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MEDIAN FILTERS OF PSEUDO-COMPLEMENTED DISTRIBUTIVE LATTICES

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Abstract

Coherent filters, strongly coherent filters, and τ-closed filters are introduced in pseudo-complemented distributive lattices and their characterization theorems are derived. A set of equivalent conditions is derived for every filter of a pseudo-complemented distributive lattice to become a coherent filter. The notion of median filters is introduced and some equivalent conditions are derived for every maximal filter of a pseudo-complemented distributive lattice to become a median filter which leads to a characterization of Stone lattices.

Keywords: coherent filter, strongly coherent filter, median filter, minimal prime filter, maximal filter, Stone lattice.

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1. Introduction

The theory of pseudo-complements in lattices, and particularly in distributive lattices was developed by Stone [10], Frink [5], and George Gratzer [6]. Later many authors like Balbes [1], Speed [9], and Frink [5] etc., extended the study of pseudo-complements to characterize Stone lattices. In [3], Chajda, Halaš and Kühr extensively studied the structure of pseudo-complemented semilattices. In [7], the concept of δ-ideals was introduced in pseudo-complemented distributive lattices and then Stone lattices were characterized in terms of δ-ideals. In [8], the authors investigated the properties of D-filters and prime D-filters of distributive lattices and characterized the minimal prime D-filters of distributive lattices.

In this note, the concepts of coherent filters and strongly coherent filters are introduced in pseudo-complemented distributive lattices. A set of equivalent conditions is derived for every filter of a pseudo-complemented distributive lattice to become a coherent filter which characterizes a Boolean algebras. It is showed that every strongly coherent f ilter of a pseudo-complemented distributive lattice is coherent. The concepts of τ-closed filters and semi Stone lattices are introduced within pseudo-complemented distributive lattices and the class of all semi Stone lattices is characterized using the τ-closed filters. It is observed that the classes of maximal filters and prime D-filters coincide in a pseudo-complemented lattice. This observation precisely motivates to investigate the properties of certain class of filters under the name median filters as a special subclass of maximal filters of pseudo-complemented distributive lattices. Median filters are characterized and it is shown that every median filter of a pseudo-complemented distributive lattice is a coherent filter. A set of equivalent conditions is derived for every maximal filter of a pseudo-complemented distributive lattice to become a strongly coherent filter. Some equivalent conditions are stated for maximal filters of a pseudo-complemented distributive lattice to become median filters which leads to a characterization of Stone lattices.

2. Preliminaries

The reader is referred to [2, 3] and [8] for the elementary notions and notations of pseudo-complemented distributive lattices. However some of the preliminary definitions and results are presented for the ready reference of the reader.

A non-empty subset A of a lattice L is called an *ideal* (*filter*) [2] of L if $a \vee b \in A$ ($a \wedge b \in A$) and $a \wedge x \in A$ ($a \vee x \in A$) whenever $a, b \in A$ and $x \in L$. The set $(a] = \{x \in L \mid x \leq a\}$ (respectively, $[a) = \{x \in L \mid a \leq x\}$) is called a *principal ideal* (respectively, *principal filter*) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice in which $F \vee G = \{i \wedge j \mid i \in F \text{ and } j \in G\}$ for any two filters F and G. A proper filter P of a lattice L is said to be *prime* if for any $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. A proper filter P of a lattice L is called *maximal* if there exists no proper filter Q of L such that $P \subset Q$. A proper filter P of a distributive lattice is *minimal* if there exists no prime filter Q of L such that $Q \subset P$.

The *pseudo-complement* b^* of an element b is the element satisfying

$$a \wedge b = 0 \;\Leftrightarrow\; a \wedge b^* = a \;\Leftrightarrow\; a \leq b^*$$

where $\leq$ is the induced order of L.

A distributive lattice L in which every element has a pseudo-complement is called a *pseudo-complemented distributive lattice*. For any two elements a, b of a pseudo-complemented semilattice [3], we have the following.

(1) $a \leq b$ implies $b^* \leq a^*$,

(2) $a \leq a^{**}$,

(3) $a^{***} = a^*$,

(4) $(a \vee b)^* = a^* \wedge b^*$,

(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element a of a pseudo-complemented distributive lattice L is called a *dense* if $a^* = 0$ and the set D of all dense elements of L forms a filter in L.

Definition [2]**.** A pseudo-complemented distributive lattice L is called a *Stone lattice* if $x^* \vee x^{**} = 1$ for all $x \in L$.

Theorem 1 [2]**.** *The following assertions are equivalent in a pseudo-complemented distributive lattice L.*

(1) *L is a Stone lattice;*

(2) *for $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$;*

(3) *for $x, y \in L$, $(x \vee y)^{**} = x^{**} \vee y^{**}$.*

A filter F of a distributive lattice L is called a *D-filter* [8] if $D \subseteq F$. For any non-empty subset A of a distributive lattice L, the set $A^\circ = \{x \in L \mid x \vee a \in D \text{ for all } a \in A\}$ is a D-filter of L. In case of $A = \{a\}$, we simply represent $\{a\}^\circ$ by $(a)^\circ$. A prime D-filter of a distributive lattice is minimal if it is the minimal element in the poset of all prime D-filters. A prime D-filter of a distributive lattice is minimal [8] if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$. Throughout this note, all lattices are considered to be bounded pseudo-complemented distributive lattices unless otherwise mentioned.

3. Coherent filters

In this section, the concepts of coherent filters and strongly coherent filter are introduced. Stone lattices are characterized with the help of coherent filters. A set of equivalent conditions is derived for every filter of a lattice to become a coherent filter which leads to a characterization of a Boolean algebras.

