
- (ii) A = (A], that is, for any x in A and y in S, $y \le x$ implies $y \in A$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal*, or simply an *ideal* of S. It is known that the union or intersection of two ideals of S is an ideal of S.

Let (S, \cdot, \leq) be an ordered semigroup. A left ideal A of S is said to be *proper* if $A \subset S$. The symbol \subset stands for proper subset of sets. A proper right and two-sided ideals are defined similarly. S is said to be *left (resp., right) simple* if S does not contain proper left (resp., right) ideals. If S does not contain proper *ideals* then we call S *simple*. A proper ideal A of S is said to be *maximal* if for any ideal B of S, if $A \subset B \subseteq S$, then B = S.

For any element a of an ordered semigroup (S, \cdot, \leq) , the principal ideal generated by a is of the form $I(a) = (a \cup Sa \cup aS \cup SaS]$.

A nonempty subset B is called a *bi-ideal* of S if

- (i) $BSB \subseteq B$;
- (ii) for any x in B and y in S, $y \le x$ implies $y \in B$ (see [5]).

For any element a of an ordered semigroup (S, \cdot, \leq) the *bi-ideal generated* by a is of the form $B(a) = (\{a\} \cup aSa\}$.

An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semi-lattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$. An ordered semigroup S is called a *semilattice of Archimedean semigroups* (resp., complete semilattice congruence) if there exists a semilattice congruence

(resp., complete semilattice congruence) σ on S such that the σ -class $(x)_{\sigma}$ of S containing x is a Archimedean subsemigroup of S for every $x \in S$.

- A subsemigroup F is called a *filter* of S if
- (i) $a, b \in S, ab \in F$ implies $a \in F$ and $b \in F$;
- (ii) if $a \in F$ and b in S, $a \leq b$, then $b \in F$ (see [6]).

For an element x of S, we denote by N(x) the filter of S generated by x and \mathcal{N} the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$. The relation \mathcal{N} is the least complete semilattice congruence on S. An element e of an ordered semigroup (S, \cdot, \leq) is called an ordered idempotent if $e \leq e^2$. We call an ordered semigroup S idempotent ordered semigroup if every element of S is an ordered idempotent (see [1]). The set of all ordered idempotent of an ordered semigroup S denoted by E(S) and the set of all positive integers denoted by N.

An ordered semigroup (S, \cdot, \leq) is called *Archimedean* if for any a, b in S there exists a positive integer n such that $a^n \in (SbS]$ (see [8]). An ordered semigroup S is called *regular* if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for any $A \subseteq S$ or (2) $a \in (aSa]$ for any $a \in S$ (see [7]). An ordered semigroup S is said to be *normal* if (xS] = (Sx] for all $x \in S$. An ordered semigroup S is said to be *weakly commutative* if for any $a, b \in S$, then there exists positive integer n such that $(ab)^n \in (bSa]$ (see [4]). An ideal A of an ordered semigroup S is called *globally idempotent* if $A = (A^2]$ (see [9]). An ideal A of an ordered semigroup S is called *complete* if A = (AS] = (SA] (see [9]).

Definition. Let (S, \cdot, \leq) be an ordered semigroup. A bi-ideal A of S is said to be B-pure if $A \cap (xS] = (xA]$ and $A \cap (Sx] = (Ax]$ for all $x \in S$. An ordered semigroup S is said to be B^* -pure if every bi-ideal of S is B-pure.

Example 1. Let $S = \{a, b\}$, xy = b for all $x, y \in S$, $\leq = \{(a, a), (b, b), (a, b)\}$. It is clear that S is an ordered semigroup. We show that S is B^* -pure. We determine all bi-ideals in S. We have two candidates: $\{a\}$ and S. Of course, S is a bi-ideal, but $\{a\}$ is not a bi-ideal, because $\{a\}S\{a\} = \{b\}$. So there exists only one bi-ideal in S, namely S. Bi-deal S is B-pure, because $S \cap (Sx] = (Sx]$ and $S \cap (xS] = (xS]$ for all $x \in S$.

2. Main results

First, we have the following lemma.

Lemma 2. Any normal ordered semigroups are weakly commutative.

