Discussiones Mathematicae
General Algebra and Applications xx (xxxx) 1–11

A CONSTRUCTION OF SEMIGROUPS CONTAINING MIDDLE UNITS

Attila Nagy

Institute of Mathematics
Department of Algebra
Budapest University of Technology and Economics
Műegyetem rkp. 3., Budapest, 1111 Hungary

e-mail: nagyat@math.bme.hu

Abstract

In this paper, we show that semigroups containing middle units can be constructed from semigroups containing one-sided identity elements. Moreover, we show that regular semigroups containing middle units can be obtained from regular monoids.

Keywords: semigroup, regular semigroup, middle unit.

2020 Mathematics Subject Classification: 20M17.

1. Introduction and motivation

An element e of a semigroup S is called a *middle unit* of S if $xey = xy$ is satisfied for every $x, y \in S$. A semigroup S is called a *regular semigroup* if, for every element $a \in S$, there is an element $a' \in S$ such that $aa'a = a$. Semigroups, especially regular semigroups containing middle units are examined in many papers. See, for example, [1–4, 8, 9, 15]. It is clear that every one-sided identity element of a semigroup S is a middle unit of S. In this paper we show that semigroups containing middle units can be constructed by the help of semigroups containing left identity elements as well as semigroups containing right identity elements. Because of the duality, we focus only on the role of semigroups containing left identity elements. Results corresponding to semigroups containing right identity elements are also presented, but not proved. Applying the construction defined in Theorem 2 of [11] and its dual, we define the right regular extension and the left regular extension of semigroups, and show that a semigroup has a middle unit if and only if it is a right regular extension (respectively, left regular extension) of a semigroup containing a left identity element (respectively, a right identity element). As the middle units play an important role in the examination of regular semigroups, we examine how to construct medial semigroups having middle units. Introducing the notions of the strongly right regular extension and the strongly left regular extension of semigroups, we show that a semigroup is regular and contains a middle unit if and only if it is a strongly right regular extension (respectively, strongly left regular extension) of a regular semigroup which is a strongly left regular extension (respectively, strongly right regular extension) of a regular monoid.

2. Preliminaries

We focus our attention only on the concepts that are related to the examination of the role of semigroups containing left identity elements. The dual concepts are also used, but we do not define all of them.

Let S be a semigroup. A transformation ϱ of S acting on the right is called a *right translation* of S if $(xy)\varrho = x(y\varrho)$ is satisfied for every $x, y \in S$. Let a be an arbitrary element of S. The transformation ϱ_a of S defined by $\varrho_a : s \mapsto sa$ is a right translation of S. It is known that $\Phi : a \mapsto \varrho_a$ is a homomorphisms of the semigroup S into the semigroup of all right translations of S. Φ is called the *right regular representation* of S. Let $\theta(S)$ denote the kernel of Φ. It is obvious that $\theta(S) = \{(a, b) \in S \times S : \ (\forall x \in S)\ xa = xb\}$. A semigroup S is said to be *left reductive* if, for arbitrary elements a and b of S, $sa = sb$ for all $s \in S$ implies $a = b$. The right regular representation Φ of a semigroup S is *faithful* (that is, injective) if and only if S is *left reductive*, that is, $\theta(S)$ is the identity relation on S. The concept of the right regular representation of semigroups occurs in several studies. See, for example, [5, 6, 11, 12] and [13]. Among them, [11] plays an important role in our present paper. Examinations of semigroups containing middle units are based on the construction given in Theorem 2 of [11]. Since this construction will be used several times, Theorem 2 of [11] is quoted in detail here.

Theorem 1 ([11], Theorem 2)**.** (a) *Let S be a semigroup. For each $x \in S$, associate a set $T_x \neq \emptyset$. Assume $T_x \cap T_y = \emptyset$ for all $x, y \in S$ with $x \neq y$. Assume that, for each couple $(x, y) \in S \times S$ is associated a mapping*

$$f_{(x,y,xy)} : T_x \mapsto T_{xy}$$

acting on the left. Suppose further

$$(\forall x, y, z \in S) \quad f_{(xy,z,xyz)} \circ f_{(x,y,xy)} = f_{(x,yz,xyz)}. \tag{1}$$

Let

$$T = \bigcup_{x \in S} T_x.$$

On the set T, define the following operation. If $a \in T_x$ and $b \in T_y$, then let

$$a \star b = f_{(x,y,xy)}(a). \tag{2}$$

Then $(T;\star)$ is a semigroup, and each set T_x is contained in a $\theta(T)$-class of T. If S is left reductive then the $\theta(T)$-classes of the semigroup $(T;\star)$ are exactly the sets T_x $(x \in S)$.