Definition. For any non-empty subset A of a lattice L, define

$$A^\tau = \{x \in L \mid a^{**} \vee x^{**} = 1 \text{ for all } a \in A\}.$$

Clearly $D^\tau = D$ and $L^\tau = D$. For any $a \in L$, we denote $(\{a\})^\tau$ simply by $(a)^\tau$. It is obvious that $(0)^\tau = D$ and $(1)^\tau = L$. For any $\emptyset \neq A \subseteq L$, $A \cap A^\tau \subseteq D$.

Proposition 2. *For any non-empty subset A of L, A^τ is a D-filter in L.*

Proof. Clearly $D \subseteq A^\tau$. Let $x, y \in A^\tau$. For any $a \in A$, we get $(x \wedge y)^{**} \vee a^{**} = (x^{**} \wedge y^{**}) \vee a^{**} = (x^{**} \vee a^{**}) \wedge (y^{**} \vee a^{**}) = 1 \wedge 1 = 1$. Hence $x \wedge y \in A^\tau$. Again, let $x \in A^\tau$ and $x \leq y$. Then $x^{**} \vee a^{**} = 1$ for any $a \in A$. Since $x \leq y$, we get $x^{**} \leq y^{**}$. For any $c \in A$, we get $1 = x^{**} \vee c^{**} \leq y^{**} \vee c^{**}$. Thus $y^{**} \vee c^{**} = 1$. Hence $y \in A^\tau$. Therefore A^τ is a D-filter of L. ■

The following lemma is a direct consequence of the above definition.

Lemma 3. *For any two non-empty subsets A and B of a lattice L, we have*

(1) $A \subseteq B$ *implies* $B^\tau \subseteq A^\tau$,

(2) $A \subseteq A^{\tau\tau}$,

(3) $A^{\tau\tau\tau} = A^\tau$,

(4) $A^\tau = L$ *if and only if* $A = D$.

Proposition 4. *For any two filters F, G of a lattice L, $(F \vee G)^\tau = F^\tau \cap G^\tau$.*

Proof. Clearly $(F \vee G)^\tau \subseteq F^\tau \cap G^\tau$. Conversely, let $x \in F^\tau \cap G^\tau$. Let $c \in F \vee G$ be an arbitrary element. Then $c = i \wedge j$ for some $i \in F$ and $j \in G$. Now $x^{**} \vee c^{**} = x^{**} \vee (i \wedge j)^{**} = x^{**} \vee (i^{**} \wedge j^{**}) = (x^{**} \vee i^{**}) \wedge (x^{**} \vee j^{**}) = 1 \wedge 1 = 1$. Thus $x \in (F \vee G)^\tau$ and therefore $(F \vee G)^\tau = F^\tau \cap G^\tau$. ■

The following corollary is a direct consequence of the above results.

Corollary 5. *Let L be a lattice. For any $a, b \in L$, the following properties hold.*

(1) $a \leq b$ *implies* $(a)^\tau \subseteq (b)^\tau$,

(2) $(a \wedge b)^\tau = (a)^\tau \cap (b)^\tau$,

(3) $(a)^\tau = L$ *if and only if a is dense,*

(4) $a \in (b)^\tau$ *implies* $a \vee b \in D$,

(5) $a^* = b^*$ *implies* $(a)^\tau = (b)^\tau$.

Clearly $A^\tau \subseteq A^\circ$. We derive a set of equivalent conditions for a filter to satisfy the reverse inclusion which leads to a characterization of Stone lattices.

Theorem 6. *The following assertions are equivalent in a lattice L:*

(1) *L is a Stone lattice,*

(2) *for any filter F of L, $F^\tau = F^\circ$,*

(3) *for any $a \in L$, $(a)^\tau = (a)^\circ$,*

(4) *for any two filters F, G of L, $F \cap G \subseteq D$ if and only if $F \subseteq G^\tau$,*

(5) *for $a, b \in L, a \vee b \in D$ implies $a^{**} \vee b^{**} = 1$,*

(6) *for $a \in L$, $(a)^{\tau\tau} = (a^*)^\tau$.*

Proof. (1)⇒(2) Assume that L is a Stone lattice. Let F be a filter of L. Clearly $F^\tau \subseteq F^\circ$. Conversely, let $x \in F^\circ$. Then $x \vee y \in D$ for all $y \in F$. Since L is Stone, $x^{**} \vee y^{**} = (x \vee y)^{**} = 0^* = 1$ for all $y \in F$. Therefore $x \in F^\tau$.

(2)⇒(3) It is clear.

(3)⇒(4) Assume condition (3). Let F, G be two filters of L. Suppose $F \cap G \subseteq D$. Let $x \in F$. For any $y \in G$, we get $x \vee y \in F \cap G \subseteq D$. Hence $x \vee y \in D$. Now

$$\begin{aligned} x \vee y \in D \text{ for all } y \in G &\Rightarrow x \in (y)^\circ \text{ for all } y \in G \\ &\Rightarrow x \in (y)^\tau \text{ for all } y \in G \\ &\Rightarrow x^{**} \vee y^{**} = 1 \text{ for all } y \in G \end{aligned}$$

which yields that $x \in G^\tau$. Conversely, suppose that $F \subseteq G^\tau$. Let $x \in F \cap G$. Then $x \in F \subseteq G^\tau$ and $x \in G$. Hence $x \in G \cap G^\tau \subseteq D$. Therefore $F \cap G \subseteq D$.

