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THE PARTIAL MANY-SORTED ALGEBRAS OF TERMS AND FORMULAS WITH FIXED VARIABLES COUNT

Thodsaporn Kumduang

and

Sorasak Leeratanavalee*

Research Group in Mathematics and Applied Mathematics
Department of Mathematics, Faculty of Science
Chiang Mai University
Chiang Mai 50200, Thailand

e-mail: kumduang01@gmail.com
sorasak.l@cmu.ac.th

Abstract

Terms and formulas, which are formal expressions in first and second order languages obtained by alphabets, operation symbols, and relation symbols, are used to study algebras and algebraic systems. In this paper, we introduce the notion of terms with fixed variables count. The partial many-sorted superposition operations of such terms and their partial many-sorted algebra satisfying clone axioms as weak identities are presented. We also extend our structures from algebras to algebraic systems via the concept of formulas with fixed variables count. Conditions for the set of such formulas to be closed under taking of superposition of formulas are determined. We construct the partial many-sorted algebra of formulas with fixed variables count and investigate its satisfaction by clone axioms. Finally, we prove that such partial structure is isomorphic to some Menger systems of the same rank of partial multiplace functions.

Keywords: partial many-sorted algebra, term, formula, partial operation, representation.

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*Corresponding author.

1. Introduction and preliminaries

For $n \in \mathbb{N} := \{1, 2, \ldots\}$, by $O^n(A)$ we denote the set of n-ary operations on a nonempty set A and $O(A) := \bigcup_{n\in\mathbb{N}} O^n(A)$. Any subset of $O(A)$ is called a clone (or clone of operations) if it contains all n-ary proje ction operations pr_i^n on A which are defined by $\mathrm{pr}_i^n(a_1, \ldots, a_n) = a_i$ for all $i = 1, \ldots, n$ and is closed under the following composition of operations: For $f \in O^n(A)$, $g_1, \ldots, g_n \in O^m(A)$, the m-ary operation $f(g_1, \ldots, g_n)$ (sometimes written as $f \circ (g_1, \ldots, g_n)$, $\mathcal{O}_m^n(f, g_1, \ldots, g_n)$, or $f[g_1 \cdots g_n]$), which is defined by

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)).$$

Actually, the notion of a clone is essential in many parts of universal algebra and theoretical computer science. Two elementary examples of clones are now proposed. The first one is the full clone $O(A)$ of all finitary non-nullary operations on A, particularly, the trivial clone which consists of the set $J(A)$ of all projection operations defined on A. Another one is the clone $(X, \mathcal{T})$ of all continuous operations on a given topological space $(X, \mathcal{T})$. For a brief knowledge on the study of clones, see [16]. Other applications of clones in different aspects were investigated, for example, in [17, 36].

One of the important concepts connecting with clones is the concept of many-sorted algebras. To attain this concept, the S-sorted sets $A = (A_s)_{s\in S}$ are essential. In this case, the set S is called a set of sorts. Moreover, the sort mapping $\phi : A \to B$ from an S-sorted set $A = (A_s)_{s\in S}$ to an S-sorted set $B = (B_s)_{s\in S}$ is an S-sorted family $\phi : (\phi_s)_{s\in S}$ of mappings $\phi_s : A_s \to B_s$ where $s \in S$. Recent developments in many-sorted algebras may be seen in [6, 24]. Accordingly, we can describe clones in the sense of many-sorted algebras as follows: Let $\mathbb{N}$ be the set of sorts and let $(O^n(A))_{n\in\mathbb{N}}$ be a sorted set. A composition operation $\mathcal{O}_m^n$ where $n, m \in \mathbb{N}$,

$$\mathcal{O}_m^n : O^n(A) \times (O^m(A))^m \to O^m(A)$$

can be defined by $\mathcal{O}_m^n(f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$ where $f(g_1, \ldots, g_n)$ was already known. The projection operations pr_i^n, $i \leq n, n \in \mathbb{N}$ act as nullary operations. Consequently, the many-sorted algebra

$$\text{clone } A = ((O^n(A))_{n\in\mathbb{N}}, (\mathcal{O}_m^n)_{n,m\in\mathbb{N}}, (\mathrm{pr}_i^n)_{i\leq n, n\in\mathbb{N}}),$$

is formed.

The *variety of all abstract clones*, denoted by K_0, is a family of $\mathbb{N}$-sorted algebras satisfying the following three identities:

(C1) $\tilde{S}_m^n(\tilde{S}_n^p(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p), \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{S}_m^p(\tilde{Z}, \tilde{S}_m^n(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}_m^n(\tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n)), m, n, p \in \mathbb{N}$;

(C2) $\tilde{S}_m^n(\lambda_j, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_j, n, m \in \mathbb{N}, 1 \leq j \leq n$;

(C3) $\tilde{S}^n_n(\tilde{Y}, \lambda_1, \ldots, \lambda_n) \approx \tilde{Y}, n \in \mathbb{N}$;

where $\tilde{S}^n_m$, $\tilde{S}^p_n$, $\tilde{S}^p_m$, $\tilde{S}^n_n$ are operation symbols, $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_p, \tilde{X}_1, \ldots, \tilde{X}_n, \tilde{Y}$ are variables for terms, and λ_j are symbols for variables. In general, (C1) is said to be the *superassociative law* since it generalizes the associative law. In fact, if we set $m = n = p = 1$, one can reduce it to the associative law of the form $\cdot(\cdot(a, b), c) = \cdot(a, \cdot(b, c))$. Each member of the variety K_0 is called an abstract clone. Note that every concrete clone can be considered as an abstract clone. For the converse, every abstract clone is isomorphic to a concrete clone. This result may be regarded as a generalization of Cayley's theorem for groups and semigroups, which were stated that every group is isomorphic to a permutation group and every semigroup can be isomorphically embedded into some transformation semigroup.

From an applied point of view, clones are important in both, computational complexity due to their connection to constraint satisfaction problems (CSPs) and computer science. In universal algebras, clones were naturally studied in various aspects. One of outstanding structures is the clone of term operations or term algebras. Actually, the wide application of terms or trees as a natural tool in computer science leads us to consider its theoretical basics. To mention them, we present several basic ideas of terms concerning their definitions, operations and structures which will be used in the sequel. Terms may be regarded as words formed by letters. A construction of tree expressions by terms were proposed in [14]. An equational approach to tree transformations induced by terms were explored in [2]. Let I be a nonempty indexed set and $(f_i)_{i\in I}$ be a sequence of operation symbols. To every operation symbol f_i, we assign a natural number $n_i \in \mathbb{N}$, called the arity of f_i. The sequence $\tau := (n_i)_{i\in I}$ is called a *type*. We denote by $X_n := \{x_1, \ldots, x_n\}$ a finite set called an *alphabet* and its elements are called *variables*. The set of all *n-ary terms of type* τ is the smallest set $W_\tau(X_n)$ inductively defined by these two items:

(1) $X_n \subseteq W_\tau(X_n)$ and

(2) If $t_1, \ldots, t_{n_i} \in W_\tau(X_n)$ and f_i is an operation symbol of the arity n_i then $f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X_n)$. We denote by $W_\tau(X) := \bigcup_{n\in\mathbb{N}} W_\tau(X_n)$ the set of all terms of type τ. The most important operation on terms is a superposition operation. For each natural numbers $m, n \geq 1$, the superposition operation is a many-sorted mapping

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

defined on the structure of $s \in W_\tau(X_n)$ by

(1) for $s = x_i, 1 \leq j \leq n$, $S^n_m(x_j, t_1, \ldots, t_n) := t_j$,

(2) for $s = f_i(s_1, \ldots, s_{n_i}), S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_n), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n))$.

Then the many-sorted algebra can be defined by

$$\text{clone}(\tau)= \big((W_\tau(X_n))_{n\in\mathbb{N}^+}, (S^n_m)_{n,m\in\mathbb{N}^+}, (x_i)_{i\leq n\in\mathbb{N}^+}\big),$$

which is called the *clone of all terms of type* τ. In this case, the variables $x_1,\ldots,x_n$ act as the nullary operations. It turns out that clone(τ) satisfies the axioms (C1)–(C3), and hence it belongs to the variety K_0. Current topics in the construction of new terms have been extensively studied, for example, linear terms [9] and their extensions [25], full terms induced by order-decreasing transformations [34] and terms with a fixed variable [33]. Stable varieties of semigroups characterized by terms were obtained in [30]. Furthermore, a mapping whose range is the set of different kinds of terms was presented in [5, 22, 23].

In [12], functional measurements of a complexity of terms and various formulas for measuring the complexity of a new term were introduced. The variable count and the operation symbol count are two fundamental tools used to study of measurements of terms. The *variables count of a term* t is the total number of occurring variables in t, and is denoted by $\text{vb}(t)$. If t is a variable, then $\text{vb}(t) = 1$ and if $t = f_i(t_1,\ldots,t_{n_i})$, then $\text{vb}(t) = \sum_{j=1}^{n_i}\text{vb}(t_j)$. The *operation symbol count of a term* t, denoted by *op*(t), is the total number of occurring operation symbols in t. If t is a variable, then $\text{op}(t) = 0$ and if $t = f_i(t_1,\ldots,t_{n_i})$, then $\text{op}(t) = 1 + \sum_{j=1}^{n_i}\text{op}(t_j)$. The x_k*-variable count* of t for $k \in \{1,\ldots,n\}$, denoted by $\text{vb}_k(t)$, is defined by $\text{vb}_k(x_k) = 1$; if t is a variable or if x_k does not occur in t, then $\text{vb}_k(t) = 0$, and $\text{vb}_k(t) = \sum_{j=1}^{n_i}\text{vb}_k(t_j)$ if $t = f_i(t_1,\ldots,t_{n_i})$ and $t_1,\ldots,t_{n_i} \in W_\tau(X_n)$. Several formulas for the complexity of the input terms $s,t_1,\ldots,t_n$ under the many-sorted superposition operation of terms were constructed. In fact, it was proved that $\text{vb}(S^n_m(s,t_1,\ldots,t_n)) = \sum_{j=1}^{n}\text{vb}_j(s)\text{vb}(t_j)$ and $\text{vb}_k(S^n_m(s,t_1,\ldots,t_n)) = \sum_{j=1}^{n}\text{vb}_j(s)\text{vb}_k(t_j)$, see [7]. Recently, in [1], the concept of the total number of both occurring variables and occurring operation symbols in a term t which is called the *length of the term* t, and denoted by $\text{len}(t)$. It can be normally defined inductively by $\text{len}(t) = 1$ if t is a variable and $\text{len}(t) = \sum_{j=1}^{n_i}\text{len}(t_j) + 1$ if $t = f_i(t_1,\ldots,t_{n_i})$.