(b) *Let $(T;\cdot)$ be a semigroup. Let θ denote the congruence $\theta(T)$. Let*

$$S = T/\theta.$$

For an element $[x]\theta \in S$, let $T_{[x]\theta} = [x]_\theta$. For arbitrary couple $([x]\theta, [y]\theta)$ and arbitrary $a \in [x]\theta$, let

$$f_{([x]\theta,[y]\theta,[xy]\theta)}:\ a \mapsto a \cdot b, \tag{3}$$

where b is an arbitrary element of $[y]\theta$. Then, for all $c, d \in T$,

$$c \star d = f_{([c]\theta,[d]\theta,[cd]\theta)}(c) = c \cdot d.$$

Consequently, the semigroup $(T;\cdot)$ is isomorphic to the semigroup $(T;\star)$ derived from the semigroup $S = T/\theta$ by applying the construction in part (a) *of Theorem* 1.

The semigroup $T = (T;\star)$ defined in part (a) of Theorem 1 will be called a *right regular extension* of the semigroup S if the sets T_s $(s \in S)$ are the classes of the kernel of the right regular representation of $(T;\star)$. We shall say that a right regular extension $(T;\star)$ of a semigroup S is a *strongly right regular extension* if, for every $s \in S$ and $a \in T_s$, there is an element $x \in S$ such that $f_{(s,xs,sxs)}(a) = a$.

Remark 2. If $T = (T;\star)$ is a right regular extension of a semigroup S, then $S \cong T/\theta(T)$.

Remark 3. We note that if S is a left reductive semigroup, then the semigroup $(T;\star)$ defined in part (a) of Theorem 1 is a right regular extension of S. This is the case, when S has a left identity element. By part (b) of Theorem 1, every semigroup T is isomorphic to a right regular extension of the factor semigroup $T/\theta(T)$.

Remark 4. We note that if $(T;\star)$ is a strongly right regular extension of a semigroup S containing a left identity, then S is regular, because $f_{(s,xs,sxs)}(a) = a$ and $a \in T_s$ imply $sxs = s$.

In this paper we also refer to the dual of Theorem 1. We note that, in the dual of Theorem 1, for each couple $(x, y) \in S \times S$ is associated a mapping

$$f_{(y,x,xy)} : T_y \mapsto T_{xy}$$

acting on the right, and is supposed

$$(\forall x, y, z \in S) \quad f_{(z,y,yz)} \circ f_{(yz,x,xyz)} = f_{(z,xy,xyz)}. \tag{4}$$

On the set $T = \bigcup_{x \in S} T_x$, an associative operation $\diamond$ is defined: if $a \in T_x$ and $b \in T_y$, then

$$a \diamond b = (b) f_{(y,x,xy)}. \tag{5}$$

Starting from the dual of Theorem 1, we can define the dual concepts of the right regular extension and the strongly right regular extension of semigroups. A semigroup $T = (T; \diamond)$ defined above is called a *left regular extension* of the semigroup S if the sets T_s ($s \in S$) are the classes of the kernel of the left regular representation of $(T; \diamond)$. We say that a left regular extension $(T; \diamond)$ of a semigroup S is a *strongly left regular extension* if, for every $s \in S$ and $a \in T$, there is an element $x \in S$ such that $(a) f_{(s,sx,sxs)} = a$.

For notions and notations not defined but used in this paper, we refer the reader to the books [7, 10] and [14].

3. Results

Theorem 5. *The following assertions on an arbitrary semigroup T are equivalent.*

(i) *T has a middle unit.*

(ii) *T is isomorphic to a right regular extension of a semigroup containing a left identity element.*

(iii) *T is isomorphic to a left regular extension of a semigroup containing a right identity element.*

Proof. Only the equivalence of (i) and (ii) is proved. The equivalence of (i) and (iii) can be proved similarly.

(i)$\Rightarrow$(ii) Assume that T is a semigroup which has a middle unit. By Remark 3, the semigroup $(T; \star)$ defined in part (b) of Theorem 1 is a right regular extension of the factor semigroup $S = T/\theta(T)$, and T is isomorphic to $(T; \star)$. Let e be a middle unit of T. Then, for every $x, y \in T$, we have $xey = xy$. This implies that $(ey, y) \in \theta(T)$, that is, $[e]_{\theta(T)}[y]_{\theta(T)} = [y]_{\theta(T)}$ in $S = T/\theta(T)$ for every $y \in T$. Thus $[e]_{\theta(T)}$ is a left identity element of $S = T/\theta(T)$.