(4)⇒(5) Assume condition (4). Let $a, b \in L$ be such that $a \vee b \in D$. Then

$$\begin{aligned} a \vee b \in D &\Rightarrow [a) \cap [b) \subseteq D \\ &\Rightarrow [a) \subseteq [b)^\tau \qquad \text{by (4)} \\ &\Rightarrow a \in [b)^\tau \\ &\Rightarrow a^{**} \vee b^{**} = 1. \end{aligned}$$

(5)⇒(6) Assume condition (5). Let $a \in L$. Clearly, we have $a \vee a^* \in D$. By assumption (5), we get that $a^{**} \vee a^{***} = 1$. Hence $a^* \in (a)^\tau$. Thus $(a)^{\tau\tau} \subseteq (a^*)^\tau$. Conversely, let $x \in (a^*)^\tau$ and $t \in (a)^\tau$. Since $t \in (a)^\tau$, we get that $a^{**} \vee t^{**} = 1$. Hence $a^* \wedge t^* = 0$. Thus $t^* \leq a^{**}$. Now

$$\begin{aligned} x \in (a^*)^\tau &\Rightarrow a^* \vee x^{**} = 1 \\ &\Rightarrow a^{**} \wedge x^* = 0 \\ &\Rightarrow t^* \wedge x^* = 0 \qquad \text{since } t^* \leq a^{**} \\ &\Rightarrow t \vee x \in D \\ &\Rightarrow t^{**} \vee x^{**} = 1 \qquad \text{by (5)} \end{aligned}$$

which holds for all $t \in (a)^\tau$. Hence $x \in (a)^{\tau\tau}$. Therefore $(a^*)^\tau \subseteq (a)^{\tau\tau}$.

(6)⇒(1) Assume condition (6). Let $a \in L$. Since $a \in (a)^{\tau\tau} = (a^*)^\tau$, we get $a^* \vee a^{**} = a^{***} \vee a^{**} = 1$. Therefore L is a Stone lattice. ∎

Definition. A filter F of a lattice L is called a *coherent filter* if for all $x, y \in L$, $(x)^\tau = (y)^\tau$ and $x \in F$ imply that $y \in F$.

Clearly each $(x)^\tau$, $x \in L$ is a coherent filter. It is evident that any filter F is a coherent filter if it satisfies $(x)^{\tau\tau} \subseteq F$ for all $x \in F$.

Theorem 7. *The following assertions are equivalent in a lattice L:*

(1) *L is a Boolean algebra,*

(2) *every element is closed,*

(3) *for any filter F, $x^{**} \in F$ implies $x \in F$,*

(4) *every principal filter is a coherent filter,*

(5) *every filter is a coherent filter,*

(6) *every prime filter is a coherent filter,*

(7) *for $a, b \in L, (a)^{\tau} = (b)^{\tau}$ implies $a = b$,*

(8) *for $a, b \in L, a^{*} = b^{*}$ implies $a = b$.*

Proof. (1)⇒(2) It is proved in [[7], Theorem 2.15].

(2)⇒(3) It is clear.

(3)⇒(4) Assume that every element of L is closed. Let $[x)$ be a principal filter of L. Since $x \vee x^{*} \in D$, we get $(x \vee x^{*})^{**} = 1 \in [1)$. By (3), we get $x \vee x^{*} \in [1)$, which gives $x \vee x^{*} = 1$. Let $a, b \in L$ be such that $(a)^{\tau} = (b)^{\tau}$ and $a \in [x)$. Then

$$\begin{aligned} x \vee x^{*} = 1 &\Rightarrow a \vee x^{*} = 1 \quad \text{since } a \in [x) \\ &\Rightarrow a^{**} \vee x^{***} = 1 \\ &\Rightarrow x^{*} \in (a)^{\tau} = (b)^{\tau} \\ &\Rightarrow b^{**} \vee x^{*} = 1 \\ &\Rightarrow (b^{**} \vee x^{*})^{*} = 0 \\ &\Rightarrow b^{*} \wedge x^{**} = 0 \\ &\Rightarrow b^{*} \wedge x = 0 \quad \text{since } x \leq x^{**} \\ &\Rightarrow x \leq b^{**} \end{aligned}$$

which yields $b^{**} \in [x)$. By (3), we get $b \in [x)$. Hence $[x)$ is a coherent filter.

(4)⇒(5) Assume condition (4). Let F be a filter of L. Choose $a, b \in L$. Suppose $(a)^{\tau} = (b)^{\tau}$ and $a \in F$. Then clearly $[a) \subseteq F$. Since $(a)^{\tau} = (b)^{\tau}$ and $[a)$ is a coherent filter, we get that $b \in [a) \subseteq F$. Therefore F is a coherent filter.

(5)⇒(6) It is clear.

(6)⇒(7) Assume that every prime filter of L is a coherent filter. Let $a, b \in L$ such that $(a)^{\tau} = (b)^{\tau}$. Suppose $a \neq b$. Then there exists a prime filter P such that $a \in P$ and $b \notin P$. By the hypothesis, P is a coherent filter of L. Since $(a)^{\tau} = (b)^{\tau}$ and $a \in P$, we get $b \in P$, which is a contradiction. Therefore $a = b$.

(7)⇒(8) By Corollary 5(5), it is direct.

(8)⇒(1) Assume condition (8). Then L has a unique dense element. Therefore L is a Boolean algebra. ■

Definition. For any filter F of a lattice L, define $\pi(F)$ as follows

$$\pi(F) = \{x \in L \mid (x)^{\tau} \vee F = L\}.$$

The following lemma is an immediate consequence of the above definition.

Lemma 8. *For any two filters F, G of a lattice L, the following properties hold.*

(1) $F \subseteq G$ *implies* $\pi(F) \subseteq \pi(G)$

(2) $\pi(F \cap G) = \pi(F) \cap \pi(G)$.