One of outstanding structures that plays a vital role in the first and second order languages considering in theoretical computer science is an algebraic system. It is a triplet consisting of a nonempty set A, a sequence of n_i-ary operations defined on A, and a sequence of n_j-ary relations on A. Normally, we may write $\mathcal{A} = (A,(f_i^{\mathcal{A}})_{i\in I},(\gamma_j^{\mathcal{A}})_{j\in J})$ for an algebraic system of type (τ,τ') where $\tau = (n_i)_{i\in I}$ and $f_i^{\mathcal{A}} : A^{n_i} \to A$ for each $i \in I$ and $\tau' = (n_j)_{j\in J}$ and $\gamma_j^{\mathcal{A}} \subseteq A^{n_j}$ for each $j \in J$. We remark here that if a sequence of n_j-ary relations on A is not defined this structure is reduced to an original algebra of type τ, i.e., $\mathcal{A} = (A,(f_i^{\mathcal{A}})_{i\in I})$. For extensive information of algebraic systems, the reader is referred to the monograph of Malcev [26]. Among recent contributions in algebraic systems are [27, 28, 29, 37]. To investigate several properties of algebraic systems of type (τ, τ'), we need the concept of formulas. Recall from [11] that for $n \in \mathbb{N}$ an *n-ary formula of type* (τ, τ') is defined in the following way.

(1) If t_1, t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ').

(2) If $j \in J$ and $t_1, \ldots, t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \ldots, t_{n_j})$ is an n-ary formula of type (τ, τ').

(3) If F is an n-ary formula of type (τ, τ'), then $\neg F$ is an n-ary formula of type (τ, τ').

(4) If F_1 and F_2 are n-ary formulas of type (τ, τ'), then $F_1 \vee F_2$ is an n-ary formula of type (τ, τ').

(5) If F is an n-ary formula of type (τ, τ') and $x_i \in X_n$, then $\exists x_i(F)$ is an n-ary formula of type (τ, τ').

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $\mathcal{F}_{(\tau,\tau')}(W_\tau(X)) := \bigcup_{n\in\mathbb{N}} \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ, τ') and the set of all formulas of type (τ, τ'), respectively. Actually, several classes of formulas of a given type and their related topics were presented by many authors, for instance, linear formulas [8], C-formulas [4], and formulas with fixed variables [32]. The links among formulas, semantic and syntactic families were described in [31].

This paper is organized as follows: First, in Section 2, we introduce the concept of terms in which variables count is equal and then we investigate the conditions under which these defined terms are closed with respect to the superposition operation. Additionally, the many-sorted algebra of such terms is constructed. We continue our study in Section 3 with describing the total number of occurring variables in any formula. This allows us to form the partial clone consisting of a sequence of such formulas, a sequence of the partial operations defined on those sets, and a sequence of nullary operations. Finally, a representation theorem in our obtained structure by partial multiplace functions is given.

2. Terms with fixed variables count and their partial operations

This section aims to introduce a new term defined by variables count under some conditions.

Definition. An n-ary term with fixed variables count of type τ is inductively defined by:

(1) Every variable x_i in X_n is an n-ary term with fixed variables count of type τ.

(2) If $t_1, \ldots, t_{n_i}$ are n-ary terms with fixed variables count of type τ, and if $\mathrm{vb}(t_k) = \mathrm{vb}(t_l)$ for all $1 \leq k < l \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term with fixed variables count of type τ.

Let $W_\tau^{fvc}(X_n)$ be the set of all n-ary terms with fixed variables count of type τ and let $W_\tau^{fvc}(X) := \bigcup_{n\in\mathbb{N}} W_\tau^{fvc}(X_n)$ be the set of all terms with fixed variables count of type τ.

Now we will present some examples of terms with fixed variables count of some types.

Example 1. Let $\tau = (2)$ be the type with one binary operation symbol f and the set of variables X_3. Then some examples of ternary terms with fixed variables count of type (2) are $x_1, x_2, x_3, f(x_1, x_3), f(f(x_1, x_1), f(x_2, x_2))$. But the following are not $f(x_1, f(x_3, x_2)), f(f(f(x_1, x_1), f(x_2, x_2)), f(x_2, x_3))$.

Example 2. We consider the type $\tau = (3, 2)$ with one ternary operation symbol and one binary operation symbol, say f and g, repectively. Then

$$x_1, x_2, x_3, f(x_3, x_2, x_1), f(g(x_1, x_1), g(x_3, x_2), g(x_1, x_3)) \in W_{(3,2)}^{fvc}(X_3),$$

$$x_1, x_2, x_3, x_4, f(x_4, x_1, x_2), g(x_3, x_2), g(f(x_4, x_3, x_3), f(x_1, x_2, x_2)) \in W_{(3,2)}^{fvc}(X_4),$$

but

$$f(g(x_1, x_1), x_2, x_3), g(x_2, f(x_1, x_3, x_3)) \notin W_{(3,2)}^{fvc}(X_3),$$

$$f(x_2, g(x_4, x_1), x_2), g(f(x_1, g(x_4, x_1), x_4), f(x_1, x_2, x_2)) \notin W_{(3,2)}^{fvc}(X_4).$$

Generally, it turns out that the set $W_\tau^{fvc}(X_n)$ of all n-ary terms with fixed variables count of type τ is not closed under the usual superposition operations S_m^n of terms. As an example we consider the type $\tau = (2)$ with the binary operation symbol f and the superposition S_2^2. Then $S_2^2(f(x_2, x_1), f(x_1, x_2), x_1) = f(x_1, f(x_1, x_2))$ is not a binary term with fixed variables count of type (2), although $f(x_1, x_2)$ and x_1 are binary terms with fixed variables count.

In order to ensure that the superposition operation of terms can be applied to the set $W_\tau^{fvc}(X_n)$, for all $n \geq 1$, some essential conditions are considered. Clearly, $S_m^n(t, s_1, \ldots, s_n)$ is again an m-ary term with fixed variables count of type τ if t is a variable from X_n. Otherwise, we prove

Lemma 3. *If $f_i(t_1, \ldots, t_{n_i}) \in W_\tau^{fvc}(X_n), s_1, \ldots, s_n \in W_\tau^{fvc}(X_m)$, and $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, then*

$$S_m^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n) \in W_\tau^{fvc}(X_m).$$

Proof. Following the definition of the usual superposition S_m^n, we have

$$S_m^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n) = f_i(S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_{n_i}, s_1, \ldots, s_n)).$$

We first prove that each term $S_m^n(t_j, s_1, \ldots, s_n)$ belongs to the set $W_\tau^{fvc}(X_m)$ for $1 \leq j \leq n_i$. For this, let $1 \leq j \leq n_i$, we substitute the terms from $\{s_1, \ldots, s_n\}$

for the variables that appear in t_k, for $1 \leq k \leq n_i$. Since $t_j \in W_\tau^{fvc}(X_n)$, by the assumption, and $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, we have that $S_m^n(t_j, s_1, \ldots, s_n)$ is an m-ary term with fixed variables count of type τ.

Now we prove that the equations $\mathrm{vb}(S_m^n(t_j, s_1, \ldots, s_n)) = \mathrm{vb}(S_m^n(t_k, s_1, \ldots, s_n))$ hold for all $1 \leq j < k \leq n_i$. According to the hypothesis, we have $f_i(t_1, \ldots, t_{n_i}) \in W_\tau^{fvc}(X_n)$, which means $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$ for all $1 \leq j < k \leq n_i$. Then, we obtain $\sum_{i=1}^n \mathrm{vb}_i(t_j) = \sum_{i=1}^n \mathrm{vb}_i(t_k)$. Applying the condition $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, we have $\sum_{i=1}^n \mathrm{vb}_i(t_j)\mathrm{vb}(s_i) = \sum_{i=1}^n \mathrm{vb}_i(t_k)\mathrm{vb}(s_i)$ for all $1 \leq j < k \leq n$. It follows directly from the formula for counting a number of occurring variables in a composition S_m^n that $\mathrm{vb}(S_m^n(t_j, s_1, \ldots, s_n)) = \mathrm{vb}(S_m^n(t_k, s_1, \ldots, s_n))$. As a consequence, the resulting term $S_m^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n)$ belongs to the set $W_\tau^{fvc}(X_m)$. ■

The previous result presented in Lemma 3 allows us to define a many-sorted partial mapping on the sequence $(W_\tau^{fvc}(x_n))_{n\geq 1}$. In fact, for $t \in W_\tau^{fvc}(X_n)$ and $s_1, \ldots, s_n \in W_\tau^{fvc}(X_m)$, the many-sorted partial mapping

$$\overline{S}_m^n : W_\tau^{fvc}(X_n) \times \left(W_\tau^{fvc}(X_m)\right)^n \multimap\rightarrow W_\tau^{fvc}(X_m)$$

is inductively defined by

$$\overline{S}_m^n(t, s_1, \ldots, s_n) := \begin{cases} S_m^n(t, s_1, \ldots, s_n), & \text{if } \mathrm{vb}(s_j) = \mathrm{vb}(s_k) \\ & \text{for all } 1 \leq j < k \leq n; \\ \text{not defined}, & \text{otherwise.} \end{cases}$$

As a consequence, by applying all of these preparations, the many-sorted partial algebra

$$\mathrm{clone}^{fvc}(\tau) := \left(\left(W_\tau^{fvc}(x_n)\right)_{n\in\mathbb{N}}, (\overline{S}_m^n)_{n,m\in\mathbb{N}}, (x_i)_{i\leq n, n\in\mathbb{N}}\right),$$

which is called the *partial clone of terms with fixed variables count* of type τ, is constructed.

Our next aim is to examine the fact that the superassociativity and other clone axioms are valid in this partial algebra. For this, the concept of weak identities is given. We recall from [7] that an equation $s \approx t$ is said to be a weak identity in an algebra $\mathcal{A}$ if one side is defined then another side is also defined and both sides are equal. An excellent overview of partial algebras can be found in [3, 15].

