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ON THE NUMBER OF GROUP HOMOMORPHISMS BETWEEN CERTAIN GROUPS

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Abstract

Let H be a finite abelian group and $Dih(H) = \langle H, b \mid b^2 = 1 \ \& \ bhb^{-1} = h^{-1};\ \forall h \in H\rangle$ be the generalized dihedral group of H. The aim of this paper is to compute the number of group homomorphisms between two generalized dihedral groups and a generalized dihedral group and an abelian group. One of these results generalized an earlier work by J.W. Johnson published in 2013.

Keywords: group homomorphism, generalized dihedral group, abelian group.

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1. Introduction

Throughout this paper we will make this assumption that all groups are assumed to be finite. If G and H are such groups then we interest the number of homomorphisms from G into H, denoted by $\gamma(H, G) = |Hom(H, G)|$. In the case that $H = G$, we will use the notation $\gamma(G)$ as $\gamma(G, G)$. The problem of computing the number of homomorphisms between two groups is so difficult in general, and so some mathematicians presented methods to compute $\gamma(H, G)$ for certain groups.

With the best of our knowledge, the first published paper in which the number of homomorphisms between two finite groups is considered into account was the

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joint paper of Gallian and Van Buskirk [3]. In the mentioned paper, the authors obtained closed formulas for the number of group homomorphisms, and also ring homomorphisms, from $\mathbb{Z}_n$ into $\mathbb{Z}_m$. If (m, n) denotes the greatest common divisor of two positive integers m and n, then they proved that:

Theorem 1. $\gamma(\mathbb{Z}_m, \mathbb{Z}_n) = (m, n)$.

Johnson [4], found the number of group homomorphisms from the dihedral group D_{2m} into the dihedral group D_{2n}. He proved that

Theorem 2.

$$\gamma(D_{2m}, D_{2n}) = \begin{cases} n(m,n)+1 & 2 \nmid mn \\ n(m,n)+2 & 2 \nmid m \ \& \ 2 \mid n \\ n(m,n)+4n+4 & 2 \mid m \ \& \ 2 \mid n \\ n(m,n)+2n+1 & 2 \mid m \ \& \ 2 \nmid n. \end{cases}$$

The most important works on the problem of counting group homomorphisms were given by Takegahara and his co-authors. Chigira and Takegahara [2], studied the number of homomorphisms from a finite group to a general linear group over a finite field, and the authors of [5, 9, 10] investigated the number of homomorphisms from a finite abelian group to a symmetric or alternating groups. Liebeck and Shalev [6] have been estimated the number of homomorphisms from a finite group A to the general linear group $GL(n, q)$, where q is a prime power coprime to $|A|$.

Bate [1] provided upper and lower bounds for the number of completely reducible homomorphisms from a finite group to general linear and unitary groups over arbitrary finite fields, and to orthogonal and symplectic groups over finite fields of odd characteristic. Matei and Suciu [7] presented a method for computing the number of epimorphisms from a finitely presented group to a finite solvable group, which generalizes a formula of Gaschütz.

An elementary abelian group of order p^n, p is prime, is denoted by $E(p^n)$. Suppose G is an abelian group with decomposition $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_d}$ in which $n_{i+1} \,|\, n_i$, $1 \le i \le d-1$, and for all j, $n_j \ge 2$. Then we define $\mathcal{S}(G) = \{n_1, \ldots, n_d\}$. Note that this decomposition is unique for each abelian group and so our definition for $S(G)$ is well-defined. The number of factors of even orders in this decomposition of H into cyclic groups is denoted by $\varepsilon(H)$. If n is a positive integer, then $\phi(n)$ denotes the Euler totient function evaluated at n.

Throughout this paper our notations are standard and we refer to the famous book of Robinson [8] for concepts and notations not presented here. Our results are checked by the computer algebra package Gap [11].