Proof. Routine verification. ■

Proposition 9. *For any filter F of a lattice L, $\pi(F)$ is a D-filter of L.*

Proof. Clearly $D \subseteq \pi(F)$. Let $x, y \in \pi(F)$. Then $(x)^{\tau} \vee F = (y)^{\tau} \vee F = L$. Hence $(x \wedge y)^{\tau} \vee F = \{(x)^{\tau} \cap (y)^{\tau}\} \vee F = \{(x)^{\tau} \vee F\} \cap \{(y)^{\tau} \vee F\} = L$. Hence $x \wedge y \in \pi(F)$. Again let $x, \in \pi(F)$ and $x \leq y$. Then $L = (x)^{\tau} \vee F \subseteq (y)^{\tau} \vee F$. Thus $y \in \pi(F)$. Therefore $\pi(F)$ is a D-filter in L. ■

Lemma 10. *Let F be a filter of a lattice L. Then F is a D-filter of L if and only if $\pi(F) \subseteq F$.*

Proof. Assume that F is a D-filter of L. Let $x \in \pi(F)$. Then $(x)^{\tau} \vee F = L$. Hence $x = a \wedge b$ for some $a \in (x)^{\tau} \subseteq (x)^{\circ}$ and $b \in F$. Then $x \vee a \in D \subseteq F$ and $x \vee b \in F$. Thus $x = x \vee x = x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) \in F$. Therefore $\pi(F) \subseteq F$. Converse is clear because of $D \subseteq \pi(F) \subseteq F$. ■

Definition. A filter F of a lattice L is called *strongly coherent* if $F = \pi(F)$.

Proposition 11. *Every strongly coherent filter of a lattice is a coherent filter.*

Proof. Let F be a strongly coherent filter of a lattice L. Clearly F is a D-filter of L. Let $x, y \in L$ be such that $(x)^{\tau} = (y)^{\tau}$ and $x \in F = \pi(F)$. Then $(x)^{\tau} \vee F = L$. Hence $(y)^{\tau} \vee F = L$ and so $y \in \pi(F) = F$. Thus F is a coherent filter of L. ■

For any filter F of a lattice L, it can be noted that $F \subseteq D$ if and only if $F^{\tau\tau} = D$. A D-filter F of a lattice L is called a τ-closed if $F = F^{\tau\tau}$. Clearly D is the smallest τ-closed filter and L is the largest τ-closed filter of the lattice L.

Proposition 12. *Every τ-closed filter of a lattice is a coherent filter.*

Proof. Let F be a τ-closed filter of a lattice L. Let $x, y \in L$ be such that $(x)^{\tau} = (y)^{\tau}$. Suppose $x \in F$. Then, we get that $y \in (y)^{\tau\tau} = (x)^{\tau\tau} \subseteq F^{\tau\tau} = F$. Therefore F is a coherent filter of L. ■

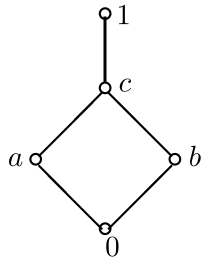
Definition. A lattice L is called a *semi Stone lattice* if $(x)^{\tau} \vee (x)^{\tau\tau} = L$ for all $x \in L$.

Theorem 13. *Every Stone lattice is a semi Stone lattice.*

Proof. Assume that L is a Stone lattice. Let $x \in L$. Suppose $(x)^\tau \vee (x)^{\tau\tau} \neq L$. Then there exists a maximal filter M such that $(x)^\tau \vee (x)^{\tau\tau} \subseteq M$. Then $(x)^\tau \subseteq M$ and $x \in (x)^{\tau\tau} \subseteq M$. Since M is maximal, we get $x^* \notin M$. Since L is Stone, we get $x^* \vee x^{**} = 1$. Hence $x^* \in (x)^\tau$. Thus $(x)^\tau \nsubseteq M$, which is a contradiction. Hence $(x)^\tau \vee (x)^{\tau\tau} = L$. Therefore L is a semi Stone lattice. ∎

The converse of the above theorem is not true. For, consider

Example 14. Consider the following bounded and finite distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



Clearly L is a pseudo-complemented lattice. It can be easily observed that $(a)^\tau = (b)^\tau = D$ and $(c)^\tau = (1)^\tau = L$. Hence $(a)^{\tau\tau} = (b)^{\tau\tau} = L$. Observe that L is a semi Stone lattice. But L is not a Stone lattice because of $a^* \vee a^{**} = b \vee a = c \neq 1$.

Theorem 15. *The following assertions are equivalent in a lattice L:*

(1) *L is a semi Stone lattice,*

(2) *every τ-closed filter is strongly coherent,*

(3) *for each $x \in L, (x)^{\tau\tau}$ is strongly coherent.*

Proof. (1)⇒(2) Assume that L is a semi Stone lattice. Let F be a τ-closed filter of L. Then F is a D-filter with $F^{\tau\tau} = F$. Clearly $\pi(F) \subseteq F$. Conversely, let $x \in F$. It can be easily verified that $(x)^{\tau\tau} \subseteq F^{\tau\tau}$. Hence $L = (x)^\tau \vee (x)^{\tau\tau} \subseteq (x)^\tau \vee F^{\tau\tau} = (x)^\tau \vee F$. Thus $x \in \pi(F)$. Therefore F is strongly coherent.

(2)⇒(3) Since each $(x)^{\tau\tau}$ is τ-closed, it is obvious.

(3)⇒(1) Assume condition (3). Let $x \in L$. Then we get $\pi((x)^{\tau\tau}) = (x)^{\tau\tau}$. Since $x \in (x)^{\tau\tau}$, we get $(x)^\tau \vee (x)^{\tau\tau} = L$. Therefore L is a semi Stone lattice. ∎

4. Median filters

In this section, the notion of a median filter is introduced in lattices. Characterization theorems of median filters are derived for every prime D-filter to become median and every maximal filter to become median.