Theorem 4. *The many-sorted partial operations* $(\overline{S}_m^n)_{n,m\in\mathbb{N}}$ *on the many-sorted partial algebra* $\mathrm{clone}^{fvc}(\tau)$ *satisfy* (C1)–(C3) *as weak identities.*

Proof. To prove (C1), we let $t_1,\ldots,t_p \in W_\tau^{fvc}(X_n), s_1,\ldots,s_n \in W_\tau^{fvc}(X_m)$ and $t \in W_\tau^{fvc}(X_p)$. Suppose that the left-hand side is defined. We have $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$ for all $1 \le j < k \le p$ and $\mathrm{vb}(s_l) = \mathrm{vb}(s_r)$ for all $1 \le l < r \le n$. It follows that $\overline{S}^n_m(\overline{S}^p_n(t,t_1,\ldots,t_p),s_1,\ldots,s_n)$ equal to $S^n_m(S^p_n(t,t_1,\ldots,t_p),s_1,\ldots,s_n)$. Furthermore, for each $1 \le j \le p$, the partial superposition $\overline{S}^n_m(t_j,s_1,\ldots,s_n)$ is defined and equal to $S^n_m(t_j,s_1,\ldots,s_n)$. For $1 \le j < k \le p$, by our assumption, we obtain $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$, which means $\sum_{i=1}^n \mathrm{vb}_i(t_j) = \sum_{i=1}^n \mathrm{vb}_i(t_k)$. Particularly, we obtain $\sum_{i=1}^n \mathrm{vb}_i(t_j)\mathrm{vb}(s_i) = \sum_{i=1}^n \mathrm{vb}_i(t_k)\mathrm{vb}(s_i)$. As a consequence, we have $\mathrm{vb}(S^n_m(t_j,s_1,\ldots,s_n)) = \mathrm{vb}(S^n_m(t_k,s_1,\ldots,s_n))$. Therefore, the right-hand side of (C1) is defined. In fact, we have

$$\begin{aligned}&\overline{S}^p_m(t,\overline{S}^n_m(t_1,s_1,\ldots,s_n),\ldots,\overline{S}^n_m(t_p,s_1,\ldots,s_n))\\&= S^p_m(t,S^n_m(t_1,s_1,\ldots,s_n),\ldots,S^n_m(t_p,s_1,\ldots,s_n)).\end{aligned}$$

Because S^n_m satisfies (C1), our aim is directly obtained. To prove (C2), assume that $t_1,\ldots,t_p$ are elements in $W_\tau^{fvc}(x_n)$ and $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$ for all $1 \le j < k \le p$. Then the left-hand side of (C2) is defined. Thus, according to the definition of S^n_m, we conclude $\overline{S}^n_m(x_i,t_1,\ldots,t_n) = S^n_m(x_i,t_1,\ldots,t_n) = t_i$. Finally,(C3) is proved. Observe that $\mathrm{vb}(x_1) = \cdots = \mathrm{vb}(x_n) = 1$. Then $\overline{S}^n_n(t,x_1,\ldots,x_n) = S^n_n(t,x_1,\ldots,x_n) = t$. The proof is finished. ■

3. A particular class of formulas and its partial structures

The reader may have noticed that the study of algebras presented in the previous section were all defined via terms with fixed variables count. In this section, we aim to extend our study to formulas with fixed variables count in algebraic systems. To continue this intention, the superposition operations R^n_m, $n,m \ge 1$ of formulas are recalled.

The operations on sets of formulas were introduced in [11]

$$R^n_m : W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)) \times (W_\tau(X_m))^n \to W_\tau(X_m) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_m))$$

are defined in the following way:

(1) If $t \in W_\tau(X_n)$, then $R^n_m(t,s_1,\ldots,s_n) := S^n_m(t,s_1,\ldots,s_n)$.

(2) If $t_1 \approx t_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n_m(t_1 \approx t_2,s_1,\ldots,s_n)$ is the formula

$$R^n_m(t_1,s_1,\ldots,s_n) \approx R^n_m(t_2,s_1,\ldots,s_n).$$

(3) If $\gamma_j(t_1,\ldots,t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n_m(\gamma_j(t_1,\ldots,t_{n_j}),s_1,\ldots,s_n)$ is the formula $\gamma_j(R^n_m(t_1,s_1,\ldots,s_n),\ldots,R^n_m(t_{n_j},s_1,\ldots,s_n))$.

(4) If $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n_m(\neg F, s_1, \ldots, s_n)$ is the formula

$$\neg R^n_m(F, s_1, \ldots, s_n).$$

(5) If $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n_m(F_1 \vee F_2, s_1, \ldots, s_n)$ is the formula

$$R^n_m(F_1, s_1, \ldots, s_n) \vee R^n_m(F_2, s_1, \ldots, s_n).$$

(6) If $\exists\, x_i(F) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n_m(\exists\, x_i(F), s_1, \ldots, s_n)$ is the formula

$$\exists\, x_i(R^n_m(F, s_1, \ldots, s_n)).$$

These operations induce the many-sorted algebra

$$\text{Formclone}(\tau,\tau') := ((W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)))_{n\in\mathbb{N}}, (R^n_m)_{m,n\in\mathbb{N}}, (x_i)_{1\le i\le n, i, n\in\mathbb{N}}),$$

which is called the *formula-term clone of type* (τ,τ'). It was proved that the algebra Formclone(τ,τ') satisfies the equations (C1)–(C3). Thus it belongs to the variety K_0.

We seperate the presentation of this section into two parts: Construction of the partial many-sorted algebra of formulas with fixed variables count and their representations via partial multiplace functions.

3.1. Partial clone of formulas with fixed variables count

The first aim of this section is to set formulas with fixed variables count which have not been properly defined before and then to define their operations.

Based on a useful concept of $\text{vb}(t)$ where t is a term, one can present its generalizations in algebraic systems as follows:

Definition. For any formula F, the *variable count* of F, denoted by $\text{vb}(F)$, is the total number of occurring variables in F. This can be defined inductively by $\text{vb}(F) = \sum_{k=1}^{2} \text{vb}(t_k)$ if F is an equation $t_1 \approx t_2$, $\text{vb}(F) = \sum_{k=1}^{n_j} \text{vb}(t_k)$ if $F = \gamma_j(t_1, \ldots, t_{n_j})$, $\text{vb}(\neg F) = \text{vb}(F)$, $\text{vb}(F_1 \vee F_2) = \sum_{k=1}^{2} \text{vb}(F_k)$, and $\text{vb}(\exists x_i(F)) = \text{vb}(F)$.

Example 5. We now consider the type $(\tau,\tau') = ((3),(3,2))$ with a ternary operation symbol f and two relation symbols of arities 3 and 2, say γ_1 and γ_2, respectively. Then we have $\text{vb}(f(x_1,x_5,x_2) \approx x_2) = 4$, $\text{vb}(F) = 5$ if $F = \gamma_1(x_1, f(x_2,x_2,x_1), x_4)$, $\text{vb}(F) = 2$ if $F = \neg(\gamma_2(x_3,x_3))$, $\text{vb}(F) = 11$ if $F = (f(x_2,x_2,x_3) \approx f(x_1,x_2,x_8)) \vee \neg(\gamma_1(f(x_1,x_2,x_3),x_4,x_6))$.

Similarly, the situation for the x_k-variable count of a formula F is extended.

Definition. Let F be an n-ary formula of type (τ,τ'). For each variable x_k, the *x_k-variable count* $\mathrm{vb}_k(F)$ of F is defined by inductively as follows: $\mathrm{vb}_k(t_1 \approx t_2) = \sum_{i=1}^{2} \mathrm{vb}_k(t_i)$, $\mathrm{vb}_k(F) = \sum_{i=1}^{n_j} \mathrm{vb}_k(t_i)$ if $F = \gamma_j(t_1,\ldots,t_{n_j})$, $\mathrm{vb}_k(\neg F) = \mathrm{vb}_k(F)$, $\mathrm{vb}_k(F_1 \vee F_2) = \sum_{i=1}^{2} \mathrm{vb}_k(F_i)$, and $\mathrm{vb}(\exists x_i(F)) = \mathrm{vb}_k(F)$.

Example 6. Let $(\tau,\tau') = ((2),(3,1))$ be a type with a binary operation symbol g and two relation symbols of arities 3 and 1, say β_1,β_2, repectively. We first consider the following two formulas $F := g(x_4, g(x_1,x_2)) \approx x_2$ and $Q = \neg(\beta_1(x_1, g(x_4,x_4), x_4)) \vee \beta_2(g(g(x_3,x_1), g(x_2,x_3)))$. Then $\mathrm{vb}_1(F) = 1, \mathrm{vb}_2(F) = 2, \mathrm{vb}_3(F) = 0$ and $\mathrm{vb}_4(F) = 1$. On the other hand, $\mathrm{vb}_1(Q) = 2, \mathrm{vb}_2(Q) = 1, \mathrm{vb}_3(Q) = 2$ and $\mathrm{vb}_4(Q) = 3$.

