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S-k-PRIME AND S-k-SEMIPRIME IDEALS OF SEMIRINGS

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Abstract

Let R be a commutative ring and S a multiplicatively closed subset of R. Hamed and Malek[7] defined an ideal P of R disjoint with S to be an S-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$, then $sa \in P$ or $sb \in P$. In this paper, we introduce the notions of S-k-prime and S-k-semiprime ideals of semirings, S-k-m-system, and S-k-p-system. We study some properties and characterizations for S-k-prime and S-k-semiprime ideals of semirings in terms of S-k-m-system and S-k-p-system respectively. We also introduce the concepts of S-prime semiring and S-semiprime semiring and study the characterizations for S-k-prime and S-k-semiprime ideals in these two semirings.

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1. Introduction

Semiring theory has emerged as an intriguing research topic in recent years. Semiring theory has numerous applications in computer science, automata theory, control theory, quantum mechanics, and a variety of other fields. In a similar manner as ring theory, semiring theory relies heavily on ideals, which aids in the study of structure theory and other topics.

Golan [6] was the first to develop the terminologies prime ideals and semiprime ideals of semirings and he has contributed a significant number of results in these aspects. After Golan, the studies on prime ideals and semiprime ideals of semirings has been continued by Dubey [4], Leskot [10], Atani et. al. [2], and many others. The k-ideal is one of the basic ideals in semiring theory. Sen and Adhikari [12, 13] studied k-ideal of semiring and its properties. The k-prime(k-semiprime) ideal is a class of ideals in semiring that are equivalent to prime (semiprime) ideals in a ring. A prime (semiprime) ideal becomes a k-prime (k-semiprime) ideal if it coincides with its k-closure. Kar et. al. [11] have done extensive work on the k-prime ideal and k-semiprime ideal in a semiring.

The concept of the S-prime ideal of a commutative ring has been introduced by Hamed and Malek in [7] and established many remarkable results. For a commutative ring R and a multiplicatively closed set $S \subseteq R$, an ideal P of R is said to be S-prime ideal if there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in P$, then $sa \in P$ or $sb \in P$. Later on, Almahdi et. al. [1] and Visweswaran [14] studied weakly S-prime ideals and S-primary ideals of a commutative ring respectively.

In this paper, we define S-prime ideal and S-semiprime ideal in a semiring. We introduce the concepts of S-m-system and S-p-system, as well as some analogous results. Furthermore, we introduce the notions of S-k-prime and S-k-semiprime ideals of semirings and study their properties and characterizations in terms of S-k-m-system and S-k-p-system respectively. Finally, we also introduce the concepts of S-prime semiring and S-semiprime semiring and study the characterizations for S-k-prime and S-k-semiprime ideals in these two semirings.

2. PRELIMINARIES

In this section, we recall some basic terminology and preliminary results of semiring theory that will be useful in later sections of the paper.

A non-empty set R with two binary operations '$+$' and '$\cdot$' is said to be a semiring [8] if (i) $(R, +)$ be a commutative semigroup; (ii) $(R, \cdot)$ be a semigroup and (iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in R$.

Throughout this paper we consider semiring $(R, +, \cdot)$ with zero element 0 and nonzero identity 1.

Let J be an ideal of a semiring R. Then the k-closure [13] of ideal J is denoted by $\overline{J}$ and is given by $\overline{J} = \{x \in R | x + y = z \text{ for some } y, z \in J\}$.

We say a left ideal (respectively right ideal, ideal) J of a semiring R to be a left k-ideal (respectively right k-ideal, k-ideal) if for any $a \in R$ and $b \in J$, $a + b \in J$ implies that $a \in J$. For any k-ideal J, we have $J = \overline{J}$.

A non-empty subset S of a semiring R is said to be a multiplicatively closed set if (i) $1 \in S$ and (ii) for $a, b \in S$ implies $ab \in S$.

A non-zero element a of semiring R is said to be a zero divisor if there exists a non-zero element $b \in R$ such that $ab = 0$.

A proper ideal I of a commutative semiring R is said to be a 2-absorbing ideal[3] if $a, b, c \in R$ and $abc \in I$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

The following lemma will be useful in the next section.