Proposition 16. *Let P be a prime filter of a lattice L. Then the following assertions are equivalent:*

(1) $D \subseteq P$,

(2) *for any* $x \in L$, $x \in P$ *if and only if* $x^* \notin P$,

(3) *for any* $x \in L$, $x^{**} \in P$ *if and only if* $x \in P$,

(4) *for any* $x, y \in L$ *with* $x^* = y^*$, $x \in P$ *implies that* $y \in P$,

(5) $D \cap (L - P) = \emptyset$.

Proof. (1)⇒(2) Assume that $D \subseteq P$. Suppose $x \in P$. If $x^* \in P$, then $0 = x \wedge x^* \in P$, which is a contradiction. Hence $x^* \notin P$. Conversely, let $x^* \notin P$. Clearly $x \vee x^* \in D \subseteq P$. Since P is prime and $x^* \notin P$, we get $x \in P$.

(2)⇒(3) It is clear.

(3)⇒(4) It is clear.

(4)⇒(5) Assume condition (4). Suppose $x \in D \cap (L - P)$. Then, we get $x^* = 0 = 1^*$ and $x \notin P$. Since $1 \in P$, by (4), we get that $x \in P$ which is a contradiction. Therefore $D \cap (L - P) = \emptyset$.

(5)⇒(1) It is obvious. ∎

Theorem 17. *Let M be a proper filter of a lattice L. The following assertions are equivalent:*

(1) *M is maximal,*

(2) *M is a prime D-filter,*

(3) $x \notin M$ *implies* $x^* \in M$.

Proof. (1)⇒(2) Assume that M is a maximal filter of L. Clearly M is a prime filter. Let $x \in D$. Then, we get $x^* = 0$. Suppose $x \notin M$. Then $M \vee [x) = L$. Hence $0 = m \wedge x$ for some $0 \neq m \in M$. Then $m \leq x^* = 0$, which is a contradiction. Hence $x \in M$. Thus $D \subseteq M$. Therefore M is a prime D-filter.

(2)⇒(3) Assume that M is a prime D-filter of L. Suppose $x \notin M$. Clearly $x \vee x^* \in D \subseteq M$. Since M is prime and $x \notin M$, we have $x^* \in M$.

(3)⇒(1) Assume condition (3). Suppose M is not maximal. Let Q be a proper filter such that $M \subset Q$. Choose $x \in Q - M$. Since $x \notin M$, by (3), we get that $x^* \in M \subset Q$. Therefore $0 = x \wedge x^* \in Q$, which is a contradiction. ∎

From Theorem 17, one can notice that the class of all maximal filters and the class of all prime D-filters are the same. Since every prime D-filter is maximal, we can conclude that every prime D-filter is minimal. Therefore maximal filters, prime D-filter, and minimal prime D-filters are the same in a pseudo-complemented distributive lattice. The notion of median filters is now introduced.

Definition. A maximal filter M of a lattice L is called *median* if to each $x \in M$, there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$.

From Example 14, we initially observe that a maximal filter of a lattice need not be median, For consider the maximal filter $M = \{1, a, c\}$ of L. Notice that for $a \in M$, there is no $x \notin M$ such that $a^{**} \vee x^{**} = 1$. Therefore M is not median.

Lemma 18. *Let M be a maximal filter of a lattice L. For any $x \in L$, it holds*

$$x \notin M \quad \textit{implies} \quad (x)^{\tau} \subseteq M.$$

Proof. Suppose $x \notin M$. Let $a \in (x)^{\tau}$. Then $a^{**} \vee x^{**} = 1$. Hence $(a \vee x)^{**} = 1$ and so $a \vee x \in D \subseteq M$. Since $x \notin M$, we get $a \in M$. Therefore $(x)^{\tau} \subseteq M$. ■

Lemma 19. *Let M be a median filter of a lattice L. For any $x \in L$, we have*

$$x \in M \ \ \textit{if and only if} \ \ (x)^{\tau\tau} \subseteq M.$$

Proof. Suppose that $x \in M$. Let $a \in (x)^{\tau\tau}$. Then, we get $(x)^{\tau} \subseteq (a)^{\tau}$. Since $x \in M$ and M is a median filter, there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$. Then, we get $y \in (x)^{\tau} \subseteq (a)^{\tau}$. Since $y \notin M$, we must have $(y)^{\tau} \subseteq M$. Hence $a \in (a)^{\tau\tau} \subseteq (y)^{\tau} \subseteq M$. Therefore $(x)^{\tau\tau} \subseteq M$. ■

In the following, we derive a characterization theorem of median filters.

Theorem 20. *Let M be a maximal filter of a lattice L. For each $x \in L$, the following assertions are equivalent:*

(1) *M is median,*

(2) *$x \notin M$ if and only if $(x)^{\tau} \subseteq M$,*

(3) *$x^{**} \in M$ implies $(x)^{\tau} \nsubseteq M$.*

Proof. (1)⇒(2) Assume that M is a median filter of L and $x \in L$. Suppose $x \notin M$. By Lemma 18, we have $(x)^{\tau} \subseteq M$. Conversely, assume that $(x)^{\tau} \subseteq M$. Suppose $x \in M$. Since M is median, there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$. Hence $y \in (x)^{\tau} \subseteq M$, which is a contradiction. Therefore $x \notin M$.

(2)⇒(3) Assume condition (2). Let $x \in L$. Suppose $x^{**} \in M$. By Proposition 16, we get $x \in M$. By (2), we get $(x)^{\tau} \nsubseteq M$.