2. Preliminaries

Suppose H is an abelian group. The generalized dihedral group $Dih(H)$ can be presented by $Dih(H) = \langle H, b_H \mid b_H^2 = 1 \ \&\ b_H h b_H^{-1} = h^{-1};\ \forall h \in H\rangle$. It is well-known that this group is the semidirect product of H by the cyclic group of order 2. In an exact phrase, $Dih(H) = H \rtimes_\alpha Z_2$ in which $\alpha(0)$ is the identity element of $Aut(H)$ and $\alpha(1) = f$ in which $f(x) = x^{-1}$, for arbitrary element $x \in H$. Note that for each subgroup M of H, $\overline{M} = \{(m, 0) \mid m \in M\}$ is a subgroup of $Dih(H)$ isomorphic to M. On the other hand, the set of all elements in the form of $(h, 0), h \in H$, constitutes a subgroup of index 2 in $Dih(H)$ isomorphic to H.

The first main result of this paper is as follows.

Theorem 3. *Let H and G be two finite abelian groups. The number of homomorphisms from $Dih(H)$ into $Dih(G)$ can be computed by the following formula:*

$$\begin{aligned}\gamma(Dih(H), Dih(G)) \ &= |G|\gamma(H, G) + \frac{|G|}{2^{\varepsilon(G)}}\left(2^{(1+\varepsilon(H))(1+\varepsilon(G))} - 2^{(1+\varepsilon(H))\varepsilon(G)}\right) \\ &- \ |G|2^{\varepsilon(H)\varepsilon(G)} + 2^{(1+\varepsilon(H))\varepsilon(G)}.\end{aligned}$$

In particular, $\gamma(Dih(H)) = |H||End(H)| + \frac{|H|}{2^{\varepsilon H}}\left(2^{(\varepsilon(H))^2} - 2^{(\varepsilon(H)+1)\varepsilon(H)}\right) - |H|2^{\varepsilon(H)^2} + 2^{(1+\varepsilon(H))\varepsilon(H)}$.

We now apply this theorem to present a simple proof for Theorem 2 which is the main result of [4].

***New Proof for Theorem* 2.** Since for each natural number r, $Dih(\mathbb{Z}_r)$ is a dihedral group of order $2r$, it is enough to apply Theorem 3. By Theorem 1, $\gamma(\mathbb{Z}_m, \mathbb{Z}_n) = (m, n)$ and it is easy to see that for each cyclic group A, $\varepsilon(A) \in \{0, 1\}$ and $\varepsilon(A) = 0$ if and only if A has odd order. Therefore,

$$\begin{aligned}\gamma(D_{2m}, D_{2n}) \ &= \begin{cases} n(m,n) + (2-1)n - n + 1 & 2 \nmid mn \\ n(m,n) + (2^2 - 2^1)\frac{n}{2} - n + 2^1 & 2 \nmid m \ \&\ 2 \mid n \\ n(m,n) + (2^4 - 2^2)\frac{n}{2} - 2n + 2^2 & 2 \mid m \ \&\ 2 \mid n \\ n(m,n) + (2^2 - 1)n - n + 1 & 2 \mid m \ \&\ 2 \nmid n \end{cases} \\ &= \begin{cases} n(m,n) + 1 & 2 \nmid mn \\ n(m,n) + 2 & 2 \nmid m \ \&\ 2 \mid n \\ n(m,n) + 4n + 4 & 2 \mid m \ \&\ 2 \mid n \\ n(m,n) + 2n + 1 & 2 \mid m \ \&\ 2 \nmid n. \end{cases}\end{aligned}$$

We are now ready to state our second main result which can be proved in a similar way as the proof of Theorem 3.

Theorem 4. *Let H and G be finite abelian groups. Then the following hold.*

1. $\gamma(Dih(H), G) = 2^{(\varepsilon(H)+1)\varepsilon(G)}$;
2. $\gamma(H, Dih(G)) = \gamma(H, G) + \frac{|G|}{2^{\varepsilon(G)}}(2^{\varepsilon(H)(1+\varepsilon(G))} + 2^{\varepsilon(H)\varepsilon(G)})$.

3. Main results

Suppose S is a minimal generating set for H, then $Dih(H) = \langle S, b_H\rangle$. If G is an abelian group of even order, then we use the notation $E(G)$ to denote the set of all involutions together with the identity element of G. It is easy to see that $E(G)$ is the largest elementary abelian 2-subgroup of G.