(ii)⇒(i) Let S be a semigroup containing a left identity element l, and let $T = (T; \star)$ be a semigroup defined by S as in part (a) of Theorem 1. Since S has a left identity element, then S is left reductive, and so the $\theta(T)$-classes of the semigroup $(T; \star)$ are exactly the sets T_x ($x \in S$). Thus (by Remark 3) the semigroup $(T; \star)$ is a right regular extension of S. We show that $(T; \star)$ has a middle unit. Let $e \in T_l$ and $b \in T$ be arbitrary elements. Assume $b \in T_y$ ($y \in S$). Since l is a left identity element of S, then $e \star b \in T_l T_y \subseteq T_{ly} = T_y$, and so $(e \star b, b) \in \theta(T)$, because the $\theta(T)$-classes of $(T; \star)$ are the sets T_s ($s \in S$). Thus $a \star e \star b = a \star b$ for every $a \in T$. Consequently e is a middle unit of the semigroup $T = (T; \star)$. ■

Remark 6. From the proof of Theorem 5 it follows that if l is a left identity element of a semigroup S, then every element of T_l is a middle unit of the right regular extension $(T; \star)$ of S.

Example 7. In this example, we construct a semigroup which contains a middle unit. We use the notations of Theorem 1. Let S be a semigroup containing a left identity element. For every $x \in S$, let $T_x = \{(x, s) : \ s \in S\}$. For every $x, y \in S$, let $f_{(x,y,xy)}$ be the mapping of T_x into T_{xy} defined by the following way: for $(x, s) \in T_x$, let $f_{(x,y,xy)}((x, s)) = (xy, sy)$. Then, for every $x, y, z \in S$, $f_{(xy,z,xyz)} \circ f_{(x,y,xy)}((x, s)) = ((xy)z, (sy)z) = (x(yz), s(yz)) = f_{(x,yz,xyz)}((x, s))$. Thus condition (1) of Theorem 1 is satisfied, and so $(T; \star)$ is a semigroup under the operation: $(x, s) \star (y, t) = (xy, sy)$. If l denotes a left identity element of S, then (by Theorem 5 and Remark 6) (l, z) is a middle unit of T for every $z \in S$. Indeed, $(x, s)(l, z)(y, t) = (xly, sly) = (xy, sy) = (x, s)(y, t)$ for every $x, y, s, z, t \in S$.

A semigroup satisfying the identity $ab = b$ is called a *right zero semigroup.* A semigroup is a right zero semigroup if and only if its every element is a left identity element.

Corollary 8 [13]. *A semigroup has the property that its every element is a middle unit if and only if it is isomorphic to a right regular extension of a right zero semigroup.*

Proof. By Theorem 5, if every element of a semigroup T is a middle unit, then T is a right regular extension of a semigroup S containing a left identity element, and so $S \cong T/\theta(T)$ by Remark 2. Since the equation $xay = xy$ is satisfied for every elements $x, y, a \in T$, then $(ay, y) \in \theta(T)$ for every $a, y \in S$. Thus every element of $S \cong T/\theta(T)$ is a left identity element, that is, S is a right zero semigroup.

By Theorem 5 and Remark 6, if S is a right zero semigroup, then every element of a right regular extension $(T; \star)$ of S is a middle unit of T. ■

In the next example, we give a semigroup in which every element is a middle unit.

Example 9. Let S be a right zero semigroup. Let T_x $(x \in S)$ be pairwise disjoint nonempty sets. Fix an element t_s^* in T_s for every $s \in S$. For every $x, y \in S$, let $f_{(x,y,xy)}$ be the mapping of T_x into $T_{xy} = T_y$ defined by $f_{(x,y,y)}(a) = t_y^*$. It is easy to see that this system of mappings satisfies condition (1) of Theorem 1. Let $T = \bigcup_{s \in S} T_s$, and define an operation $\star$ on T as in (2) of Theorem 5. Then the semigroup $(T : \star)$ is a semigroup in which every element is a middle unit. For example, if $S = \{x, y\}$, $T_x = \{a, b\}$, $t_x^* = a$, $T_y = \{c\}$, $t_y^* = c$, then the Cayley multiplicative table of $(T; \star)$ is Table 1.