Then we prove

Theorem 7. *Let $Q \in W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $s_1,\ldots,s_n \in W_\tau(X_m)$. Then*

$$\mathrm{vb}\big(R^n_m(Q, s_1,\ldots,s_n)\big) = \sum_{k=1}^{n} \mathrm{vb}_k(Q)\cdot \mathrm{vb}(s_k).$$

Proof. We give a proof on the complexity of Q. Firstly, the theorem is proved if Q is an n-ary term of type τ. Now we consider only the case when $Q \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$. To do this, we begin with the case when Q is a formula $s \approx t$ and assume that the theorem is satisfied for s,t. Then

$$\begin{aligned}\mathrm{vb}(R^n_m(s\approx t, s_1,\ldots,s_n)) &= \sum_{k=1}^{n}\mathrm{vb}_k(s)\cdot\mathrm{vb}(s_k) + \sum_{k=1}^{n}\mathrm{vb}_k(t)\cdot\mathrm{vb}(s_k)\\ &= \sum_{k=1}^{n}\Big(\mathrm{vb}_k(s)+\mathrm{vb}_k(t)\Big)\cdot\mathrm{vb}(s_k)\\ &= \sum_{k=1}^{n}\mathrm{vb}_k(s\approx t)\cdot\mathrm{vb}(s_k).\end{aligned}$$

If $Q = \gamma_j(t_1,\ldots,t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and assume that the formula is satisfied for $t_1,\ldots,t_{n_j}$. Then $\mathrm{vb}(R^n_m(F,s_1,\ldots,s_n)) = \sum_{i=1}^{n_j}\mathrm{vb}(S^n_m(t_i,s_1,\ldots,s_n)) = \sum_{i=1}^{n_j}\big(\sum_{k=1}^{n}\mathrm{vb}_k(t_i)\mathrm{vb}(s_k)\big) = \sum_{k=1}^{n}\big(\big(\sum_{i=1}^{n_j}\mathrm{vb}_k(t_i)\big)\mathrm{vb}(s_k)\big) = \sum_{k=1}^{n}\mathrm{vb}_k(F)\,\mathrm{vb}(s_k)$. In the case $Q = \neg F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and $Q = \exists\, x_i(F) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, we inductively assume that

$$\mathrm{vb}(R^n_m(F,s_1,\ldots,s_n)) = \sum_{k=1}^{n}\mathrm{vb}_k(F)\cdot\mathrm{vb}(s_k).$$

Then we have

$$\begin{aligned}&\mathrm{vb}(R_m^n(\neg F, s_1, \dots, s_n)) = \mathrm{vb}(\neg(R_m^n(F, s_1, \dots, s_n)))\\&= \mathrm{vb}(R_m^n(F, s_1, \dots, s_n)) = \sum_{k=1}^{n} \mathrm{vb}_k(F)\cdot \mathrm{vb}(s_k) = \sum_{k=1}^{n} \mathrm{vb}_k(\neg F)\cdot \mathrm{vb}(s_k)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{vb}(R_m^n(\exists x_i(F), s_1, \dots, s_n)) = \mathrm{vb}(\exists x_i(R_m^n(F, s_1, \dots, s_n)))\\&= \mathrm{vb}(R_m^n(F, s_1, \dots, s_n)) = \sum_{k=1}^{n} \mathrm{vb}_k(F)\cdot \mathrm{vb}(s_k) = \sum_{k=1}^{n} \mathrm{vb}_k(\exists x_i)\cdot \mathrm{vb}(s_k).\end{aligned}$$

Finally, the theorem is satisfied if $Q = F_1 \vee F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$. In fact, we have

$$\begin{aligned}&\mathrm{vb}(R_m^n(F_1 \vee F_2, s_1, \dots, s_n)) = \mathrm{vb}(R_m^n(F_1, s_1, \dots, s_n) \vee R_m^n(F_2, s_1, \dots, s_n))\\&= \mathrm{vb}(R_m^n(F_1, s_1, \dots, s_n)) + \mathrm{vb}(R_m^n(F_2, s_1, \dots, s_n)) = \sum_{k=1}^{n} \mathrm{vb}_k(F_1)\cdot \mathrm{vb}(s_k)\\&+ \sum_{k=1}^{n} \mathrm{vb}_k(F_2)\cdot \mathrm{vb}(s_k) = \left(\sum_{k=1}^{n} \mathrm{vb}_k(F_1) + \sum_{k=1}^{n} \mathrm{vb}_k(F_2)\right)\mathrm{vb}(s_k)\\&= \sum_{k=1}^{n} \big(\mathrm{vb}_k(F_1) + \mathrm{vb}_k(F_2)\big)\mathrm{vb}(s_k) = \sum_{k=1}^{n} \mathrm{vb}_k(F_1 \vee F_2)\mathrm{vb}(s_k).\end{aligned}$$

This finishes a proof. ∎

We now ready to define our main definition.

Definition. Let $n \geq 1$. *An n-ary formula with fixed variables count of type* (τ, τ') is inductively defined in the following way.

(1) If t_1, t_2 are n-ary terms with fixed variables count of type τ, then the equation $t_1 \approx t_2$ is an n-ary formula with fixed variables count of type (τ, τ').

(2) If $j \in J$ and $t_1, \dots, t_{n_j}$ are n-ary terms with fixed variables count of type τ, if $\mathrm{vb}(t_k) = \mathrm{vb}(t_l)$ for all $1 \leq k < l \leq n_j$ and if γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1, \dots, t_{n_j})$ is an n-ary formula with fixed variables count of type (τ, τ').

(3) If F is an n-ary formula with fixed variables count of type (τ, τ'), then $\neg F$ is an n-ary formula with fixed variables count of type (τ, τ').

(4) If F_1 and F_2 are n-ary formulas with fixed variables count of type (τ, τ'), and if $\mathrm{vb}(F_1) = \mathrm{vb}(F_2)$, then $F_1 \vee F_2$ is an n-ary formula with fixed variables count of type (τ, τ').

(5) If F is an n-ary formula with fixed variables count of type (τ,τ'), then $\exists\, x_i(F)$ is an n-ary formula with fixed variables count of type (τ,τ').

Let $\mathcal{F}^{fvc}_{(\tau,\tau')}\big(W^{fvc}_{\tau}(X_n)\big)$ be the set of all n-ary formulas with fixed variables count of type (τ,τ') and let

$$\mathcal{F}^{fvc}_{(\tau,\tau')}\big(W^{fvc}_{\tau}(X)\big) := \bigcup_{n\in\mathbb{N}} \mathcal{F}^{fvc}_{(\tau,\tau')}\big(W^{fvc}_{\tau}(X_n)\big)$$

be the set of all formulas with fixed variables count of type (τ,τ').

We can replace (1) by the following condition $(1')$.

$(1')$ If t_1, t_2 are n-ary terms with fixed variables count of type τ, and if $\mathrm{vb}(t_1) = \mathrm{vb}(t_2)$, then $t_1 \approx t_2$ is called an n-ary strong formula with fixed variables count of type (τ,τ'). It can be seen that every strong formula with fixed variables count of type (τ,τ') is a formula with fixed variables count of type (τ,τ').

Note that the formulas defined by (1) and (2) are called *atomic formulas*.

Example 8. Let $(\tau,\tau') = ((3),(2))$ be the type with a ternary operation symbol f and a binary relation symbol γ. We provide lists of some elements in $\mathcal{F}^{fvc}_{((3),(2))}(W^{fvc}_{(3)}(X_3))$. For this, some atomic formulas are firstly determined as follows: $x_1 \approx x_3, x_2 \approx x_2, f(x_1,x_2,x_3) \approx x_1, f(x_2,x_2,x_2) \approx f(x_1,x_3,x_1), \gamma(x_1,x_2), \gamma(x_3,x_3), \gamma(x_2,x_3), \gamma(f(x_3,x_3,x_2), f(x_1,x_3,x_1))$. Apart form these are obtained by using the following three logical connectors, say $\neg, \exists, \vee$.

It is not difficult to see that the operations R^n_m for $n,m \geq 1$ can not directly applied to the sets $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$ for $n \geq 1$. In fact, we consider the superposition R^2_3 on the sets $\mathcal{F}^{fvc}_{((2),(2))}(W^{fvc}_{(2)}(X_2)), \mathcal{F}^{fvc}_{((2),(2))}(W^{fvc}_{(2)}(X_3))$ and let $f(x_2,x_1) \approx x_2$ be an element in $\mathcal{F}^{fvc}_{((2),(2))}(W^{fvc}_{(2)}(X_2)), s_1 = f(x_1,x_3), s_2 = x_3, s_3 = f(x_2,x_1) \in W^{fvc}_{(2)}(X_3)$. Particularly, we have

$$R^2_3(f(x_2,x_1) \approx x_2, s_1, s_2, s_3) = f(s_2,s_1) \approx x_2 \notin \mathcal{F}^{fvc}_{((2),(2))}(W^{fvc}_{(2)}(X_3)).$$

For this reason, in order to guarantee that the set $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$ is closed with respect to R^n_m with some additional conditions, the following two theorems are needed

Theorem 9. *If Q_1 and Q_2 are two elements in $W^{fvc}_{\tau}(X_n) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}\big(W^{fvc}_{\tau}(X_n)\big)$ satisfying $\mathrm{vb}(Q_1) = \mathrm{vb}(Q_2)$ and if $s_1,\ldots,s_n$ are m-ary terms with fixed variables count of type τ satisfying $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, then*

$$\mathrm{vb}\big(R^n_m(Q_1,s_1,\ldots,s_n)\big) = \mathrm{vb}\big(R^n_m(Q_2,s_1,\ldots,s_n)\big).$$

Proof. Let $Q_1, Q_2 \in W_\tau^{fvc}(X_n) \cup \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. Assume that the conditions hold. To prove that $\mathrm{vb}(R_m^n(Q_1, s_1, \ldots, s_n)) = \mathrm{vb}(R_m^n(Q_2, s_1, \ldots, s_n))$, we consider in a few cases. Let us start by considering the case of both Q_1 and Q_2 are elements in $W_\tau^{fvc}(X_n)$. Suppose now that $Q_1 = x_i$ and $Q_2 = x_j$. Then by the definition of R_m^n and our hypothesis, we have $\mathrm{vb}(R_m^n(x_i, s_1, \ldots, s_n)) = \mathrm{vb}(s_i) = \mathrm{vb}(s_j) = \mathrm{vb}(R_m^n(x_j, s_1, \ldots, s_n))$. If $Q_1 = x_i$ and $Q_2 = f_i(t'_1, \ldots, t'_{n_i})$, then we obtain $\mathrm{vb}(x_i) = \mathrm{vb}(f_i(t'_1, \ldots, t'_{n_i})) = 1$. Thus, we have $\mathrm{vb}(R_m^n(x_i, s_1, \ldots, s_n)) = \mathrm{vb}(s_i)$ and $\mathrm{vb}(R_m^n(f_i(t'_1, \ldots, t'_{n_i}), s_1, \ldots, s_n)) = \sum_{k=1}^{n} \mathrm{vb}_k(f_i(t'_1, \ldots, t'_{n_i}))\mathrm{vb}(s_k) = \mathrm{vb}(s_k)$, which implies that our goal are obtained. Assume that $Q_1 = f_i(t_1, \ldots, t_{n_i})$ and $Q_2 = f_j(t'_1, \ldots, t'_{n_j})$ satisfying $\mathrm{vb}(f_i(t_1, \ldots, t_{n_i})) = \mathrm{vb}(f_j(t'_1, \ldots, t'_{n_j}))$. This means that we have $\sum_{k=1}^{n_i} \mathrm{vb}(t_k) = \sum_{k=1}^{n_j} \mathrm{vb}(t'_k)$. According to the definition of $\mathrm{vb}(t)$ and $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, we get

$$\sum_{k=1}^{n_i} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t_k) \right) = \sum_{k=1}^{n_j} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t'_k) \right),$$

particularly,

$$\sum_{k=1}^{n_i} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t_k)\mathrm{vb}(s_a) \right) = \sum_{k=1}^{n_j} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t'_k)\mathrm{vb}(s_a) \right),$$

and

$$\sum_{k=1}^{n_i} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t_k) \right) \mathrm{vb}(s_a) = \sum_{k=1}^{n_j} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t'_k) \right) \mathrm{vb}(s_a),$$

which means $\sum_{k=1}^{n_i} \mathrm{vb}_a(f_i(t_1, \ldots, t_{n_i}))\mathrm{vb}(s_a) = \sum_{k=1}^{n_j} \mathrm{vb}_a(f_j(t'_1, \ldots, t'_{n_j}))\mathrm{vb}(s_a)$.