Proof. Let S be a normal ordered semigroup and $a, b \in S$. We have

$$(ab)^3 = ababab \in (SbSaS] \subseteq ((Sb]S(aS]] \subseteq ((bS]S(Sa]] \subseteq (bSa].$$

Hence S is weakly commutative.

Lemma 3. Let S be a B^* -pure ordered semigroup. Then S has the following properties:

- (1) $(aS] = (a^2S]$ and $(Sa] = (Sa^2]$ for all $a \in S$;
- (2) S is normal;
- (3) S is weakly commutative;
- (4) for each $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy] \text{ for some } n \in N\};$
- (5) a^2 is regular for all $a \in S$.

Proof. (1) Let $a \in S$. Since S is B^* -pure, the bi-ideal (aS] is B-pure. Thus $(aS] = (aS] \cap (aS] = (a(aS)] \subseteq (a^2S]$. The converse is obvious. Hence $(aS] = (a^2S]$. Similarly, we have $(Sa] = (Sa^2]$.

(2) Let $a \in S$. By (1), we have

$$(aS] = (a^2S] \subseteq (SaS] \subseteq ((Sa]S] = (Sa] \cap (SS] \subseteq (Sa].$$

Similarly, we have $(Sa] \subseteq (aS]$. It follows that (aS] = (Sa]. Hence S is normal.

- (3) This follows by (2) and Lemma 2.
- (4) This follows by (3) and lemma in [4].
- (5) Let a be any element of S. By (1) and (2) we have

$$a^2 \in (aS] = (a^2S] = (a^4S] \subseteq (a^2(a^2S)] = (a^2(Sa^2)] \subseteq (a^2Sa^2).$$

Thus a^2 is regular.

The following Corollary 4 can be obtained from Lemma 2 and theorem in [4].

Corollary 4. Any normal ordered semigroups are semilattices of Archimedean semigroups.

The following Theorem 5 can be obtained from Lemma 3 and theorem in [4].

Theorem 5. Any B^* -pure ordered semigroups are semilattices of Archimedean semigroups.

Theorem 6. Let (S, \cdot, \leq) be an ordered semigroup such that $(aS] = (a^2S]$ and $(Sa] = (Sa^2]$ for all a in S. The following statements are equivalent:

- (1) (Se] = (eS] for all e in E(S);
- (2) S is normal;
- (3) S is weakly commutative;

(4) for each
$$x \in S$$
, $N(x) = \{y \in S \mid x^n \in (ySy) \text{ for some } n \in N\}$.

Proof. By Lemma 2, (2) implies (3). We have that (3) and (4) are equivalent by lemma in [4].

 $(1)\Rightarrow(2)$. Let $a \in S$. We have $a^2 \in (aS] = (a^2S] = (a^4S]$ and $a^2 \in (Sa] = (Sa^2] = (Sa^4]$. Thus $a^2 \leq a^4x$ and $a^2 \leq ya^4$ for some x, y in S. This implies that $a^4 \leq a^4xya^4$. Hence $xya^4 \in E(S)$. Let $b \in (aS] = (a^2S] = (a^4S]$. Then $b \leq a^4z$ for some z in S. We have

$$b \le a^4 z \le a^4 xya^4 z \in (a^4 xya^4 S] \subseteq (a^4 (xya^4 S]]$$

= $(a^4 (Sxya^4)]$
 $\subseteq (a^4 Sxya^4]$
 $\subseteq (Sa^4]$
 $\subset (Sa].$

Similarly, we have $(Sa] \subseteq (aS]$. Hence S is normal.

 $(3) \Rightarrow (1)$. Let $e \in E(S)$ and $x \in (eS]$. Then $x \leq ea$ for some $a \in S$. Since S is weakly commutative, then there exists a positive integer n such that $(ea)^n \in (aSe]$. It follows that

$$x \le ea \le eea \in (Sea] \subseteq (S(ea)^n] \subseteq (S(aSe]] \subseteq (SaSe] \subseteq (SSSe] \subseteq (Se].$$

Similarly, we have $(Se] \subseteq (eS]$. Hence (Se] = (eS]. This complete the proof.