$\cdot$	a	b	c
a	a	a	c
b	a	a	c
c	a	a	c

Table 1.

It is easy to see that every element of $(T; \star)$ is a middle unit.

If e is a right identity element and a is a middle unit of a semigroup S, then $sa = sae = se = s$ for every $s \in S$. Thus every middle unit of a semigroup S containing a right identity element is a right identity element of S. Similarly, if a semigroup S contains a left identity element, then every middle unit of S is a left identity element of S.

The middle units of the semigroup $(T; \star)$ defined in Example 7 are the elements (l, z), where l is a left identity and z is an arbitrary element of S. It is easy to see that if $(T; \star)$ has an identity element, then S is a monoid. This is true in general.

Lemma 10. *If a semigroup T has a right identity element* (*respectively, left identity element*) *and T is a right regular extension* (*respectively, left regular extension*) *of a semigroup S containing a left identity element* (*respectively, right identity element*), *then S is a monoid.*

Proof. If e is a right identity element of T, then $[e]_{\theta(T)}$ is a right identity element of S. As S has a left identity element by assumption, S is a monoid. The proof of the dual assertion is similar. ■

The converse assertion of Lemma 10 is true for globally idempotent semigroups T (that is, when $T^2 = T$).

Theorem 11. *A globally idempotent semigroup T contains a right identity element (respectively, left identity element) if and only if it is a right regular extension (respectively, left regular extension) of a monoid.*

Proof. We prove only the case on the right. The dual assertion can be proved similarly.

Let T be a globally idempotent semigroup which contains a right identity element e. Since e is a middle unit of T, then, by Theorem 5, T is a right regular extension of a semigroup S containing a left identity element. By Remark 2, $S \cong T/\theta(T)$. By Lemma 10, S is a monoid.

Conversely, assume that T is a globally idempotent semigroup which is a right regular extension of a monoid S. Then $S \cong T/\theta(T)$. Let e be an element of T such that $[e]_{\theta(T)}$ is the identity element of S. Let $t \in T$ be an arbitrary element. As T is globally idempotent, there are elements $a, b \in T$ such that $ab = t$. As $[b]_{\theta(T)}[e]_{\theta(T)} = [b]_{\theta(T)}$ in S, we have $(be, b) \in \theta(T)$, and so $abe = ab$, that is, $te = t$. Hence e is a right identity element of T. ∎

In Example 9, the semigroup $T = T_x \cup T_y$ ($T_x = \{a, b\}$, $T_y = \{c\}$) is a right regular extension of the right zero semigroup $S = \{x, y\}$. Every element of T is a middle unit, but T is not regular (for every $x \in T$, $bxb = a$). Theorem 5 shows that if a semigroup T a right regular extension of a regular semigroup S containing a left identity element, then T has a middle unit, but Example 9 shows that T is not regular, in general. The following theorem shows that the strongly right regular extension plays an important role in the description of the regular semigroups containing middle units.

Theorem 12. *The following assertions on a semigroup T are equivalent.*

(i) *T is regular and contains a middle unit.*

(ii) *T is isomorphic to a strongly right regular extension of a regular semigroup containing a left identity element.*

(iii) *T is isomorphic to a strongly left regular extension of a regular semigroup containing a right identity element.*

Proof. Only the equivalence of (i) and (ii) is proved. The equivalence of (i) and (iii) can be proved similarly.

(i)$\Rightarrow$(ii) Assume that T is a regular semigroup containing a middle unit. By part (b) of Theorem 1, T is isomorphic to a right regular extension $(T; \star)$ of the factor semigroup $S = T/\theta(T)$. Since T is regular, and every homomorphic image of a regular semigroup is regular, then the semigroup S is regular. Since T contains a middle unit, Theorem 5 implies that S contains a left identity element. Let $s = [t]_{\theta(T)}$ be an arbitrary element of S (and so $T_s = [t]_{\theta(T)}$). Let $a \in T_s$ be an arbitrary element. Then $[a]_{\theta(T)} = [t]_{\theta(T)}$. Since T is a regular semigroup, then there is an element $b \in T$ such that $a \star b \star a = a$. Let $x = [b]_{\theta(T)}$ (and so $T_x = [b]_{\theta(T)}$). Since

$$xs = [b]_{\theta(T)}[t]_{\theta(T)} = [b]_{\theta(T)}[a]_{\theta(T)} = [b \star a]_{\theta(T)}$$

in S, then we have $T_{xs} = [b \star a]_{\theta(T)}$. It is clear that, in S,

$$sxs = [a \star b \star a]_{\theta(T)} = [a]_{\theta(T)} = s.$$

Consider the mapping $f_{(s,xs,s)}$ of $T_s = [a]_{\theta(T)}$ into itself. By (3) in part (b) of Theorem 1, $f_{(s,xs,s)}(a) = a \star \xi$, where ξ is an arbitrary element of $T_{xs} = [b \star a]_{\theta(T)}$. Choosing $\xi = b \star a$, we get

$$f_{(s,xs,sxs)}(a) = a \star \xi = a \star b \star a = a.$$

Thus T is a strongly right regular extension of the regular semigroup S.