Now we let $Q_1 \in W_\tau^{fvc}(X_n)$ and $Q_2 \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. If $Q_1 = x_i \in X_n$ and Q_2 is a formula of the form $t'_1 \approx t'_2$, then we have $\mathrm{vb}(x_i) = \mathrm{vb}(t'_1 \approx t'_2) = 1$. As a result, the thorem is proved since $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$ and by the definition of $\mathrm{vb}(t)$ and R_m^n. In the case $Q_1 = f_i(t_1, \ldots, t_{n_i})$ and Q_2 is a formula $t'_1 \approx t'_2$, by the hypothesis, we obtain $\sum_{k=1}^{n_i} \mathrm{vb}(t_k) = \sum_{k=1}^{2} \mathrm{vb}(t'_k)$, in particular, we have $\sum_{k=1}^{n_i}(\sum_{a=1}^{n} \mathrm{vb}_a(t_k)) = \sum_{k=1}^{2}(\sum_{a=1}^{n} \mathrm{vb}_a(t'_k))$. Since we have $\mathrm{vb}(s_1) = \cdots = \mathrm{vb}(s_n)$,

$$\sum_{k=1}^{n_i} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t_k) \right) \mathrm{vb}(s_a) = \sum_{k=1}^{2} \left(\sum_{a=1}^{n} \mathrm{vb}_a(t'_k) \right) \mathrm{vb}(s_a).$$

This implies $\mathrm{vb}(R_m^n(f_i(t_1, \ldots, t_{n_i}), s_1, \ldots, s_n)) = \mathrm{vb}(R_m^n(t'_1 \approx t'_2, s_1, \ldots, s_n))$. In general, we can prove by the same process that the theorem is valid if Q_1 is a term x_i or $f_i(t_1, \ldots, t_{n_i})$ and Q_2 is a formula $\gamma_j(t_1, \ldots, t_{n_j})$. Analogously, we can prove the case $Q_1 \in \mathcal{F}_{(\tau,\tau')}^{fov}(W_\tau^{fov}(X_n))$ and $Q_2 \in W_\tau^{fov}(X_n)$.

We now consider atomic formulas Q_1 and Q_2 in $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$. Let us begin by letting Q_1 and Q_2 be any two formulas $s'_1 \approx s'_2$ and $t'_1 \approx t'_2$, respectively. Due to the presumption, we have $\mathrm{vb}(s'_1)+\mathrm{vb}(s'_2) = \mathrm{vb}(t'_1)+\mathrm{vb}(t'_2)$, which means $\sum_{k=1}^{n}(\mathrm{vb}_k(s'_1)) + \sum_{k=1}^{n}(\mathrm{vb}_k(s'_2)) = \sum_{k=1}^{n}(\mathrm{vb}_k(t'_1)) + \sum_{k=1}^{n}(\mathrm{vb}_k(t'_2))$. According to our assumption, we have

$$\left(\sum_{k=1}^{n}(\mathrm{vb}_k(s'_1)) + \sum_{k=1}^{n}(\mathrm{vb}_k(s'_2))\right)\mathrm{vb}(s_k) = \left(\sum_{k=1}^{n}(\mathrm{vb}_k(t'_1)) + \sum_{k=1}^{n}(\mathrm{vb}_k(t'_2))\right)\mathrm{vb}(s_k),$$

and hence

$$\sum_{j=1}^{2}\big(\mathrm{vb}(S^n_m(s'_j, s_1, \ldots, s_n)\big) = \sum_{j=1}^{2}(\mathrm{vb}\big(S^n_m(t'_j, s_1, \ldots, s_n)\big),$$

which gives $\mathrm{vb}(R^n_m(s'_1 \approx s'_2, s_1, \ldots, s_n)) = \mathrm{vb}(R^n_m(t'_1 \approx t'_2, s_1, \ldots, s_n))$. Similarlity, we can prove in the case of Q_1 is a formula $s \approx t$ and $Q_2 = \gamma_j(t_1, \ldots, t_{n_j})$. Moreover, the proof is obtained if we consider $\sum_{k=1}^{n_j}(\mathrm{vb}(t_j)$ and $\sum_{k=1}^{n_p}(\mathrm{vb}(t'_p)$ when $Q_1 = \gamma_j(t_1, \ldots, t_{n_j})$ and $Q_2 = \gamma_p(t'_1, \ldots, t'_{n_p})$.

In addition, it is not difficult to show that the theorem is satisfied after applying steps (3) and (5) in Definition 3.1 because of

$$\mathrm{vb}(F) = \mathrm{vb}(\neg F) = \mathrm{vb}(\exists\, x_i(F)),$$

$$R^n_m(\neg F, s_1, \ldots, s_n) = \neg R^n_m(F, s_1, \ldots, s_n)$$

$$R^n_m(\exists\, x_i(F), s_1, \ldots, s_n) = \exists\, x_i(R^n_m(F, s_1, \ldots, s_n)).$$

Finally, we give a proof of the theorem when either Q_1 or Q_2 is a formula that generated by a logical connector $\vee$ or both of them are formulas of the form $F \vee F'$ and $Q \vee Q'$. If $Q_1 \in W^{fvc}_{\tau}(X_n)$ and $Q_2 = F \vee F'$, we have $\mathrm{vb}(t) = \mathrm{vb}(F)+\mathrm{vb}(F')$, which means $\sum_{k=1}^{n}(\mathrm{vb}_k(t)) = \sum_{k=1}^{n}(\mathrm{vb}_k(F)) + \sum_{k=1}^{n}(\mathrm{vb}_k(F'))$. If follows from the hypothesis that

$$\left(\sum_{k=1}^{n}(\mathrm{vb}_k(t))\right)\mathrm{vb}(s_k) = \left(\sum_{k=1}^{n}(\mathrm{vb}_k(F)) + \sum_{k=1}^{n}(\mathrm{vb}_k(F'))\right)\mathrm{vb}(s_k),$$

which implies

$$\mathrm{vb}\big(R^n_m(t, s_1, \ldots, s_n)\big) = \mathrm{vb}\big(R^n_m(F, s_1, \ldots, s_n)\big) + \mathrm{vb}\big(R^n_m(F', s_1, \ldots, s_n)\big).$$

We can prove the case when Q_1 is a formula of the steps (1), or (2), or (3) and $Q_2 = Q \vee Q'$ in a similar way. Inductively, we finally obtain the proof both Q_1 and Q_2 are formulas of the form $F \vee F'$ and $Q_2 = Q \vee Q'$. As a consequence, the proof is finished. ∎

Consequently, we prove

Theorem 10. *If* $Q \in W_\tau^{fvc}(X_n) \cup \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$ *and if* $s_1, \ldots, s_n$ *are* m-*ary terms with fixed variables count of type* τ *satisfying* $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ *for* $1 \leq j < k \leq n$, *then*

$$R_m^n(Q, s_1, \ldots, s_n) \in W_\tau^{fvc}(X_m) \cup \mathcal{F}_{(\tau,\tau')}^{fvc}\big(W_\tau^{fvc}(X_m)\big).$$

Proof. Clearly, if Q is a variable in X_n, the theorem is proved. It follows directly from Lemma 3 that the statement is obtained if Q is an n-ary term $f_i(t_1, \ldots, t_{n_i})$ with fixed variables count of type τ. We now prove that the theorem is also valid if Q is an n-ary formula with fixed variables count of type (τ, τ'). At first, we assume that Q is an equation $s \approx t$ with $s, t \in W_\tau^{fvc}(X_n)$. According to the definition of R_m^n, we have that $R_m^n(s \approx t, s_1, \ldots, s_n)$ is the equation $S_m^n(s, s_1, \ldots, s_n) \approx S_m^n(t, s_1, \ldots, s_n)$. Because of $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$, by Lemma 3 and our presumption, variables count of the terms on both sides of the equation are equal and thus $R_m^n(Q, s_1, \ldots, s_n) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. Let Q be a formula of the form $\gamma_j(t_1, \ldots, t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. Using the definition of R_m^n, we obtain $R_m^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n) = \gamma_j(S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_{n_j}, s_1, \ldots, s_n))$. Applying the proof of Lemma 3, it is easy to show that $S_m^n(t_p, s_1, \ldots, s_n) \in W_\tau^{fvc}(X_n)$ for all $1 \leq p \leq n_j$. Now we prove that for all $1 \leq k < l \leq n_j$ the equations

$$\mathrm{vb}(S_m^n(t_k, s_1, \ldots, s_n)) = \mathrm{vb}(S_m^n(t_l, s_1, \ldots, s_n))$$

hold. Since $\gamma_j(t_1, \ldots, t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$, $\mathrm{vb}(t_k) = \mathrm{vb}(t_l)$ for all $1 \leq k < l \leq n_j$, and hence $\sum_{i=1}^n \mathrm{vb}_i(t_k) = \sum_{i=1}^n \mathrm{vb}_i(t_l)$, for all $1 \leq k < l \leq n_j$. It follows from the condition $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$ that $\sum_{i=1}^n \mathrm{vb}_i(t_k)\mathrm{vb}(s_i) = \sum_{i=1}^n \mathrm{vb}_i(t_l)\mathrm{vb}(s_i)$ for all $1 \leq k < l \leq n_j$, which gives $\mathrm{vb}(S_m^n(t_k, s_1, \ldots, s_n)) = \mathrm{vb}(S_m^n(t_l, s_1, \ldots, s_n))$. Consequently, $\gamma_j(S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_{n_j}, s_1, \ldots, s_n))$ belongs to $\mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_m))$ and thus $R_m^n(\gamma_j(t_1, \ldots, t_{n_j}), s_1, \ldots, s_n)$ is an m-ary formula with fixed variables count of type (τ, τ')Let Q be an n-ary formula with fixed variables count of type (τ, τ'). We inductively assume that $R_m^n(Q, s_1, \ldots, s_n) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. Since

$$R_m^n(\neg Q, s_1, \ldots, s_n) = \neg R_m^n(Q, s_1, \ldots, s_n)$$

and

$$R_m^n(\exists x_i(Q), s_1, \ldots, s_n) = \exists x_i(R_m^n(Q, s_1, \ldots, s_n)),$$

we also obtain

$$R_m^n(\neg Q, s_1, \ldots, s_n) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n)),$$

$$R_m^n(\exists x_i(Q), s_1, \ldots, s_n) \in \mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n)).$$