Lemma 2.1:[8] Let R be a semiring. Then for any two ideals A, B of R, we have the following results: (i) $A \subseteq \overline{A}$; (ii) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; (iii) $\overline{\overline{A}} = A$; (iv) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$ and (v) $\overline{A}$ is a k-ideal of R.

For any other undefined terminologies of semiring theory, we refer to [5, 6, 8].

3. S-k-PRIME IDEALS OF SEMIRINGS

In this section, we introduce the notion of S-prime and S-k-prime ideal of a semiring and study their basic properties. We begin with the following definitions.

Definition 3.1. Let R be a semiring, S a multiplicatively closed subset of R and P be an ideal of R disjoint with S. We say P is an S-prime ideal of R if there exists an $s \in S$ such that for all A, B two ideals of R, if $AB \subseteq P$, then $sA \subseteq P$ or $sB \subseteq P$.

Definition 3.2. An S-prime ideal P of a semiring R is said to be an S-k-prime ideal of R if $P = \overline{P}$.

Proposition 3.3. *Let R be a semiring, $S \subseteq R$ a multiplicatively closed set and P a k-ideal of R disjoint with S. Then P is an S-k-prime ideal of R if and only* *if there exists an $s \in S$ for all k-ideals I, J of R, if $IJ \subseteq P$, then $sI \subseteq P$ or $sJ \subseteq P$.*

Proof. Let P be an S-k-prime ideal of R. Then there exists an $s \in S$ such that for all I, J two k-ideals of R with $IJ \subseteq P$ then $sI \subseteq P$ or $sJ \subseteq P$.

To prove the converse, let I, J be any two k-ideals of R with $IJ \subseteq P$ such that $sI \subseteq P$ or $sJ \subseteq P$ for some $s \in S$. We have $\overline{I}\ \overline{J} \subseteq \overline{\overline{I}\ \overline{J}} = \overline{IJ} \subseteq \overline{P} = P$. Then $s\overline{I} \subseteq P$ or $s\overline{J} \subseteq P$ which implies that $sI \subseteq P$ or $sJ \subseteq P$. Hence P is an S-k-prime ideal of R. ■

Corollary 3.4. *Let R be a semiring, $S \subseteq R$ a multiplicatively closed set and P a k-ideal of R disjoint with S. Then P is an S-k-prime ideal of R if and only if there exists an $s \in S$ such that for all k-ideals J_i of R with $J_1 J_2 \cdots J_n \subseteq P$, then $sJ_i \subseteq P$ for some $i \in \{1, 2, \ldots, n\}$.*

A characterization theorem for an S-k-prime ideal of a semiring will be introduced here. Golan[6] first established the characterization theorem for a prime ideal, and subsequently Kar et. al.[11] proved it for the k-prime ideal of a semiring.

Theorem 3.5 [6]**.** *The following statements are equivalent for an ideal P of a semiring R:*

1. *P is a prime ideal of a semiring R.*
2. *For any $a, b \in R, aRb \subseteq P$ if and only if $a \in P$ or $b \in P$.*

Theorem 3.6 [11]**.** *The following statements are equivalent for an ideal P of a semiring R:*

1. *P is a k-prime ideal of a semiring R.*
2. *For any $a, b \in R, aRb \subseteq \overline{P}$ if and only if $a \in P$ or $b \in P$.*

Theorem 3.7. *Let R be a semiring, $S \subseteq R$ be a multiplicatively closed set and P an ideal of R disjoint with S. Then the following statements are equivalent:*

1. *P is an S-prime ideal of a semiring R.*
2. *there exists an $s \in S$ such that for all $a, b \in R$, if $aRb \subseteq P$, then $sa \in P$ or $sb \in P$.*

Proof. (1) $\Rightarrow$ (2): Let P be an S-prime ideal of R. Consider $a, b \in R$ and $A = < a >$ and $B = < b >$. Then A and B are ideals of R with $aRb \subseteq AB$. Also, AB is contained in any ideal which contains aRb. Thus $aRb \subseteq P$ implies that $AB \subseteq P$ and hence $sA \subseteq P$ or $sB \subseteq P$ for some $s \in S$. Thus $sa \in P$ or $sb \in P$.