(3)⇒(1) Assume that condition (3) holds. Suppose $x \in M$. Clearly $x^{**} \in M$. By the assumed condition, we get that $(x)^{\tau} \nsubseteq M$. Then there exists $y \in (x)^{\tau}$ such that $y \notin M$. Hence $x^{**} \vee y^{**} = 1$ where $y \notin M$. Therefore M is median. ■

Theorem 21. *Every median filter of a lattice is a coherent filter.*

Proof. Let M be a median filter of a lattice L. Suppose $x, y \in L$ be such that $(x)^\tau = (y)^\tau$ and $x \in M$. Since M is median, there exists $a \notin M$ such that $x^{**} \vee a^{**} = 1$. Hence $a \in (x)^\tau = (y)^\tau$. Thus $1 = y^{**} \vee a^{**} \leq (y \vee a)^{**}$. Hence $(y \vee a)^{**} = 1$, which gives that $(y \vee a)^* = 0$. Thus $y \vee a \in D \subseteq M$. Since M is prime and $a \notin M$, it yields that $y \in M$. Therefore M is a coherent filter. ∎

In the following theorem, we derive a set of equivalent conditions for a Stone lattice to become a Boolean algebra in terms of median filters and maximal filters.

Theorem 22. *Let L be a Stone lattice. Then the following are equivalent:*

(1) *L is a Boolean algebra,*

(2) *every prime filter is maximal,*

(3) *every prime filter is median,*

(4) *every prime filter is a D-filter.*

Proof. (1)⇒(2) It is well known.

(2)⇒(3) Since L is a Stone lattice, it is through.

(3)⇒(4) Since every median filter is a D-filter, it is clear.

(4)⇒(1) Assume condition (3). Then $D \subseteq \bigcap\{P \mid P \text{ is a prime filter}\} = \{1\}$. Hence $D = \{1\}$, which gives $x \vee x^* \in D = \{1\}$. Thus it is through. ∎

Definition. For any maximal filter M of a lattice L, define

$$\Omega(M) = \{x \in L \mid (x)^\tau \nsubseteq M\}.$$

Lemma 23. *For any maximal filter M, $\Omega(M)$ is a D-filter contained in M.*

Proof. Clearly $D \subseteq \Omega(M)$. Let $x, y \in \Omega(M)$. Then $(x)^\tau \nsubseteq M$ and $(y)^\tau \nsubseteq M$. Since M is prime, we get $(x \wedge y)^\tau = (x)^\tau \cap (y)^\tau \nsubseteq M$. Hence $x \wedge y \in \Omega(M)$. Let $x \in \Omega(M)$ and $x \leq y$. Then $(x)^\tau \nsubseteq M$ and $(x)^\tau \subseteq (y)^\tau$. Since $(x)^\tau \nsubseteq M$, we get $(y)^\tau \nsubseteq M$. Thus $y \in \Omega(M)$. Therefore $\Omega(M)$ is a D-filter of L. Now, let $x \in \Omega(M)$. Then, we get $(x)^\tau \nsubseteq M$. Hence there exists $a \in (x)^\tau$ such that $a \notin M$. Since $a \in (x)^\tau$, we get $1 = a^{**} \vee x^{**} \leq (a \vee x)^{**}$. Thus $a \vee x \in D \subseteq M$. Since $a \notin M$, we must have $x \in M$. Therefore $\Omega(M) \subseteq M$. ∎

Let us denote that $\mathcal{M}$ is the set of all maximal filters of a lattice L. For any $a \in L$, we also denote $\mathcal{M}_{a^*} = \{M \in \mathcal{M} \mid a^* \in M\}$.

Theorem 24. *Let L be a lattice and $a \in L$. Then $(a)^\tau \subseteq \bigcap_{M \in \mathcal{M}_{a^*}} \Omega(M)$.*

Proof. Let $x \in (a)^\tau$ and $M \in \mathcal{M}_{a^*}$. Then $x^{**} \vee a^{**} = 1$ and $a^* \in M$. Suppose $a \in M$. Then $0 = a \wedge a^* \in M$, which is a contradiction Hence $a \notin M$. Hence $a \in (x)^\tau$ such that $a \notin M$. Thus $(x)^\tau \nsubseteq M$. Hence $x \in \Omega(M)$. Thus $(a)^\tau \subseteq \Omega(M)$ which is true for all $M \in \mathcal{M}_{a^*}$. Therefore $(a)^\tau \subseteq \bigcap_{M \in \mathcal{M}_{a^*}} \Omega(M)$. ∎

Corollary 25. *Let L be a lattice and $a \in L$. Then $a^* \in M$ implies $(a)^\tau \subseteq \Omega(M)$.*

In Example 14, consider $P = \{1, a, c\}$. Clearly $D = \{1, c\}$ and P is a prime D-filter. For any element $x \in P$, there exists no $y \notin P$ such that $x^{**} \vee y^{**} = 1$. Hence P is not median. However, in the following, some equivalent conditions are derived for every prime D-filter of a lattice to become a median filter.