Finally, we let F and Q be any two elements in $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$. Because of

$$R^n_m(F \vee Q, s_1, \ldots, s_n) = R^n_m(F, s_1, \ldots, s_n) \vee R^n_m(Q, s_1, \ldots, s_n),$$

and the satisfaction of F and Q, using the result of Theorem 9, we also have that the formula $R^n_m(F \vee Q, s_1, \ldots, s_n)$ belongs to the set $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_m))$. This completes the proof. ∎

As a consequence, using the fact that terms and formulas with fixed variables count are closed with respect to the many-sorted operations R^n_m for $n, m \geq 1$ in Theorem 10, we define the partial operations denoted by $\overline{R}^n_m$ as follows.

For any $Q \in W^{fvc}_{\tau}(x_n) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n)), s_1, \ldots, s_n \in W^{fvc}_{\tau}(X_m)$, the many-sorted partial mapping

$$\overline{R}^n_m : \Big(W^{fvc}_{\tau}(X_n) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}\big(W^{fvc}_{\tau}(X_n)\big)\Big) \times \Big(W^{fvc}_{\tau}(X_m)\Big)^n \\ \multimap W^{fvc}_{\tau}(X_m) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}\Big(W^{fvc}_{\tau}(X_m)\Big)$$

is inductively defined by:

$$\overline{R}^n_m(Q, s_1, \ldots, s_n) := \begin{cases} R^n_m(Q, s_1, \ldots, s_n), & \text{if } \mathrm{vb}(s_j) = \mathrm{vb}(s_k) \\ & \text{for all } 1 \leq j < k \leq n; \\ \text{not defined}, & \text{otherwise.} \end{cases}$$

The following many-sorted algebra consisting of a sequence of the union between $W^{fvc}_{\tau}(X_n)$ and $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$, a sequence of the partial operations $\overline{R}^n_m$, and a sequence of variables acting as projections, i.e.,

$$\Big(\big(W^{fvc}_{\tau}(X_n) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))\big)_{n\in\mathbb{N}}, (\overline{R}^n_m)_{n,m\in\mathbb{N}}, (x_i)_{i\leq n, n\in\mathbb{N}}\Big),$$

which is called the *formulas clone with fixed variables count of type* (τ, τ'), is formed and denoted by $\mathrm{Formclone}^{fvc}(\tau, \tau')$.

Theorem 11. *The axioms* (C1), (C2), *and* (C3) *are weak identities in the many-sorted partial algebra* $\mathrm{Formclone}^{fvc}(\tau, \tau')$.

Proof. To prove (C1), we replace $\tilde{Z}$ in (C1) by an arbitrary element $Q \in W^{fvc}_{\tau}(X_n) \cup \mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_{\tau}(X_n))$, $\tilde{Y}_1, \ldots, \tilde{Y}_p$ by n-ary terms $t_1, \ldots, t_p$ with fixed variables count of type τ, $\tilde{X}_1, \ldots, \tilde{X}_n$ by m-ary terms $s_1, \ldots, s_n$ with fixed variables count of type τ, and the operation symbols by the following partial operations $\overline{R}^n_m, \overline{R}^p_n$ and $\overline{R}^p_m$. As a result, we obtain

$$\overline{R}^n_m(\overline{R}^p_n(Q, t_1, \ldots, t_p), s_1, \ldots, s_n) \\ = \overline{R}^p_m(Q, \overline{R}^n_m(t_1, s_1, \ldots, s_n), \ldots, \overline{R}^n_m(t_p, s_1, \ldots, s_n)).$$

It follows immediately from Theorem 4 that the above equation is satisfied if Q is an element of $W_\tau^{fvc}(X_n)$. Otherwise, let Q be an arbitrary formula in $\mathcal{F}_{(\tau,\tau')}^{fvc}(W_\tau^{fvc}(X_n))$. Suppose first that $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$ for all $1 \leq j < k \leq p$ and $\mathrm{vb}(s_l) = \mathrm{vb}(s_r)$ for all $1 \leq l < r \leq n$. Then the left-hand side of above equation is defined and equals to $R_m^n(R_n^p(Q, t_1, \ldots, t_p), s_1, \ldots, s_n)$. Moreover, for each $1 \leq j \leq p$, the partial superposition $\overline{R}_m^n(t_j, s_1, \ldots, s_n)$ is defined and equals to $S_m^n(t_j, s_1, \ldots, s_n)$. For $1 \leq j < k \leq p$, according to our presumption, we have $\mathrm{vb}(t_j) = \mathrm{vb}(t_k)$, which means $\sum_{i=1}^n \mathrm{vb}_i(t_j) = \sum_{i=1}^n \mathrm{vb}_i(t_k)$. Consequently, we have

$$\sum_{i=1}^n \mathrm{vb}_i(t_j)\mathrm{vb}(s_i) = \sum_{i=1}^n \mathrm{vb}_i(t_k)\mathrm{vb}(s_i),$$

and hence $\mathrm{vb}(S_m^n(t_j, s_1, \ldots, s_n)) = \mathrm{vb}(S_m^n(t_k, s_1, \ldots, s_n))$. Therefore, the right-hand side of above equation is defined and equals to

$$R_m^p(Q, S_m^n(t_1, s_1, \ldots, s_n), \ldots, S_m^n(t_p, s_1, \ldots, s_n)).$$

Actually, it was proved in [11] that the operations R_m^n satisfy (C1). Thus, our equation is obtained.

In order to show (C2), we replace λ_j in (C2) by a variable x_j in $W_\tau^{fvc}(X_n)$, $\tilde{X}_1, \ldots, \tilde{X}_n$ by terms $s_1, \ldots, s_n$ from $W_\tau^{fvc}(X_n)$, and the symbol $\tilde{S}_m^n$ by $\overline{R}_m^n$ and obtain $\overline{R}_m^n(x_i, s_1, \ldots, s_n) = s_i$. Since $\mathrm{vb}(s_l) = \mathrm{vb}(s_r)$ for all $1 \leq l < r \leq n$, $\overline{R}_m^n(x_i, s_1, \ldots, s_n)$ is defined and thus $\overline{R}_m^n(x_i, s_1, \ldots, s_n) = R_m^n(x_i, s_1, \ldots, s_n) = s_i$.

Finally, if we replace $\tilde{S}_n^n, \tilde{Y}, \lambda_1, \ldots, \lambda_n$ in (C3) by $\overline{R}_n^n, Q, x_1 \ldots, x_n$, repectively, we have $\overline{R}_n^n(Q, x_1, \ldots, x_n) = R_n^n(Q, x_1, \ldots, x_n) = Q$ by the satisfaction of R_m^n which was proved in [11]. This completes the proof. ∎

3.2. Representation of formulas with fixed variables count

This section is devoted to a new construction of partial multiplace functions which correspond to formulas with fixed variables count. To achieve this, the notions of Menger systems and algebras of functions are essential. In [13], algebras represented by some kinds of operations were discussed. The authors defined left translations of n-ary operations in [19]. Menger hyperalgebras generalizing Menger algebras were introduced by the authors in [20]. Algebras of weak-near unanimity functions were recently studied by the authors in [21]. A nice connection of Menger algebras and terms was mentioned in [10, 35]. Let $(G_n)_{n\in I}$ be a family of nonempty sets and n, m be positive integers belonging to nonempty index set I of positive integers. Consider the many-sorted operations

$$\circ_m^n : G_n \times (G_m)^n \to G_m$$

satisfying the following conditions:

(1) if $n, m \in I, x \in G_n, y_1, \ldots, y_n \in G_m$, then $\circ_m^n(x, y_1, \ldots, y_n)$ is defined and it belongs to G_m,

(2) the *superassociative law* holds:

$$\begin{aligned}&\circ_m^n(\circ_n^p(x, y_1, \ldots, y_n), z_1, \ldots, z_n)\\&= \circ_m^p(x, \circ_m^n(y_1, z_1, \ldots, z_n), \ldots, \circ_m^n(y_p, z_1, \ldots, z_n)),\end{aligned}$$

for all $n, m, p \in I, x \in G_p, y_1, \ldots, y_p \in G_n, z_1, \ldots, z_n \in G_m$.

The many-sorted algebra $\mathcal{G} = ((G_n)_{n\in I}, (\circ_m^n)_{n,m\in I})$ is called a *Menger system of rank I* or a *many-sorted Menger algebra of rank I*. In general, this structure can be regarded as a natural generalization of Menger algebras and semigroups if $I = \{n\}$ and $I = \{1\}$, respectively.

For an index set I, by $|I|$ we mean the cardinality of I. Furthermore, the symbol $(x_{ji})_{i\in I}$, we refer to the $|I|$-tuple $(x_{ja}, \ldots, x_{jb})$ where j is a positive integer, a and b is the minimum element and the maximum element in I, respectively. Usually, we mention that a number i that appears in $(x_{ji})_{i\in I}$ is indicated by i^{th}-sorted set. Let us consider, for instance, if $I = \{2, 3\}$, then $(x_{ji})_{i\in I} = (x_{ji})_{i\in\{2,3\}} = (x_{j2}, x_{j3})$, consequently, $x_{j2} \in G_2, x_{j3} \in G_3$.