(ii)⇒(i) Assume that the semigroup T is isomorphic to a strongly right regular extension $(T;\star)$ of a regular semigroup S defined in part (a) of Theorem 1, where S has a left identity element. Then T has a middle unit by Theorem 5. Let $a \in T_s$ ($s \in S$) be an arbitrary element. Then there is an element $x \in S$ such that

$$a = f_{(s,xs,sxs)}(a).$$

Let $b \in T_x$ be an arbitrary element. Then $b \star a \in T_{xs}$. By (3) in part (b) of Theorem 1,

$$f_{(s,xs,sxs)}(a) = a \star \xi,$$

where ξ is an arbitrary element of T_{xs}. Since $b \star a \in T_{xs}$, we have

$$a = f_{(s,xs,sxs)}(a) = a \star b \star a.$$

Hence a is a regular element of T. Consequently T is a regular semigroup. ■

Example 13. In this example we construct a regular semigroup which contains middle units. Let $S = \{a, b, c, d\}$ be a semigroup, where the multiplication is defined by Table 2.

	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	d	c	d
d	c	d	c	d

Table 2.

S is a regular semigroup, because every element of S is an idempotent element. For every $x \in S$, let $T_x = \{x^{(1)}, x^{(2)}\}$ be a two-element set. Assume $T_x \cap T_y = \emptyset$ for every $x, y \in S$ with $x \neq y$. For every $x, y \in S$, let $f_{(x,y,xy)}$ be a mapping of T_x into T_{xy} defined by $f_{(x,y,xy)} : x^{(i)} \mapsto (xy)^{(i)}$ for $i = 1, 2$. It is a matter of checking to see that this family of mappings satisfies condition (1) of Theorem 1. Thus $T = \bigcup_{x \in S} T_x$ is a semigroup under the operation $\star$ defined by $x^{(i)} \star y^{(j)} = f_{(x,y,xy)}(x^{(i)}) = (xy)^{(i)}$. The Cayley-table of the semigroup $(T; \star)$ is Table 3.

$\cdot$	$a^{(1)}$	$a^{(2)}$	$b^{(1)}$	$b^{(2)}$	$c^{(1)}$	$c^{(2)}$	$d^{(1)}$	$d^{(2)}$
$a^{(1)}$	$a^{(1)}$	$a^{(1)}$	$b^{(1)}$	$b^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$
$a^{(2)}$	$a^{(2)}$	$a^{(2)}$	$b^{(2)}$	$b^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$
$b^{(1)}$	$a^{(1)}$	$a^{(1)}$	$b^{(1)}$	$b^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$
$b^{(2)}$	$a^{(2)}$	$a^{(2)}$	$b^{(2)}$	$b^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$
$c^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$
$c^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$
$d^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$	$c^{(1)}$	$c^{(1)}$	$d^{(1)}$	$d^{(1)}$
$d^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$	$c^{(2)}$	$c^{(2)}$	$d^{(2)}$	$d^{(2)}$

Table 3.

Since the semigroup S contains left identity elements (a and b), then the semigroup $(T; \star)$ is a right regular extension of S by Remark 3. Then, by Theorem 5, $(T; \star)$ contains middle units. It is easy to see that $a^{(1)}, a^{(2)}, b^{(1)}, b^{(2)}$ are the middle units of $(T; \star)$. As every element of S is an idempotent, we have $f(x, x, x)(x^{(i)}) = x^{(i)}$ for every $x \in S$ and every $i = 1, 2$. Thus $(T; \star)$ is a strongly right regular extension of S, and so $(T; \star)$ is a regular semigroup by Theorem 12. We note that the regularity of $(T; \star)$ can be checked by its Cayley-table: since every element of $(T; \star)$ is an idempotent element, then $(T; \star)$ is a regular semigroup.