Theorem 26. *The following conditions are equivalent in a lattice L:*

(1) *L is a Stone lattice,*

(2) *every D-filter is strongly coherent,*

(3) *every maximal filter is strongly coherent,*

(4) *every maximal filter is median,*

(5) *for any $M \in \mathcal{M}$, $\Omega(M)$ is median,*

(6) *for any $a, b \in L$, $a \vee b \in D$ implies $(a)^\tau \vee (b)^\tau = L$,*

(7) *for any $a \in L$, $(a)^\tau \vee (a^*)^\tau = L$.*

Proof. (1)⇒(2) Assume that L is a Stone lattice. Let F be a D-filter of L. Clearly $\pi(F) \subseteq F$. Conversely, let $x \in F$. Since L is a Stone lattice, we get $x^* \vee x^{**} = 1$. Suppose $(x)^\tau \vee F \neq L$. Then there exists a maximal filter M of L such that $(x)^\tau \vee F \subseteq M$. Hence $(x)^\tau \subseteq M$ and $x \in F \subseteq M$. Since M is a prime, we get $x^* \notin M$. Since $x^{***} \vee x^{**} = 1$, we get $x^* \in (x)^\tau \subseteq M$ which is a contradiction. Thus $(x)^\tau \vee F = L$. Therefore F is a strongly coherent filter.

(2)⇒(3) It is obvious.

(3)⇒(4) Assume that every maximal filter is strongly coherent. Let M be a maximal filter of L. Then by our assumption, $\pi(M) = M$. Let $x \in M$. Then $(x)^\tau \vee M = L$. Hence $a \wedge b = 0$ for some $a \in (x)^\tau$ and $b \in M$. Since $a \in (x)^\tau$, we get $a^{**} \vee x^{**} = 1$. Suppose $a \in M$. Then $0 = a \wedge b \in M$, which is a contradiction. Hence $a \notin M$. Therefore M is median.

(4)⇒(5) Assume condition (4). Let $M \in \mathcal{M}$. Clearly $\Omega(M) \subseteq M$. Conversely, let $x \in M$. Since M is median, there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$. Hence $(x)^\tau \nsubseteq M$. Thus $x \in \Omega(M)$. Therefore $\Omega(M) = M$ is a median filter.

(5)⇒(6) Assume condition (5). Let $a, b \in L$ be such that $a \vee b \in D$. Suppose $(a)^\tau \vee (b)^\tau \neq L$. Then there exists a maximal filter M such that $(a)^\tau \vee (b)^\tau \subseteq M$. Since $\Omega(M)$ is median, by Theorem 20, we get

$$\begin{aligned} (a)^\tau \vee (b)^\tau \subseteq M &\Rightarrow (a)^\tau \subseteq M \text{ and } (b)^\tau \subseteq M \\ &\Rightarrow (a)^\tau \subseteq \Omega(M) \text{ and } (b)^\tau \subseteq \Omega(M) \\ &\Rightarrow a \notin \Omega(M) \text{ and } b \notin \Omega(M) \qquad \text{since } \Omega(M) \text{ is median} \\ &\Rightarrow a \vee b \notin M \end{aligned}$$

which is a contradiction to that $a \vee b \in D \subseteq M$. Therefore $(a)^\tau \vee (b)^\tau = L$.

(6)⇒(7) Let $a \in L$. Since $a \vee a^* \in D$, by (6), we are through.

(7)⇒(1) Assume condition (7). Let $x \in L$. Then by (7), we have $(x)^\tau \vee (x^*)^\tau = L$. Hence $0 \in (x)^\tau \vee (x^*)^\tau$. Then $0 = a \wedge b$ for some $a \in (x)^\tau$ and $b \in (x^*)^\tau$. Since $b \in (x^*)^\tau$, we get $b^{**} \vee x^* = 1$, and so $b^* \wedge x^{**} = 0$. Thus $b^* \leq x^*$. Now

$$\begin{aligned} 1 &= a^{**} \vee x^{**} && \text{since } a \in (x)^\tau \\ &\leq b^* \vee x^{**} && \text{since } a \wedge b = 0 \\ &\leq x^* \vee x^{**} && \text{since } b^* \leq x^* \end{aligned}$$

which gives that $x^* \vee x^{**} = 1$. Therefore L is a Stone lattice. ∎

For any filter F of a lattice L, we denote $\mathcal{M}_F = \{M \in \mathcal{M} \mid F \subseteq M\}$.

Theorem 27. *For any filter F of a lattice $L, \pi(F) = \bigcap_{M \in \mathcal{M}_F} \Omega(M)$.*

Proof. Let $x \in \pi(F)$ and $F \subseteq M$ where $M \in \mathcal{M}$. Then $L = (x)^\tau \vee F \subseteq (x)^\tau \vee M$. Suppose $(x)^\tau \subseteq M$, then $M = L$, which is a contradiction. Hence $(x)^\tau \nsubseteq M$. Thus $x \in \Omega(M)$ for all $M \in \mathcal{M}_F$. Therefore $\pi(F) \subseteq \bigcap_{M \in \mathcal{M}_F} \Omega(M)$. Conversely, let $x \in \bigcap_{M \in \mathcal{M}_F} \Omega(M)$. Then, we get $x \in \Omega(M)$ for all $M \in \mathcal{M}_F$. Suppose $(x)^\tau \vee F \neq L$. Then there exists a maximal filter M_0 such that $(x)^\tau \vee F \subseteq M_0$. Hence $(x)^\tau \subseteq M_0$ and $F \subseteq M_0$. Since $F \subseteq M_0$, by hypothesis, we get $x \in \Omega(M_0)$. Thus $(x)^\tau \nsubseteq M_0$, which is a contradiction. Therefore $(x)^\tau \vee F = L$. Thus $x \in \pi(F)$. Therefore $\bigcap_{M \in \mathcal{M}_F} \Omega(M) \subseteq \pi(F)$. ∎

Theorem 28. *The following assertions are equivalent in a lattice L:*

(1) *L is a Stone lattice,*

(2) *for any $M \in \mathcal{M}$, $\Omega(M)$ is maximal,*

(3) *for any $F, G \in \mathcal{F}(L)$, $F \vee G = L$ implies $\pi(F) \vee \pi(G) = L$,*