(2) $\Rightarrow$ (1): Let A and B be ideals of R such that $AB \subseteq P$. Let us assume that $sA \nsubseteq P$ and let $a \in A - P$. Then for each $b \in B$ we have $aRb \subseteq AB \subseteq P$ which implies that $sb \in P$ and hence $sB \subseteq P$. So P is an S-prime ideal of R. ■

Theorem 3.8. *Let R be a semiring, $S \subseteq R$ be a multiplicatively closed set and P an ideal of R disjoint with S. Then the following statements are equivalent:*

1. *P is an S-k-prime ideal of a semiring R.*
2. *There exists an $s \in S$ such that for all $a, b \in R$, if $aRb \subseteq \overline{P}$, then $sa \in P$ or $sb \in P$.*

Proof. (1) $\Rightarrow$ (2): Let P be an S-k-prime ideal of R so $P = \overline{P}$. Consider $a, b \in R$ such that $aRb \subseteq \overline{P}$. We take $A =< a >$ and $B =< b >$. Then A and B are ideals of R with $aRb \subseteq AB$. Also, AB is contained in any ideal which contains aRb. Thus $aRb \subseteq \overline{P}$ implies that $AB \subseteq \overline{P} = P$ and hence $sA \subseteq P$ or $sB \subseteq P$ for some $s \in S$. Thus $sa \in P$ or $sb \in P$.

(2) $\Rightarrow$ (1): Let A and B be ideals of R such that $AB \subseteq P$. Let us assume that $sA \nsubseteq P$ and let $a \in A - P$. Then for each $b \in B$ we have $aRb \subseteq AB \subseteq P \subseteq \overline{P}$ which implies that $sb \in P$ and hence $sB \subseteq P$. So P is an S-prime ideal of R.

We have $P \subseteq \overline{P}$. Consider $x \in \overline{P}$. Then $x + b = c$ for some $b, c \in P$. Let $y \in xRx$. So $y = xrx$ for some $r \in R$. This implies that $y + xrb = xrx + xrb = xr(x + b)$. It follows that $y + xrb = xrc$. As P is an ideal of R so $xrb, xrc \in P$, and thus $y \in \overline{P}$. So $xRx \subseteq \overline{P}$ and $x \in P$ which implies that $\overline{P} \subseteq P$ and hence $P = \overline{P}$.

Therefore, P is an S-k-prime ideal of R. ■

Corollary 3.9. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and P an ideal of R disjoint with S. Then P is an S-k-prime ideal of R if and only if there exists an $s \in S$ such that for all $a, b \in R$, if $ab \in \overline{P}$, then $sa \in P$ or $sb \in P$.*

Proof. In a commutative semiring R, we have $ab \in P$ if and only if $arb \in P$ for all $r \in R$. The result follows from Theorem 3.8. ■

Remark 3.10. It is obvious that every prime ideal of a semiring is also an S-prime ideal of that semiring and every k-prime ideal of a semiring is also an S-k-prime ideal of that semiring. But the converse of the above may not hold which can be observed in the following example.

Example 3.11. Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define, $P = < 6 >$. Then

P is a k-ideal of R[13]. Then, $P \cap S = \emptyset$. Now, $ab \in P =< 6 > \Rightarrow ab = 6m$, for some m. Then either a or b must be even. So, there exists $s = 3 \in S$ such that $3a \in P$ or $3b \in P$. Hence, P is an S-k-prime ideal. Moreover, $2.3 \in < 6 >$ but $2 \notin < 6 >$ and $3 \notin < 6 >$ which implies that P is not a k-prime ideal of $R = \mathbb{Z}_0^+$.

In the next example, we can observe that an S-prime ideal of a semiring may not be an S-k-prime ideal of that semiring.

Example 3.12. Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define, $P = 2\mathbb{Z}_0^+ \backslash \{2\}$. Then P is an S-prime ideal of R but not an S-k-prime ideal of R.

Now let I be an ideal of a commutative semiring R and $s \in R$. We define, $I : s = \{\ x \in R :\ sx \in I\}$. Then for all $s \in R$, $I : s$ is an ideal of R.