Suppose G, H and K are three finite groups. It is well-known that $\gamma(G, H \times K) = \gamma(G, H)\gamma(G, K)$, see [7, p. 168]. Also, if A and B are abelian groups then it is well-known that $\gamma(A, B) = \gamma(B, A)$. The following lemma is an immediate consequence of these known results.

Lemma 5. *Let $G_1, \ldots, G_n$ and $H_1, \ldots, H_m$ be abelian groups. Then*

$$\gamma(G_1 \times \cdots \times G_n, H_1 \times \cdots \times H_m) = \prod_{i=1}^{n}\prod_{j=1}^{m}\gamma(G_i, H_j).$$

Corollary 6. *Let G and H be finite abelian groups. Then*

$$\gamma(H, G) = \prod_{i\in S(G)}\prod_{j\in S(H)}(i, j).$$

Suppose U_n denotes the unit group of the ring $\mathbb{Z}_n$ of integers modulo n. If $n = 2^{\alpha}{p_1}^{n_1}\cdots{p_r}^{n_r}$, where p_i's are different odd primes, then by the Chinese remainder theorem $U_n \cong U_{2^\alpha} \times U_{p_1^{n_1}} \times \cdots \times U_{p_r^{n_r}}$. Moreover, U_2 is trivial group, $U_4 \cong \mathbb{Z}_2$, $U_{2^n} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, $n > 2$ is an integer, and for each odd prime p and positive integer m, $U_{p^m} \cong \mathbb{Z}_{p^m - p^{m-1}}$.

Corollary 7. *Let $n > 2$ and $m > 2$ be two positive integers with prime factorizations $n = 2^{\alpha}{p_1}^{n_1}\cdots{p_r}^{n_r}$ and $m = 2^{\beta}{q_1}^{l_1}\cdots{q_s}^{l_s}$, where p_i, $1 \le i \le r$, as well as q_j, $1 \le j \le s$ are different odd primes. Moreover, α, β, r, s, n_i, $1 \le i \le r$, and m_j, $1 \le j \le s$ are non-negative integers. Without loss of generality we can assume that $\beta \le \alpha$. Then the following hold:*

1. *if $\alpha, \beta \in \{0, 1\}$, then $\gamma(U_n, U_m) = \prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*
2. *if $\alpha = 2$ and $\beta \in \{0, 1\}$, then $\gamma(U_n, U_m) = 2^s\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*
3. *if $\alpha = \beta = 2$ then $\gamma(U_n, U_m) = 2^{r+s+1}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;*

4. *if $\alpha > 2$ and $\beta \in \{0,1\}$, then*

$$\gamma(U_n, U_m) = 2^s \prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j)) \prod_{j=1}^{s}(2^{\alpha-2}, \phi(q_j)).$$

In particular, if $\alpha = 3$, then $\gamma(U_n, U_m) = 2^{2s}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;

5. *if $\alpha > 2$ and $\beta = 2$, then*

$$\gamma(U_n, U_m) = 2^{r+s+2} \prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j)) \prod_{j=1}^{s}(2^{\alpha-2}, \phi(q_j)).$$

In particular, if $\alpha = 3$, then $\gamma(U_n, U_m) = 2^{r+2s+2}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$;

6. *if $\alpha > 2$ and $\beta > 2$, then $\gamma(U_n, U_m) = 2^{r+s+\beta+1}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))\prod_{j=1}^{s}(2^{\alpha-2}, \phi(q_j)) \times \prod_{i=1}^{r}(2^{\beta-2}, \phi(p_i))$. In the special case that $\alpha = \beta = 3$ we will have $\gamma(U_n, U_m) = 2^{2r+2s+4}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))$ and if $\alpha > 3$ and $\beta = 3$ then $\gamma(U_n, U_m) = 2^{2r+s+4}\prod_{i=1}^{r}\prod_{j=1}^{s}(\phi(p_i), \phi(q_j))\prod_{j=1}^{s}(2^{\alpha-2}, \phi(q_i))$.*

Corollary 8. *Let G and H be abelian groups. Then*

$$\gamma(G, H) = \prod_{i\in S(G)}\prod_{i\in S(H)}(i, j).$$

In particular, $\gamma(H, \mathbb{Z}_2) = 2^{\varepsilon(H)}$.