Now we have shown that if an ordered semigroup S is B^* -pure, then the converse of Lemma 2 holds.

The following Theorem 7 can be obtained from Lemma 3 and Theorem 6.

Theorem 7. For a B^* -pure ordered semigroup S. The following statements are equivalent:

- (1) (Se] = (eS] for all e in E(S);
- (2) S is normal;
- (3) S is weakly commutative;
- (4) for each $x \in S$, $N(x) = \{y \in S \mid x^n \in (ySy) \text{ for some } n \in N\}$.

Theorem 8. For a B^* -pure ordered semigroup S. The following statements are equivalent:

- (1) every ideal of S is globally idempotent;
- (2) every ideal of S is complete.

Proof. By Theorem 2.3 in [9], (1) implies (2).

 $(2) \Rightarrow (1)$. Let A be any ideal of S and $b \in A$. Since A is complete, A = (AS]. We have $b \in (aS]$ for some $a \in A$. Since S is B^* -pure and every ideal is a bi-ideal, $A \cap (aS] = (aA]$. We have

$$b \in A \cap (aS] = (aA] \subseteq (A^2].$$

Thus $A \subseteq (A^2]$. As is easily seen, $(A^2] \subseteq A$. Hence $A = (A^2]$.

Theorem 9. For an idempotent ordered semigroup S. The following statements are equivalent:

- (1) S is B^* -pure;
- (2) S is normal and $(Sa] = (Sa^2)$ for all $a \in S$.

Proof. By Lemma 3, (1) implies (2).

 $(2) \Rightarrow (1)$. Let A be any bi-ideal of S, $x \in S$. Let $a \in A \cap (Sx] = A \cap (Sx^2]$. Then $a \leq yx^2$ for some $y \in S$. Since $ay \in (aS] = (Sa] = (Sa^2]$, $ay \leq za^2$ for some $z \in S$. We have

$$a \le a^2 \le ayx^2 \le za^2x^2 \in (SaaSx]$$

$$\subseteq ((Sa](aS]x]$$

$$= ((aS](Sa]x]$$

$$\subseteq (aSSax]$$

$$\subseteq (ASSAx]$$

$$\subseteq (Ax].$$

Thus $A \cap (Sx] \subseteq (Ax]$. Let $b \in (Ax]$. Then $b \leq ax$ for some a in A. We have

$$b \le ax \in (aS] = (Sa] = (Sa^2] \subseteq (aSa] \subseteq (ASA] \subseteq A,$$

and so $(Ax] \subseteq A$. Since $(Ax] \subseteq (Sx]$, then $(Ax] \subseteq A \cap (Sx]$. Thus $A \cap (Sx] = (Ax]$. Similarly, we have $A \cap (xS] = (xA]$. Hence A is B-pure.

Theorem 10. Any normal regular ordered semigroups are B^* -pure.

Proof. Let S be a normal regular ordered semigroup, A be a bi-deal of S and $x \in S$. Let $b \in (xA]$. Then $b \leq xa$ for some a in A. Since S is regular, then $a \leq aya$ for some y in S. We have

$$b \leq xa \leq xaya \in (SaSa] \subseteq ((Sa]Sa] = ((aS]Sa]$$
$$\subseteq (aSSa]$$
$$\subseteq (aSa]$$
$$\subseteq (ASA] \subseteq A.$$

Thus $(xA] \subseteq A$. Since $(xA] \subseteq (xS]$, then $(xA] \subseteq A \cap (xS]$. Let $a \in A \cap (xS]$. Then $a \leq xb$ for some b in S. Since S is regular, then $a \leq aya$ for some y in S. We have

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$$a \leq aya \leq ayaya \leq xbyaya = x(by)aya \in (xSaya]$$

$$\subseteq (x(Sa]ya]$$

$$\subseteq (x(aS]SA]$$

$$\subseteq (xaSSA]$$

$$\subseteq (xASSA]$$

$$\subset (xA].$$

Thus $A \cap (xS] = (xA]$. Similarly, we have $A \cap (Sx] = (Ax]$. Hence A is a B-pure.

The following Corollary 11 can be obtained from Lemma 3 and Theorem 10.