Proposition 3.13. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set consisting of nonzero divisors and P a k-ideal of R disjoint with S. Then P is an S-k-prime ideal of R if and only if $P : s$ is a k-prime ideal of R for some $s \in S$.*

Proof. As P is an S-k-prime, there exists an $s \in S$ such that for all $a, b \in R$ with $ab \in P$ then either $sa \in P$ or $sb \in P$. We show $P : s$ is k-prime ideal of R. Let $a, b \in R$ and $ab \in P : s$ which implies that $sab \in P$ so we get $s^2a \in P$ or $sb \in P$. Thus $sa \in P$ or $sb \in P$ and hence $a \in P : s$ or $b \in P : s$. Thus $P : s$ is a prime ideal of R.

Then, $P : s \subseteq \overline{P : s}$. Now let $x \in \overline{P : s}$ which implies that $x \in R$ and $x + y \in P : s$ for some $y \in P : s$. Thus $x \in R$ and $s(x + y) \in P$ for some $sy \in P$. So $x \in R$ and $sx + sy \in P$ for some $sy \in P$. Therefore $sx \in P$ and hence $x \in P : s$. So, $P : s = \overline{P : s}$. Thus, $P : s$ is a k-ideal of R and hence $P : s$ is a k-prime ideal of R.

Conversely, let $ab \in P$ then $sab \in P$ and so $ab \in P : s$. Since $P : s$ is a k-prime ideal of R so $a \in P : s$ or $b \in P : s$ and hence $sa \in P$ or $sb \in P$. Thus, P is a S-prime ideal which implies P is a S-k-prime ideal of R since P is a k-ideal of R. ■

Example 3.14. Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define, $P = < 6 >$. Then P is an S-k-prime ideal of R. Now $P : 3 = \{x \in R | 3x \in P\}$. We see that $P : 3$ is the set of all positive even integers. Then $P : 3$ is a k-ideal. If $xy \in P : 3$ then either x or y must be a positive even integer. Hence $x \in P : 3$ or $y \in P : 3$. Thus $P : 3$ is a k-prime ideal.

Proposition 3.15. *Let R be a commutative semiring and S a multiplicatively closed subset of R disjoint with a k-ideal P of R. If $R \subseteq T$ be an extension of commutative semirings, P an S-k-prime ideal of T then $P \cap R$ is an S-k-prime ideal of R.*

Proof. Let P be an S-k-prime ideal of T. For every $a, b \in T$ with $ab \in P$ implies that $sa \in P$ or $sb \in P$. Now let $xy \in P \cap R$; $x, y \in R \subseteq T$. Then $xy \in P$ which implies that $sx \in P$ or $sy \in P$. So $sx \in P \cap R$ or $sy \in P \cap R$ which implies that $P \cap R$ is S-prime ideal of R. We have $P \cap R \subseteq \overline{P \cap R}$. Let $x \in \overline{P \cap R}$ then $x \in R, x + y \in P \cap R, y \in P \cap R$. This implies that $x \in T, x + y \in P, y \in P$. Since P is k-ideal of R so $x \in P$ and hence $x \in P \cap R$. Therefore $P \cap R = \overline{P \cap R}$. Hence $P \cap R$ is S-k-prime ideal of R. ∎

Let R be a commutative semiring, S a multiplicatively closed subset of R and I be an ideal of R disjoint with S. Let $s \in S$, we denote by $\hat{s}$ the equivalent class of s in R/I. Let $\hat{S} = \{\hat{s} | s \in S\}$, then $\hat{S}$ is a multiplicatively closed subset of R/I.

Proposition 3.16. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and I a k-ideal of R disjoint with S. Let P be a proper k-ideal of R containing I such that $P/I \cap \hat{S} = \emptyset$. Then P is an S-k-prime ideal of R if and only if P/I is an $\hat{S}$-k-prime ideal of R/I.*

Proof. Let P is an S-k-prime ideal of R. There exists an $s \in S$ such that for all $a, b \in R$, if $ab \in P$ then $sa \in P$ or $sb \in P$ and $P = \overline{P}$. Let $\hat{a}, \hat{b} \in R/I$ such that $\hat{a}\hat{b} \in P/I$, then $\widehat{ab} \in P/I$. Since P is a k-ideal so $ab \in P$ and thus $sa \in P$ or $sb \in P$ and therefore $\widehat{sa} \in P/I$ or $\widehat{sb} \in P/I$. Since $P/I \subseteq \overline{P/I}$ so consider that $\hat{x} \in \overline{P/I}$ which implies that $\hat{x} \in R/I, \hat{x} + \hat{y} \in P/I, \hat{y} \in P/I$. Then $x \in R, x + y \in P, y \in P$ and so $x \in P$. Thus we get $\hat{x} \in P/I$. Therefore P/I is an $\hat{S} - k$-prime ideal of R/I.