Proof. By our definition $S(\mathbb{Z}_2) = \{2\}$ and so $\gamma(H, \mathbb{Z}_2) = \prod_{i\in S(H)}(i, 2) = 2^{\varepsilon(H)}$, proving the result. ■

Suppose G is a finite group. It is clear that there is a one to one correspondence between the set of all subgroups of index 2 in G and non-zero homomorphisms from G into the cyclic group $\mathbb{Z}_2$. This proves that there is exactly $\gamma(G, \mathbb{Z}_2) - 1$ subgroups of index 2 in G. We now apply this simple result to prove the following lemma.

Lemma 9. *Let H be an abelian group. Then $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$.*

Proof. To prove the lemma, it is enough to count the number of subgroup of index 2 in $Dih(H)$. By definition of the generalized dihedral group, H is a subgroup of index 2 in $Dih(H)$. Choose a subgroup H' of index 2 in H, $x \in Dih(H) \setminus H$ and $y \in Dih(H) \setminus (Dih(H') \cup H)$. It can be easily seen that H, $\langle H', x\rangle$ and $\langle H', y\rangle$ are the only proper subgroups of $Dih(H)$ containing H' and the last two subgroups are isomorphic to $Dih(H')$. Therefore, $\gamma(Dih(H), \mathbb{Z}_2) = 2|\{K \leq H \mid |H : K| = 2\}| + 2 = 2(\gamma(H, \mathbb{Z}_2) - 1) + 2 = 2\gamma(H, \mathbb{Z}_2)$. By Corollary 8, $\gamma(H, \mathbb{Z}_2) = 2^{\varepsilon(H)}$ and so $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$, proving the lemma. ■

Corollary 10. $\gamma(Dih(H), \mathbb{Z}_2^n) = 2^{n(\varepsilon(H)+1)}$.

Proof. By Lemma 9, $\gamma(Dih(H), \mathbb{Z}_2) = 2^{\varepsilon(H)+1}$ and by Lemma 5, $\gamma(Dih(H), \mathbb{Z}_2^n) = 2^{n(\varepsilon(H)+1)}$. ■

We are now ready to present the proof of our first main result.

***Proof of Theorem* 3.** To calculate the number of homomorphisms $h : Dih(H) \longrightarrow Dih(G)$, we consider four cases that in which the order of G or H are odd or even.

(i) *Both of G and H have odd orders.* Note that if we have the image of h under b_H and each element of S, then the homomorphism h will be completely determined. It is clear that all elements $b_G g \in Dih(G)$ are involutions. Since G has odd order, these are all elements of even order in $Dih(G)$ which shows that $h(b_H) = e_G$ or $h(b_H) = b_G g$, for some $g \in G$. If $h(b_H) = e_G$, then h is the zero homomorphism, and so we can assume that there exists $g \in G$ such that $h(b_H) = b_G g$. Furthermore, $h(H) \subseteq G$ and so h induces a homomorphism from H into G. On the other hand, we assume that $h_1 : H \longrightarrow G$ is a group homomorphism. We extend h_1 to the homomorphism $\overline{h_1} : Dih(H) \longrightarrow Dih(G)$ by $\overline{h_1}(b_H x) = b_G y h_1(x)$, where $y \in H$ is arbitrary. Therefore, we will have $|G|$ different choices for defining $h(b_H)$ and $\gamma(H, G)$ different choices for the group homomorphism h_1. This proves that there are $|G|\gamma(H, G) + 1$ homomorphisms from $Dih(H)$ into $Dih(G)$.