As mentioned in [21], a Menger system $\mathcal{G}$ of rank I is called *unitary* if it contains a *complete collection of selectors* of all arities, i.e., for every $n, m \in I$ there are elements $e_{1m}, \ldots, e_{nm} \in G_m$ and $e_{in} \in G_n$, called *selectors*, such that

$$\begin{aligned}&\circ_m^n(x, e_{1m}, \ldots, e_{nm}) = x,\\&\circ_m^n(e_{in}, y_1, \ldots, y_n) = y_i,\end{aligned}$$

for all $i = 1, \ldots, n$, and $x_n \in G_n, y_1, \ldots, y_n \in G_m$.

Let A be a nonempty set, I a nonempty set of positive intergers. By $F_n(A)$ we denote the set of all n-ary partial functions on A. On the family of sets $(F_n(A))_{n\in I}$, an $(n+1)$-ary operation (also called *composition of partial functions*) $\mathcal{O}_m^n$, where $n, m \in I$, can be defined by setting.

If $f \in F_n(A), g_1, \ldots, g_n \in F_m(A)$ and $(a_1, \ldots, a_m) \in A^m$, then by the symbol $\mathcal{O}_m^n(f, g_1, \ldots, g_n)$ we denote the partial function

$$\mathcal{O}_m^n : F_n(A) \times (F_m(A))^n \multimap\!\!\to F_m(A)$$

defined by

$$\begin{aligned}&\text{dom }(\mathcal{O}_m^n) = \left\{(a_1, \ldots, a_m) \in A^m | (a_1, \ldots, a_m) \in \textstyle\bigcap_{i=1}^n \text{dom}(g_i)\right\}\\&\text{and } (g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)) \in \text{dom}(f)\\&\text{and } \mathcal{O}_m^n(f, g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)).\end{aligned}$$

It is not hard to verify that this composition satisfies the superasscociativity, i.e., for any $n, m, p \in I$, $f \in F_p(A), g_1, \ldots, g_p \in F_n(A)$ and $h_1, \ldots, h_n \in F_m(A)$, we have

$$\begin{aligned}&\mathcal{O}^n_m(\mathcal{O}^p_n(f, g_1, \ldots, g_p), h_1, \ldots, h_n)\\ &= \mathcal{O}^p_m(f, \mathcal{O}^n_m(g_1, h_1, \ldots, h_n), \ldots, \mathcal{O}^n_m(g_p, h_1, \ldots, h_n)).\end{aligned}$$

The many-sorted algebra $((F_n(A))_{n\in I}, (\mathcal{O}^n_m)_{n,m\in I})$ is called the *Menger system of rank I of partial multiplace functions*. Obviously, this algebra can be considered as a natural generalization of a semigroup of partial transformations and a Menger algebra of partial n-ary functions if $I = \{1\}$ and $I = \{n\}$, respectively.

In the following, we prove the main result.

Theorem 12. *The many-sorted partial algebra* $\mathrm{Formclone}^{fvc}(\tau, \tau')$ *is isomorphic to some unitary Menger system of the same rank of partial multiplace functions such that its selectors correspond to the projection operations of this set.*

Proof. For convenience, the symbol $W\mathcal{F}^{fvc}(X_n)$ stands for the union of $W^{fvc}_\tau(X_n)$ and $\mathcal{F}^{fvc}_{(\tau,\tau')}(W^{fvc}_\tau(X_n))$. Let $((W\mathcal{F}^{fvc}(X_n))_{n\in\mathbb{N}}, (\overline{R}^n_m)_{n,m\in\mathbb{N}}, (x_i)_{i\leq n, n\in\mathbb{N}})$ be the many-sorted partial algebra, which can be regarded as a Menger system of rank $I = \mathbb{N}$. Then for each $n \in \mathbb{N}$ and each element $Q \in W\mathcal{F}^{fvc}(X_n)$, the partial multiplace function

$$\lambda_Q : \left(\prod_{i\in\mathbb{N}}\left(W\mathcal{F}^{fvc}(X_i)\right)\right)^n \to \prod_{i\in\mathbb{N}}\left(W\mathcal{F}^{fvc}(X_i)\right)$$

can be defined by

$$\lambda_Q((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}}) := \begin{cases}(\overline{R}^n_i(Q, a_{1i}, \ldots, a_{ni}))_{i\in\mathbb{N}}, & \text{if } (a_{ji})_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}}(W^{fvc}_\tau(X_i)) \\ & \text{for all } 1 \leq j \leq n; \\ \text{not defined}, & \text{otherwise}\end{cases}$$

for all $(a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i))$.

Now let $\overline{F_n(\prod\limits_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i)))} = \{\lambda_Q \mid Q \in W\mathcal{F}^{fvc}(X_n)\}$. It is not difficult to show that the system

$$\left(\left(\overline{F_n\left(\prod_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i))\right)}\right)_{n\in\mathbb{N}}, (\mathcal{O}^n_m)_{n,m\in\mathbb{N}}, (\mathrm{pr}^n_i)_{i\leq n, n\in\mathbb{N}}\right)$$

forms a unitary Menger system of rank $\mathbb{N}$ of partial multiplace functions.

Since $((W\mathcal{F}^{fvc}(X_n))_{n\in\mathbb{N}}, (\overline{R}^n_m)_{n,m\in\mathbb{N}}, (x_i)_{i\leq n,n\in\mathbb{N}})$ is a many-sorted algebra, we actually have to consider the mapping

$$(\phi_n)_{n\in\mathbb{N}} : \Big(W\mathcal{F}^{fvc}(X_n)\Big)_{n\in\mathbb{N}} \to \Bigg(\overline{F_n\bigg(\prod_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i))\bigg)}\Bigg)_{n\in\mathbb{N}},$$

defined by

$$\phi_n(Q) = \lambda_Q$$

for all $n \in \mathbb{N}$ where ϕ_n is a mapping from $W\mathcal{F}^{fvc}(X_n)$ to $\overline{F_n(\prod_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i)))}$.

We first show that for every $n, m \in \mathbb{N}$ the equation

$$\phi_m(\overline{R}^n_m(Q, s_1, \ldots, s_n)) = \mathcal{O}^n_m(\phi_n(Q), \phi_m(s_1), \ldots, \phi_m(s_n))$$

for all $Q \in W\mathcal{F}^{fvc}(X_n)$ and $s_1, \ldots, s_n \in W^{fvc}(X_m)$ are satisfied as weak identities. In fact, we let $(Q, s_1, \ldots, s_n) \in \mathrm{dom}(\overline{R}^n_m)$. This means that $\mathrm{vb}(s_j) = \mathrm{vb}(s_k)$ for $1 \leq j < k \leq n$. Applying the definition of ϕ_n, we obtain $(\phi_n(Q), \phi_m(s_1), \ldots, \phi_m(s_n)) \in \mathrm{dom}(\mathcal{O}^n_m)$ which means $\mathrm{vb}(\phi_m(s_j)) = \mathrm{vb}(\phi_m(s_k))$ for $1 \leq j < k \leq n$. As a result these equalities are equivalent to

$$\lambda_{\overline{R}^n_m(Q,s_1,\ldots,s_n)} = \mathcal{O}^n_m(\lambda_Q, \lambda_{s_1}, \ldots, \lambda_{s_n}).$$

In order to prove that these equalities are satisfied, we suppose that $(a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}}$ are arbitrary elements in $\prod_{i\in\mathbb{N}}(W\mathcal{F}^{fvc}(X_i))$. The existence of λ_Q is defined if we only consider the case of $(a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}(W^{fvc}(X_i))$. By the superassociativity of $(\overline{R}^n_m)_{n,m\in\mathbb{N}}$, we have

$$\begin{aligned}
&\lambda_{\overline{R}^n_m(Q,s_1,\ldots,s_n)}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}})\\
&= (\overline{R}^m_i(\overline{R}^n_m(Q, s_1, \ldots, s_n), a_{1i}, \ldots, a_{mi}))_{i\in\mathbb{N}}\\
&= (\overline{R}^n_m(Q, \overline{R}^m_i(s_1, a_{1i}, \ldots, a_{mi}), \ldots, \overline{R}^m_i(s_n, a_{1i}, \ldots, a_{mi})))_{i\in\mathbb{N}}\\
&= \lambda_Q((\overline{R}^m_i(s_1, a_{1i}, \ldots, a_{mi}))_{i\in\mathbb{N}}, \ldots, (\overline{R}^m_i(s_n, a_{1i}, \ldots, a_{mi}))_{i\in\mathbb{N}})\\
&= \lambda_Q(\lambda_{s_1}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}}), \ldots, \lambda_{s_n}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}}))\\
&= \mathcal{O}^n_m(\lambda_Q, \lambda_{s_1}, \ldots, \lambda_{s_n})((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{mi})_{i\in\mathbb{N}}).
\end{aligned}$$

Furthermore, it is also an isomorphism. In fact, assume that $\lambda_{Q_1} = \lambda_{Q_2}$ for some $Q_1, Q_2 \in W\mathcal{F}^{fvc}(X_n), n \in \mathbb{N}$. Then

$$\lambda_{Q_1}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}}) = \lambda_{Q_2}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}}).$$

Thus, in particular, we have

$$(\overline{R}^n_i(Q_1, a_{1i}, \ldots, a_{ni}))_{i\in\mathbb{N}} = (\overline{R}^n_i(Q_2, a_{1i}, \ldots, a_{ni}))_{i\in\mathbb{N}}.$$

For each $i \in \mathbb{N}$, replacing each element a_{ji} in this equation by variables x_{ji} of $W^{fvc}(X_n)$ for all $j = 1, \ldots, n$, we obtain

$$(\overline{R}_i^n(Q_1, x_{1i}, \ldots, x_{ni}))_{i\in\mathbb{N}} = (\overline{R}_i^n(Q_2, x_{1i}, \ldots, x_{ni}))_{i\in\mathbb{N}}.$$

Due to the satisfaction of (C3) of the partial operation $\overline{R}_m^n$ which was proved in Theorem 11, we conclude $Q_1 = Q_2$. This shows that $(\phi_n)_{n\in\mathbb{N}}$ is an isomorphism.