Conversely, if $P/I \cap \hat{S} = \emptyset$ then P must be disjoint with S. Let P/I be an $\hat{S}$-k-prime ideal of R/I. There exists $\hat{s} \in \hat{S}$ such that for all $\hat{a}, \hat{b} \in R/I$, if $\widehat{ab} \in P/I$, then $\widehat{sa} \in P/I$ or $\widehat{sb} \in P/I$. Let $a, b \in P$ with $ab \in P$ then $\widehat{ab} \in P/I$. Thus $\hat{s}\hat{a} \in P/I$ or $\hat{s}\hat{b} \in P/I$ and hence $sa \in P$ or $sb \in P$. Since $P \subseteq \overline{P}$, it is enough to show the other inclusion. Let $x \in \overline{P}$ which implies that $x \in R$ and $x + y \in P$ for some $y \in P$. Then $\hat{x} \in R/I$ and $\widehat{x+y} \in P/I$ for some $\hat{y} \in P/I$ and so $\hat{x} \in P/I$. Thus we get $x \in P$. Therefore P is an S-k-prime ideal of R. ∎

Now we define S-m-system and S-k-m-system as well as discuss the characterization theorem for the S-prime ideal and S-k-prime ideal of a semiring.

Definition 3.17. Let R be a semiring. A nonempty subset M of R containing a multiplicative closed set S is called an S-m-system if for any $x, y \in R$, there exists an $s \in S$ and $r \in R$ such that $sx, sy \in M$ implies that $xry \in M$.

Theorem 3.18. *Let R be a semiring and S a multiplicatively closed subset of R. A proper ideal P of a semiring R is an S-prime ideal of R if and only if P^c is an S-m-system.*

Proof. Let P be an S-prime ideal of R if and only if there exists an $s \in S$ such that for all $x, y \in R$ if $xRy \subseteq P$ then $sx \in P$ or $sy \in P$ if and only if $sx, sy \in P^c$ then there exists $r \in R$ such that $xry \notin P$ and so $xry \in P^c$ if and only if P^c is an S-m-system. ∎

Definition 3.19. Let R be a semiring. A nonempty subset M of R containing a multiplicative closed set S is called an S-k-m-system if (i) for any $x, y \in R$, there exists an $s \in S$ and $r \in R$ such that $sx, sy \in M$ implies that $xry \in M$ and (ii) $x \in M$ implies that $x \notin \overline{M^c}$.

Example 3.20. Let us consider the commutative semiring $R = \mathbb{Z}_0^+$ and the multiplicatively closed set $S = \{3^n | n \in \mathbb{Z}^+\}$ of R. We define $P = < 6 >$. Then P^c is an S-k-m-system.

Theorem 3.21. *Let R be a semiring and S a multiplicatively closed subset of R. A proper ideal P of a semiring R is an S-k-prime ideal of R if and only if P^c is an S-k-m-system.*

Proof. Let P be a proper ideal of R. Suppose P^c is an S-k-m-system. Let $x, y \in R$ such that $xRy \subseteq \overline{P}$. If possible let $sx \notin P$ and $sy \notin P$ for any $s \in S$ which implies that $sx, sy \in P^c$ for some $s \in S$. But P^c is an S-k-m-system so there exists $r \in R$ such that $xry \in P^c$ and $xry \notin \overline{(P^c)^c} = \overline{P}$. Which is a contradiction. Hence $sx \in P$ or $sy \in P$ and so P is an S-k-prime ideal of R.

Conversely, suppose P is an S-k-prime ideal of R. So P^c is an S-m-system. Let $x \in P^c$ which implies that $x \notin P = \overline{P}$ and thus $x \notin \overline{(P^c)^c}$. Hence P^c is an S-k-m-system. ∎

Definition 3.22. Let R be a semiring, S a multiplicatively closed subset of R not containing 0. The semiring R is said to be an S-prime semiring if and only if $< 0 >$ is an S-prime ideal of R.

Remark 3.23. The notions of S-prime semiring and S-k-prime semiring are the same, since $< 0 >$ is an S-k-prime ideal if and only if it is an S-prime ideal.