Corollary 11. For a regular ordered semigroup S. The following statements are equivalent:

- (1) S is B^* -pure;
- (2) S is normal.

Theorem 12. For a B^* -pure ordered semigroup S. The following statements are equivalent:

- (1) S is Archimedean;
- (2) (SaS] = (SbS] for all $a, b \in S$;
- (3) (aS] = (bS] for all $a, b \in S$;
- (4) (aSa] = (bSb] for all $a, b \in S$;
- (5) for any $e, f \in E(S), (e, f) \in \mathcal{N};$
- (6) every bi-ideal of S is Archimedean.

Proof. It is clear that (6) implies (1).

 $(1) \Rightarrow (2)$. Let $a, b \in S$. Since S is Archimedean, then there exists positive integer n such that $a^n \in (SbS]$. By Lemma 3, we have

$$(SaS] \subseteq (Sa^nS] \subseteq (S(SbS]S] \subseteq (SSbSS] \subseteq (SbS].$$

Similarly, we have $(SbS] \subseteq (SaS]$. Hence (SaS] = (SbS]. It follows from Lemma 3 (1) and (3) that (2) implies (3) and (3) implies (4).

 $(4) \Rightarrow (5)$. Let $e, f \in E(S)$. Then (eSe] = (fSf]. This implies that N(e) = N(f). Hence $(e, f) \in \mathcal{N}$.

 $(5)\Rightarrow(6)$. Let A be a bi-deal of S and $a, b \in A$. Since S is B^* -pure, a^2 and b^2 are regular by Lemma 3. Then $a^2 \leq a^2xa^2$ and $b^2 \leq b^2yb^2$ for some $x, y \in S$. This implies that $a^2x, b^2y \in E(S)$. We have $b^2y \in N(a^2x)$. Then $(a^2x)^n \in (b^2ySb^2y]$ for some positive integer n. Thus $(a^2x)^n \leq b^2yzb^2y$ for some $z \in S$. We have

$$a^{3} \leq aa^{2}xa^{2} \leq aa^{2}xa^{2}xa^{2} = a(a^{2}x)a^{2}xa^{2}$$
$$\leq a(a^{2}x)^{n}a^{2}$$
$$\leq a(b^{2}yzb^{2}y)a^{2}$$
$$= ab(b(yzb^{2}ya)a)$$
$$\in (Ab(ASA)]$$
$$\subseteq (AbA].$$

Hence A is Archimedean. This completes the proof of the theorem.

Theorem 13. Any B^* -pure Archimedean regular ordered semigroup S does not contain proper bi-ideals.

Proof. Let A be any bi-ideal of S. Let $a \in A$ and $b \in S$. Since S is Archimedean, then there exists positive integer n such that $b^n \in (SaS]$. Since S is B^* -pure, (aSa] is B-pure. Then by the regularity of S and Lemma 3, we have

$$b \in (bSb] \subseteq (b^n Sb^n] \subseteq ((SaS]S(SaS]]$$
$$\subseteq (SaSSSaS]$$
$$\subseteq (SaSSS(aS]]$$
$$\subseteq (SaSSS(Sa]]$$
$$\subseteq (SaSSSSa]$$
$$\subseteq (S(aSa)]$$
$$\equiv (SS] \cap (aSa]$$
$$\subseteq (ASA]$$
$$\subseteq A.$$

Thus $S \subseteq A$. Hence S = A.

The following Theorem 14 can be obtained from Theorem 13.

Theorem 14. Any B^{*}-pure Archimedean regular ordered semigroups are left and right simple.

Theorem 15. For a B^* -pure Archimedean ordered semigroup S. The following statements are equivalent:

- (1) S is regular;
- (2) S does not contain proper bi-ideals;
- (3) S are left and right simple.

Proof. By Theorem 13, (1) implies (2). It is clear that (2) implies (3).

 $(3) \Rightarrow (1)$. Let $a \in S$. As is easily seen, (Sa] is a left ideal and (aS] is a right ideal. Since S are left and right simple, then S = (Sa] and S = (aS]. We have $a \in (aS] = (a(Sa)] \subseteq (aSa]$. This completes the proof of the theorem.

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