Remark 14. We note that the semigroup T constructed in Example 7 is also regular containing a middle unit, if S is a group. Namely, if e denotes the identity element of S, then, for every $x \in S$ and $(x, s) \in T_x$, we have $f_{(x, x^{-1}x, xx^{-1}x)}(x, s) = (xx^{-1}x, sx^{-1}x) = (x, s)$, where x^{-1} denote the inverse of x. Thus T is a strongly right regular extension of S. Hence T is a regular semigroup containing middle units by Theorem 12.

The next theorem shows that how we can get regular semigroups containing middle units from regular monoids.

Theorem 15. *The following assertions on an arbitrary semigroup T are equivalent.*

(i) *T is regular and contains a middle unit.*

(ii) *T is isomorphic to a strongly right regular extension of a regular semigroup which is a strongly left regular extension of a regular monoid.*

(iii) *T is isomorphic to a strongly left regular extension of a regular semigroup which is a strongly right regular extension of a regular monoid.*

Proof. Only the equivalence of (i) and (ii) is proved. The equivalence of (i) and (iii) can be proved similarly.

(i)⇒(ii) Let T be a regular semigroup which contains a middle unit. By Theorem 12, T is isomorphic to a strongly right regular extension of a regular semigroup S containing a left identity element. Then, by Theorem 12, S is isomorphic to a strongly left regular extension of a regular semigroup Q containing a right identity element. By Lemma 10, Q is a monoid.

(ii)⇒(i) Assume that a semigroup T is isomorphic to a strongly right regular extension of a regular semigroup S which is a strongly left regular extension of a regular monoid. By Theorem 12, T is a regular semigroup. As S is globally idempotent, Theorem 11 implies that S contains a left identity element. By Theorem 5, T contains a middle unit. ∎

References

[1] J.E. Ault, *Semigroups with midunits*, Semigroup Forum **6** (1973) 346–351.
https://doi.org/10.1007/BF02389143

[2] J.E. Ault, *Semigroups with midunits*, Trans. Amer. Math. Soc. **190** (1974) 375–384.
https://doi.org/10.1090/S0002-9947-1974-0340456-5

[3] T.S. Blyth, *On middle units in orthodox semigroups*, Semigroup Forum **13-1** (1976) 261–265.
https://doi.org/10.1007/BF02194944

[4] T.S. Blyth and R. McFadden, *On the construction of a class of regular semigroups*, J. Algebra **81** (1983) 1–22.
https://doi.org/10.1016/0021-8693(83)90205-3

[5] J.L. Chrislock, *Semigroups whose regular representation is a group*, Proc. Japan Acad. **40** (1964) 799–800.
https://doi.org/10.3792/pja/1195522567

[6] J.L. Chrislock, *Semigroups whose regular representation is a right group*, Amer. Math. Monthly **74** (1967) 1097–1100.
https://doi.org/10.2307/2313623

[7] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups I, Amer. Math. Soc. (Providence R.I., 1961).
https://doi.org/10.1090/surv/007.1

[8] J.B. Hickey, *Semigroups under a sandwich operation*, Proc. Edinburgh Math. Soc. **26** (1983) 371–382.
https://doi.org/10.1017/S0013091500004442

[9] D.B. McAlister, *Regular Rees matrix semigroups and regular Dubreil-Jacotin semigroups*, J. Australian Math. Soc. **31** (1981) 325–336.
https://doi.org/10.1017/S1446788700019467

[10] A. Nagy, Special Classes of Semigroups, Kluwer Academic Publishers (Dordrecht, Boston, London, 2001).
https://doi.org/10.1007/978-1-4757-3316-7

[11] A. Nagy, *Remarks on the paper "M. Kolibiar, On a construction of semigroups"*, Periodica Math. Hungarica **71** (2015) 261–264.
https://doi.org/10.1007/s10998-015-0094-z

[12] A. Nagy, *Left equalizer simple semigroups*, Acta Math. Hungarica **148(2)** (2016) 300–311.
https://doi.org/10.1007/s10474-015-0578-6

[13] A. Nagy and O. Nagy, *A construction of semigroups whose elements are middle units*, Int. J. Algebra **14(3)** (2020) 163–169.
https://doi.org/10.12988/ija.2020.91248

[14] M. Petrich, Lectures in Semigroups (Akademie-Verlag-Berlin, 1977).

[15] M. Yamada, *A note on middle unitary semigroups*, Kodai Math. Sem. Rep. **7**(1955) 49–52.
https://doi.org/10.2996/kmj/1138843607