(ii) $|G|$ *is even and* $|H|$ *is odd.* Since $|H|$ is odd, all elements $h(s)$, $s \in S$, has odd orders. It is clear that $h(b_H) \in E(G) \cup (Dih(G) \setminus G)$. If $h(b_H) \in Dih(G) \setminus G$, then a similar argument as (i) shows that we have exactly $|G|\gamma(H, G)$ homomorphisms. We now assume that $h(b_H) \in E(G)$. Suppose there exists $s \in S$ such that $h(s) \notin E(G)$. Since $h(b_H s) = h(b_H)h(s) \in G$ and $O(b_H s) = 2$, $b_H s \in E(G)$ which leads to a contradiction. Therefore, elements of S map to elements of $E(G)$. Note that $O(h(s))|O(s)$ and $O(s)$ is odd which shows that $O(s) \neq 2$. This proves that $h(S) = \{e_G\}$. This creates $|E(G)|$ new homomorphisms and so $\gamma(Dih(H), Dih(G)) = |G|\gamma(H, G) + |2^{\varepsilon(G)}|$.

(iii) $|G|$ *is odd and* $|H|$ *is even.* In this case, G does not have an element of even order. Since $O(h(b_H)) = 1, 2$, b_H can be mapped to e_G or an element in the form of $b_G g$, $g \in G$. Suppose there are two elements $s_1, s_2 \in S$ such that $h(s_1) = b_G g_1$ and $h(s_2) = g_2$, where $g_1, g_2 \in G$ and $g_2 \neq e_G$. Since H is abelian, $b_G g_1 g_2 = h(s_1)h(s_2) = h(s_2)h(s_1) = g_2 b_G g_1 = b_G g_2^{-1} g_1$ which implies that $g_2^2 = e_G$. But G has odd order and so $g_2 = e_G$. This proves that if for an element $s_1 \in S$, $h(s_1) = b_G g_1$ then the image of all elements of S under the homomorphism h will be identity element of G or an element in the form of $b_G g$, where $g \in G$. Hence, we have one of the following cases.

1. $h(S) \subseteq G$. In this subcase, if $h(b_H) = e_G$, then h will be the zero homomorphism. Moreover, all mappings for which every element of S mapped to an element of G and b_H mapped to an element in the form of $b_G g$, $g \in G$, can be extended to a unique homomorphism from $Dih(H)$ into $Dih(G)$ and similar to (i) there are $|G|\gamma(H, G)$ of such homomorphisms.

2. *There exists $s_1 \in S$ such that $h(s_1) = b_G g_1$, for some $g_1 \in G$.* For each $s \in S$, $O(h(s)) = 1, 2$ and also $O(h(b_H)) = 1, 2$. This shows that $h(H)$ is an elementary abelian 2-subgroup of $Dih(G)$ and since $4 \nmid |Dih(G)|$, $h(H)$ is a subgroup of order 2 in $Dih(G)$. There are $\gamma(Dih(H), \mathbb{Z}_2) - 1$ non-trivial homomorphisms from $Dih(H)$ into $\mathbb{Z}_2$ and since we have $|G|$ involutions in $Dih(G)$, we will have $|G|[\gamma(Dih(H), \mathbb{Z}_2) - 1]$ homomorphisms. But there are $|G|$ homomorphisms for which $h(S) = \{e_G\}$ and b_H mapped to an element in the form of $b_G g$, $g \in G$. Therefore, the total number of homomorphisms from $Dih(H)$ into $Dih(G)$ is $|G|\gamma(H, G) + |G|[\gamma(Dih(H), \mathbb{Z}_2) - 1] - |G| + 1$.

We now apply Lemma 9 to complete the proof of (iii).

(iv) *Both of G and H have even orders.* Since $|H|$ and $|G|$ are both even and $O(h(b_H))|2$, there exists $g \in G$ such that $b_H = b_G g$ or $h(b_H) \in E(G)$. Our proof will consider two cases that $h(S) \subseteq G$ or there exists $s_1 \in S$ such that $h(s_1) = b_G g_1$.