Theorem 3.24. *Let R be a semiring, S a multiplicatively closed subset of R not containing 0. The semiring R is an S-prime semiring if and only if there exists $s \in S$ for all $a, b \in R$ with $aRb = 0$ implies that $sa = 0$ or $sb = 0$.*

Proof. Let R be an S-prime semiring. Then $< 0 >$ is an S-prime ideal of R. Let $a, b \in R$ with $aRb = 0 \in < 0 >$. This implies that $sa \in < 0 >$ or $sb \in < 0 >$

and it follows that $sa = 0$ or $sb = 0$.

Conversely, let for any $a, b \in R$ with $aRb = 0$ implies that $sa = 0$ or $sb = 0$. Let for any $a, b \in R$ we have $aRb \in <0>$. It implies that $aRb = 0$ and thus $sa = 0$ or $sb = 0$. Hence we get that $sa \in <0>$ or $sb \in <0>$. Therefore $<0>$ is an S-prime ideal and so R is an S-prime semiring. ■

Definition 3.25. Let R be a commutative semiring and S be any multiplicatively closed subset of R. There exists an $s \in S$ such that for all $a, b \in R$ with $ab = 0$ implies that $sa = 0$ or $sb = 0$ then R is called S-semidomain.

Lemma 3.26. *Center of an S-prime semiring is an S-semidomain.*

Proof. Let R be an S-prime semiring. Consider C to be the center of R. For any $a, b \in C$ with $aRb = 0$. Then $aRb \in <0>$ which implies $ab \in <0>$. Therefore, we have $ab = 0$. Since R is an S-prime semiring, so by Theorem 3.18, there exists an $s \in S$ such that $sa = 0$ or $sb = 0$. Hence C is an S-semidomain. ■

Remark 3.27. It is easier to see that S-semidomain is an S-prime semiring. For commutative semiring, the notions of S-prime semiring and S-semidomain coincide.

Proposition 3.28. *Let R be a commutative semiring, $S \subseteq R$ be a multiplicatively closed set of R and P be a k-ideal of R disjoint with S. Then P is an S-k-prime ideal of R if and only if R/P is an $\hat{S}$-semidomain.*

Proof. Let P is an S-k-prime ideal of R. Consider $\hat{a}, \hat{b} \in R/P$ such that $\hat{a}\hat{b} = \hat{0}$ which implies that $\widehat{ab} = \hat{0} = P$. Since P is a k-ideal so we get $ab \in P$. There exists $s \in S$ such that $sa \in P$ or $sb \in P$. Therefore $\widehat{sa} = P$ or $\widehat{sb} = P$ and thus $\hat{s}\hat{a} = \hat{0}$ or $\hat{s}\hat{b} = \hat{0}$. Hence R/P is an $\hat{S}$-semidomain.

Conversely, let R/P be an $\hat{S}$-semidomain. Consider $ab \in P$ which gives $\widehat{ab} = \hat{a}\hat{b} = \hat{0} = P$. There exists $\hat{s} \in \hat{S}$ such that $\hat{s}\hat{a} = P$ or $\hat{s}\hat{b} = P$ which implies $\widehat{sa} = P$ or $\widehat{sb} = P$. Consequently $sa \in P$ or $sb \in P$. Since P is a k-ideal therefore P is an S-k-prime ideal of R. ■

Let R be a commutative semiring and $S \subseteq R$ be a multiplicatively closed set. Now we consider $M_n(R)$ to be the set of all $n \times n$ matrices with entries over R and $M_n^d(S)$ to be the set of all $n \times n$ diagonal matrices with entries over S.

Lemma 3.29. *Let R be a commutative semiring. A nonempty subset S of R is a multiplicatively closed set if and only if $M_n^d(S)$ is a multiplicatively closed subset of $M_n(R)$.*

Proof. Let S be a multiplicatively closed subset of R. Then $1 \in S$ and for $x, y \in S$ implies that $xy \in S$. It follows that $I \in M_n^d(S)$ and let $A, B \in M_n^d(S)$. Then $A = diag(a_1, a_2, \ldots, a_n)$ and $B = diag(b_1, b_2, \ldots, b_n)$ where $a_i, b_i \in S$. So,

$AB = diag(a_1b_1, a_2b_2, \ldots, a_nb_n)$. Which shows that $AB \in M_n^d(S)$. Thus $M_n^d(S)$ is a multiplicatively closed set.