Finally, for each $1 \leq k \leq n$, $n \in \mathbb{N}$, and $x_k \in W_\tau^{fvc}(X_n)$, we have

$$\lambda_{x_k}((a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}}) = (R_i^n(x_k, a_{1i}, \ldots, a_{ni}))_{i\in\mathbb{N}} = (a_{ki})_{i\in\mathbb{N}}$$

for all $(a_{1i})_{i\in\mathbb{N}}, \ldots, (a_{ni})_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}}(W^{fvc}(X_i))$. Thus $(\lambda_{x_k})_{k\leq n, n\in\mathbb{N}} = (\mathrm{pr}_k^n)_{k\leq n, n\in\mathbb{N}}$. This means that selectors are transformed into projection operations. The proof is completely finished. ■

4. Concluding remarks and future perspectives

In this paper, we defined a class of terms over algebras and generalize its to a class of formulas in algebraic systems. To achieve these concepts, we applied the so-called variable count in the complexity theory which describes and determines many useful properties in algebras. The idea of counting the total number of occurrences of variables in a formula was proposed. In terms of algebraic structures, a condition under which the many-sorted superpositions of terms can be applied to our new terms was given. However, we may mention here that these conditions are not necessary in some situations. For example, consider terms with fixed variables count $t = f(x_1, x_3), s_1 = f(x_1, x_2), s_2 = x_3, s_3 = f(x_3, x_4)$. We note that the result after substituting these terms under $\overline{S}_4^3$, i.e.,

$$\overline{S}_4^3(t, s_1, s_2, s_3) = S_4^3(t, s_1, s_2, s_3) = f(f(x_1, x_2), f(x_3, x_4))$$

is again a term with fixed variables count, although $(t, s_1, s_2, s_3) \notin \mathrm{dom}(\overline{S}_4^3)$. Consequently, there are two possible ways of continuation of this paper. Firstly, it is possible to modify the partial many-sorted superpositions of terms with fixed variables count to which each variable in the first position of such operations are considered, for example, in $S_4^3(f(x_1, x_3), s_1, s_2, s_3)$ we may only consider variables occuring in $f(x_1, x_3)$ and set some conditions with respect to x_1 and x_3. Secondly, power sets of terms and formulas with fixed variables count can be extended and their many-sorted operations can be determined. For instance, $\{x_1, f(x_2, x_4)\}, \{f(x_1, x_1) \approx f(f(x_2, x_1), f(x_3, x_3)), \gamma(x_1, x_1), \neg(\gamma(x_1, x_1))\}$.

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References

[1] E. Aichinger, N. Mudrinski and J. Oprsal, *Complexity of term representations of finitary functions*, Int. J. Algebra Comput. **28** (2018) 1101–1118. https://doi.org/10.1142/S0218196718500480

[2] S. Bozapalidis, Z. Flap and G. Rahonis, *Equational tree transformations*, Theoret. Comput. Sci. **412** (99) (2011) 3676–3692. https://doi.org/10.1016/j.tcs.2011.03.028

[3] P. Burmeister, *A model theoretic oriented approach to partial algebras*, in: Introduction to Theory and Application of Partial Algebras, Mathematical Research **32** (Akademie Verlag, 1986).

[4] S. Busaman, *Unitary Menger algebra of C-quantifier free formulas of type* $(\tau_n, 2)$, Asian-Eur. J. Math. **14** (4) (2021) 2150050. https://doi.org/10.1142/S1793557121500509

[5] N. Chansuriya, *All maximal idempotent submonoids of generalized cohypersubstitutions of type* $\tau = (2)$, Discuss. Math. Gen. Algebra Appl. **41** (1) (2021) 45–46. https://doi.org/10.7151/dmgaa.1351

[6] J. Crulis, *Multi-algebras from the viewpoint of algebraic logic*, Algebra Discrete Math. **1** (2003) 20–31.

[7] K. Denecke, *Partial clones*, Asian-Eur. J. Math. **13** (8) (2020) 2050161. https://doi.org/10.1142/S1793557120501612

[8] K. Denecke, *The partial clone of linear formulas*, Sib. Math J. **60** (2019) 572–584. https://doi.org/10.1134/S0037446619040037

[9] K. Denecke, *The partial clone of linear terms*, Sib. Math J. **57** (4) (2016) 589–598. https://doi.org/10.1134/S0037446616040030

[10] K. Denecke and H. Hounnon, *Partial Menger algebras of terms*, Asian-Eur. J. Math. **14** (6) (2021) 2150092. https://doi.org/10.1142/S1793557121500923

[11] K. Denecke and D. Phusanga, *Hyperformulas and solid algebraic systems*, Studia Logica **9** (2008) 263–286. https://doi.org/10.1007/s11225-008-9152-3

[12] K. Denecke and S.L. Wismath, *Complexity of terms, composition and hypersubstitution*, Int. J. Math. Math. Sci. **15** (2003) 959–969. https://doi.org/10.1155/S0161171203202118

[13] W.A. Dudek and V.S. Trokhimenko, *Menger algebras of k commutative n-place functions*, Georgian Math. J. **28** (3) (2021) 355–361. https://doi.org/10.1515/gmj-2019-2072

[14] Y. Guellouma and H. Cherroun, *From tree automata to rational tree expressions*, Int. J. Found. Comput. Sci. **29** (6) (2018) 1045–1062. https://doi.org/10.1142/S012905411850020X

[15] H.J. Hoehnke and J. Schreckenberger, Partial Algebras and Their Theories (Shaker-Verlag, Aachen, 2007).

[16] S. Kerhoff, R. Pöschel and F.M. Schneider, *A short introduction to clones*, Electron. Notes Theoret. Comput. Sci. **303** (2014) 107–120.
https://doi.org/10.1016/j.entcs.2014.02.006

[17] K.A. Kearnes and A. Szendrei, *Clones of algebras with parallelogram terms*, Internat. J. Algebra Comput. **22** (2012) 1250005.
https://doi.org/10.1142/S0218196711006716

[18] J. Koppitz and D. Phusanga, *The monoid of hypersubstitutions for algebraic systems*, J. Announcements Union Sci Sliven **33** (2018) 120–127.

[19] T. Kumduang and S. Leeratanavalee, *Left translations and isomorphism theroems of Menger algebras*, Kyungpook Math. J. **61** (2) (2021) 223–237.
https://doi.org/10.5666/KMJ.2021.61.2.223

[20] T. Kumduang and S. Leeratanavalee, *Menger hyperalgebras and their representations*, Commun. Algebra **49** (4) (2021) 1513–1533.
https://doi.org/10.1080/00927872.2020.1839089

[21] T. Kumduang, and S. Leeratanavalee, *Menger systems of idempotent cyclic and weak near-unanimity multiplace functions*, Asian-Eur. J. Math. (2022).
https://doi.org/10.1142/S1793557122501625

[22] T. Kumduang and S. Leeratanavalee, *Semigroups of terms, tree languages, Menger algebra of n-ary functions and their embedding theorems*, Symmetry **13** (4) (2021) 558.
https://doi.org/10.3390/sym13040558

[23] P. Kunama and S. Leeratanavalee, *Green's relations on submonoids of generalized hypersubstitutions of type* (n), Discuss. Math. Gen. Algebra Appl. **41** (2) (2021) 239–248.
https://doi.org/10.7151/dmgaa.1366

[24] E. Lehtonen, R. Pöschel and T. Waldhauser, *Reflection-closed varieties of multisorted algebras and minor identities*, Algebra Univ. **79** (2018) 70.
https://doi.org/10.1007/s00012-018-0547-3

[25] N. Lekkoksung and S. Lekkoksung, *On partial clones of k-terms*, Discuss. Math. Gen. Algebra Appl. **41** (2021) 361–379.
https://doi.org/10.7151/dmgaa.1367

[26] A.I. Mal'cev, Algebraic Systems (Akademie-Verlag, Berlin, Germany, 1973).

[27] D. Phusanga, *A binary relation on sets of hypersubstitutions for algebraic systems*, South. Asian Bull. Math. **44** (2020) 255–269.

[28] D. Phusanga, J. Joomwong, S. Jino and J. Koppitz, *All idempotent and regular elements in the monoid of generalized hypersubstitutions for algebraic systems of type* $(2;2)$, Asian-Eur. J. Math. **14** (2) (2021) 2150015.
https://doi.org/10.1142/S1793557121500157

[29] D. Phusanga and J. Koppitz, *Some varieties of algebraic systems of type* $((n),(m))$, Asian-Eur. J. Math. **12** (2019) 1950005.
https://doi.org/10.1142/S1793557119500050

[30] S. Shtrakov and J. Koppitz, *Stable varieties of semigroups and groupoids*, Algebra Univers. **75** (2016) 85–106.
https://doi.org/10.1007/s00012-015-0359-7

[31] S.V. Sudoplatov, *Formulas and properties, their links and characteristics*, Mathematics **9** (2021) 1391.
https://doi.org/10.3390/math9121391

[32] N. Sungtong, *The algebraic structures of quantifer free formulas induced by terms of a fixed variable*, Int. J. Math. Comput. Sci. **16** (1) (2021) 459–469.

[33] K. Wattanatripop and T. Changphas, *Clones of terms of a fixed variable*, Mathematics **8** (2020) 260.
https://doi.org/10.3390/math8020260

[34] K. Wattanatripop and T. Changphas, *The Menger algebra of terms induced by order-decreasing transformations*, Commun. Algebra **49** (7) (2021) 3114–3123.
https://doi.org/10.1080/00927872.2021.1888385

[35] K. Wattanatripop, T. Kumduang, T. Changphas and S. Leeratanavalee, *Power Menger algebras of terms induced by order-decreasing transformations and superpositions*, Int. J. Math. Comput. Sci. **16** (4) (2021) 1697–1707.

[36] D. Zhuk, *The cardinality of the set of all clones containing a given minimal clone on three elements*, Algebra Univers. **68** (2012) 295–320.
https://doi.org/10.1007/s00012-012-0207-y

[37] P. Zusmanovich, *On the unity of Robinson Amitsur ultrafilters*, J. Algebra **388** (2013) 268–286.
https://doi.org/10.1016/j.jalgebra.2013.04.024

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