1. $h(S) \subseteq G$. We first assume that $h(b_H) = b_G g$. By an argument similar to Part (i) of the proof of Theorem 3, we will have $|G|\gamma(H, G)$ homomorphisms. Suppose that $h(b_H) \in E(G)$. Similar to Part (ii) of the proof of Theorem 3, we assume that there exists $s \in S$ such that $x = h(s) \in G \setminus E(G)$ and so $O(h(s)) = O(x) \neq 1, 2$. Since $h(b_H) \in E(G)$ and $x = h(s) \in G$, $h(b_H s) = h(b_H)h(s) \in G$, and since $O(h(b_H s))|2$, $h(b_H s) \in E(G)$. On the other hand, $h(b_H s) = h(b_H)h(s) = h(b_H)x$ and $x \notin E(G)$ which is impossible. This contradiction shows that $h(S) \subseteq E(G)$ which show that $h(Dih(H)) \subseteq E(G)$. Therefore, we have to counted the number of homomorphisms from $Dih(H)$ into $E(G)$.

2. *There exists $s_1 \in S$ such that $h(s_1) = b_G g_1$.* Similar to what we have done in (iii), we assume that for another element $s_2 \in S$, $h(s_2) = g_2$ in which $g_2 \in G$. Since H is abelian, $g_2^2 = e_G$. This proves that the image of each element of S has the form of $b_G g$ or is an element of $E(G)$. In each case, it can be easily seen that $O(h(s))|2$, $s \in S$. Also, $O(h(b_H))|2$ and hence $h(Dih(H))$ is a trivial subgroup or an elementary abelian 2-group. Thus, $h(Dih(H))$ is isomorphic to a subgroup of $Dih(E(G)) \cong \mathbb{Z}_2 \times E(G)$ and we have $\frac{|G|}{|E(G)|}$ subgroups isomorphic to $Dih(E(G))$. In the last case, we have to reduce this case by the number of homomorphisms with this condition that $h(S) \subseteq G$. Therefore,

$$\gamma(Dih(H), Dih(G)) = |G|\gamma(H, G) + \frac{|G|}{|E(G)|}(\gamma(Dih(H), E(G) \times Z_2)$$

$$- \gamma(Dih(H), E(G))) - |G|\gamma(H, E(G)) + \gamma(Dih(H), E(G)).$$

We now apply Lemmas 5, 9 and Corollary 10 to get the result.

This completes the proof of (iv).

***Proof of Theorem* 4.** Suppose G and H are abelian groups. Our proof will consider two separate cases as follows.

1. A similar argument as Part (iv)(b) of the proof of Theorem 3 shows that $h(S) \subseteq E(G)$ and so $\gamma(Dih(H), G) = \gamma(Dih(H), E(G))$. Now by Corollary 10, $\gamma(Dih(H), G) = 2^{(\varepsilon(H)+1)\varepsilon(G)}$, as desired.

2. If $h(S) \subseteq E(G)$, then there are $\gamma(H, G)$ homomorphisms. Thus, we can assume that there exists $s_1 \in S$ such that $h(s_1) = b_G g_1$, for some $g_1 \in G$. Now a similar argument as Part (ii) of the proof of Theorem 3 shows that $h(s) \in (Dih(G) \setminus G) \cup E(G)$. Hence the image of $Dih(H)$ is isomorphic to a subgroup of $Dih(E(G)) \cong E(G) \times \mathbb{Z}_2$ and we have $\frac{|G|}{|E(G)|}$ such subgroups. Since $h(s_1) = b_G g_1$, $Dih(H) \nsubseteq G$. Therefore, $\gamma(H, Dih(G)) = \gamma(H, G) + \frac{|G|}{|E(G)|}(2^{(\varepsilon(H)+1)\varepsilon(G)} - 2^{\varepsilon(H)\varepsilon(G)})$.

4. Concluding remarks

In this paper, the number of homomorphisms between two generalized dihedral groups were calculated. This gives a generalization of a result by Johnson [4]. We also compute the number of homomorphisms between an abelian group and a generalized dihedral groups, and the number of homomorphisms between the unite rings of integers modulo n and m, respectively. The next step in this program is to calculate the number of homomorphisms between two generalized dicyclic groups, an abelian group and a generalized dicyclic group, and a generalized dihedral group and a generalized dicyclic group. We checked all results of this paper by gap programs. These programs are accessible from the authors upon request.

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