(4) *for any $F, G \in \mathcal{F}(L)$, $\pi(F) \vee \pi(G) = \pi(F \vee G)$,*

(5) *for any two distinct maximal filters M, N, $\Omega(M) \vee \Omega(N) = L$,*

(6) *for any $M \in \mathcal{M}$, M is the unique member of μ such that $\Omega(M) \subseteq M$.*

Proof. (1)⇒(2) Assume that L is a Stone lattice. Let $M \in \mathcal{M}$. Clearly, we have $\Omega(M) \subseteq M$. Conversely, let $x \in M$. Since L is a Stone lattice, By Theorem 26, we get that M is a median filter. Then there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$. Hence $y \in (x)^\tau$ and $y \notin M$. Thus $(x)^\tau \nsubseteq \Omega(M)$. Hence $x \in \Omega(M)$. Therefore $\Omega(M) = M$ is a maximal filter.

(2)⇒(3) Assume condition (2). Clearly $\Omega(M) = M$ for all $M \in \mathcal{M}$. Let $F, G \in \mathcal{F}(L)$ be such that $F \vee G = L$. Suppose $\pi(F) \vee \pi(G) \neq L$. Then there exists a maximal filter M of L such that $\pi(F) \vee \pi(G) \subseteq M$. Hence $\pi(F) \subseteq M$ and $\pi(G) \subseteq M$. Now, we get

$$\begin{aligned}
\pi(F) \subseteq M &\Rightarrow \bigcap_{M \in \mathcal{M}_F} \Omega(M) \subseteq M \\
&\Rightarrow \Omega(M_i) \subseteq M \quad \text{for some } M_i \in \mathcal{M}_F \text{ (since } M \text{ is prime)} \\
&\Rightarrow M_i \subseteq M \qquad \text{By condition (2)} \\
&\Rightarrow F \subseteq M.
\end{aligned}$$

Similarly, we can get $G \subseteq M$. Hence $L = F \vee G \subseteq M$, which is a contradiction. Therefore $\pi(F) \vee \pi(G) = L$.

(3)⇒(4) Assume condition (3). Let $F, G \in \mathcal{F}(L)$. Clearly, we have $\pi(F) \vee \pi(G) \subseteq \pi(F \vee G)$. Let $x \in \pi(F \vee G)$. Then $((x)^\tau \vee F) \vee ((x)^\tau \vee G) = (x)^\tau \vee F \vee G = L$. Hence by condition (3), we get that $\pi((x)^\tau \vee F) \vee \pi((x)^\tau \vee G) = L$. Thus $x \in \pi((x)^\tau \vee F) \vee \pi((x)^\tau \vee G)$. Hence $x = r \wedge s$ for some $r \in \pi((x)^\tau \vee F)$ and $s \in \pi((x)^\tau \vee G)$. Now, we have

$$\begin{aligned}
r \in \pi((x)^\tau \vee F) &\Rightarrow (r)^\tau \vee (x)^\tau \vee F = L \\
&\Rightarrow L = ((r)^\tau \vee (x)^\tau) \vee F \subseteq (r \vee x)^\tau \vee F \\
&\Rightarrow (r \vee x)^\tau \vee F = L \\
&\Rightarrow r \vee x \in \pi(F).
\end{aligned}$$

Similarly, we can get $s \vee x \in \pi(G)$. Hence

$$\begin{aligned}
x &= x \vee x \\
&= x \vee (r \wedge s) \\
&= (x \vee r) \wedge (x \vee s) \in \pi(F) \vee \pi(G).
\end{aligned}$$

Hence $\pi(F \vee G) \subseteq \pi(F) \vee \pi(G)$. Therefore $\pi(F) \vee \pi(G) = \pi(F \vee G)$.

(4)⇒(5) Assume condition (4). Let M, N be two distinct maximal filters of L. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$ and $y \notin M$, we get $x^* \in N$ and $y^* \in M$. Hence $(x \wedge y^*) \wedge (y \wedge x^*) = (x \wedge x^*) \wedge (y \wedge y^*) = 0$. Then

$$\begin{aligned}
L &= \pi(L) \\
&= \pi([0)) \\
&= \pi([(x \wedge y^*) \wedge (y \wedge x^*))) \\
&= \pi([x \wedge y^*) \vee [y \wedge x^*)) \\
&= \pi([x \wedge y^*)) \vee \pi([y \wedge x^*)) \qquad \text{By condition (4)} \\
&\subseteq \Omega(M) \vee \Omega(N) \qquad \text{since } [x \wedge y^*) \subseteq M, [y \wedge x^*) \subseteq N.
\end{aligned}$$

Therefore $\Omega(M) \vee \Omega(N) = L$.

(5)⇒(6) Assume condition (5). Let $M \in \mathcal{M}$. Clearly $\Omega(M) \subseteq M$. Suppose $N \in \mathcal{M}$ such that $N \neq M$ and $\Omega(M) \subseteq N$. Since $\Omega(N) \subseteq N$, by hypothesis, we get $L = \Omega(M) \vee \Omega(N) \subseteq N$, which is a contradiction. Therefore M is the unique maximal filter of L such that $\Omega(M)$ is contained in M.

(6)⇒(1) Let M be a maximal filter of L. Suppose $\Omega(M) \neq M$. Then there exists a maximal filter M_0 such that $\Omega(M) \subseteq M_0$, which contradicts uniqueness of M. Hence $\Omega(M) = M$. Let $x \in M = \Omega(M)$. Then there exists $y \notin M$ such that $x^{**} \vee y^{**} = 1$. Hence M is median. By Theorem 26, L is a Stone lattice. ■

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