Conversely, let $M_n^d(S)$ is a multiplicatively closed subset of $M_n(R)$. Then for any $A, B \in M_n^d(S)$ we have $AB \in M_n^d(S)$. We have to show that for any $x, y \in S$ implies that $xy \in S$. We construct $A = diag(x, x, \ldots, x)$ and $B = diag(y, y, \ldots, y)$. This implies that $diag(xy, xy, \ldots, xy) \in M_n^d(S)$ and thus $xy \in S$. Hence S is a multiplicatively closed subset of R. ■

In the following, we establish a relationship between the S-k-prime ideal of a semiring and S-k-prime ideal of its corresponding matrix semiring.
For that, we mention the following Lemma proved in [11].

Lemma 3.30 [11]**.** *If A and B are two ideals of a semiring R then (i) $M_n(AB) = M_n(A)M_n(B)$ and (ii) $A \subseteq B$ if and only if $M_n(A) \subseteq M_n(B)$.*

Proposition 3.31. *Let R be a semiring with identity and S a multiplicatively closed subset of R. A proper k-ideal J of R is an S-k-prime ideal of R if and only if $M_n(J)$ is an $M_n^d(S)$-k-prime ideal of $M_n(R)$.*

Proof. Let J be an S-k-prime ideal of R. We know that the ideals of $M_n(R)$ are of the form $M_(J)$ for every ideal I of R. Suppose $M_n(A), M_n(B)$ be two ideals of $M_n(R)$ such that $M_n(A)M_n(B) \subseteq M_n(J)$. By the above Lemma 3.30 we have $M_n(A)M_n(B) = M_n(AB) \subseteq M_n(J)$. This implies that $AB \subseteq J$. Since J is an S-prime ideal of R so there exists an $s \in S$ such that $sA \subseteq J$ or $sB \subseteq J$. It follows that $M_n(sA) \subseteq M_n(J)$ or $M_n(sB) \subseteq M_n(J)$. Thus there exists a scalar matrix $sI \in M_n^d(S)$ such that $sIM_n(A) \subseteq M_n(J)$ or $sIM_n(B) \subseteq M_n(J)$. Hence $M_n(J)$ is an $M_n^d(S)$-prime ideal of $M_n(R)$. Now $M_n(J) \subseteq \overline{M_n(J)}$. Consider that $A = [a_{ij}], B = [b_{ij}] \in M_n(R)$ such that $A \in \overline{M_n(J)}$ which implies that $A \in M_n(R)$ and $A + B \in M_n(J)$ for some $B \in M_n(J)$. So $a_{ij} \in R, a_{ij} + b_{ij} \in J$ for some $b_{ij} \in J$. Since J is a k-ideal so $a_{ij} \in J$ and hence $A \in M_n(J)$. Thus $M_n(J)$ is an $M_n^d(S)$-k-prime ideal of $M_(R)$.

Conversely, let $M_n(J)$ is be $M_n^d(S)$-prime ideal of $M_n(R)$. Suppose A, B are two ideals of R such that $AB \subseteq J$. This implies that $M_n(A), M_n(B)$ are ideals of $M_n(R)$ and by above Lemma 3.30 we have $M_n(AB) \subseteq M_n(J)$. It follows that $M_n(A)M_n(B) \subseteq M_n(J)$. Since $M_n(J)$ is an $M_n^d(S)$-prime ideal of $M_n(R)$ so there exists $sI \in M_n^d(S)$ such that $sIM_n(A) = M_n(sA) \subseteq M_n(J)$ or $sIM_n(B) = M_n(sB) \subseteq M_n(J)$ and hence $sA \subseteq J$ or $sB \subseteq J$. Thus J is an S-prime ideal of R. As J is a k-ideal so J is an S-k-prime ideal. ■

4. S-k-SEMIPRIME IDEALS OF SEMIRING

In this section, we introduce the notion of S-semiprime and S-k-semiprime ideal of a semiring and discuss their basic properties. We begin with the following definitions.

Definition 4.1. Let R be a semiring, S a multiplicatively closed set of R and I be an ideal of R disjoint with S. We say I is an S-semiprime ideal of R if there exists an $s \in S$ such that for any ideal A of R with $A^2 \subseteq I$ implies that $sA \subseteq I$.

Definition 4.2. An S-semiprime ideal I of a semiring R is said to be an S-k-semiprime ideal of R if $I = \overline{I}$.

Proposition 4.3. *Let R be a semiring and $S \subseteq R$ be a multiplicatively closed set. A proper k-ideal I of a semiring R is an S-k-semiprime ideal of R if and only if for any k-ideal J of R with $J^2 \subseteq I$ implies that $sJ \subseteq I$.*

Proof. Let I be an S-k-semiprime ideal of R. Let J be any k-ideal of R such that $J^2 \subseteq I$ which implies that $sJ \subseteq I$.

To prove the converse, let J be a k-ideal such that $J^2 \subseteq I$ with $sJ \subseteq I$. We have $\overline{J}^2 \subseteq \overline{\overline{J}\,\overline{J}} = \overline{J^2} \subseteq \overline{I} \subseteq I$. Then $s\overline{J} \subseteq I$ which implies that $sJ \subseteq I$. Hence I is S-k-semiprime ideal of R. ■

We are going to introduce a characterization theorem for an S-k-semiprime ideal of a semiring. Initially, the characterization theorem for a semiprime ideal was given by Golan[6] and later by S. Kar et. al. [11] in case of k-semiprime ideal of a semiring. The proofs are similar to Theorem 3.7 and Theorem 3.8.

Theorem 4.4. *Let R be a semiring and S a multiplicatively closed subset of R. Then the following statements are equivalent for an ideal I of a semiring R:*

1. *I is an S-semiprime ideal of a semiring R.*
2. *There exists an $s \in S$ for all $a \in R$, if $aRa \subseteq I$, then $sa \in I$.*

Theorem 4.5. *Let R be a semiring and S a multiplicatively closed subset of R. Then the following statements are equivalent for an ideal I of a semiring R:*

1. *I is an S-k-semiprime ideal of a semiring R.*
2. *There exists an $s \in S$ for any $a \in R$, if $aRa \subseteq \overline{I}$, then $sa \in I$.*

Corollary 4.6. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and I be an ideal of R disjoint with S. Then I is an S-k-semiprime ideal of R if there exists an $s \in S$ such that for any $a \in R$ with $a^2 \in \overline{I}$ implies that $sa \in I$.*

Example 4.7. Let us consider the commutative semiring $R = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in \mathbb{Z}_{12}^{+}\}$ and
$S = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\}$ be the multiplicative subset of R. We consider the ideal $I = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}\}$. Then $I \cap S = \emptyset$ and I is a k-ideal.

Now $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in I$ but $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \notin I$. So I is not a k-semiprime ideal. But there exists $s = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \in S$ such that $s\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in I$. Hence I is an S-k-semiprime ideal of R.

Proposition 4.8. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and I a 2-absorbing k-ideal of R disjoint with S. Then I is an S-k-semiprime ideal of R if and only if $I : s$ is k-semiprime ideal of R for some $s \in S$.*

Proof. Let I be an S-k-semiprime ideal of R there exists an $s \in S$ such that for any $a \in R$ with $a^2 \in \overline{I}$ implies that $sa \in I$. We show, $I : s$ is a k-semiprime ideal of R.

Let $a \in R$ and $a^2 \in I : s$ which implies that $saa \in I$. Since I is a 2-absorbing so it follows that $sa \in I$ or $a^2 \in I$ and thus $sa \in I$. So $a \in I : s$. Thus, $I : s$ is a semiprime ideal of R. Then $I : s \subseteq \overline{I : s}$. Now, let $x \in \overline{I : s}$ imply that $x \in R$ and $x + y \in I : s$ for some $y \in I : s$. This implies that $x \in R$ and $s(x + y) \in I$ for some $sy \in I$. It follows that $sx \in I$ and thus $x \in I : s$. So, $I : s = \overline{I : s}$. Thus, $I : s$ is a k-ideal of R and hence $I : s$ is a k-semiprime ideal of R.

Conversely, let $I : s$ be a k-semiprime ideal. We show, I is an S-k-semiprime ideal. Let $a^2 \in I$ which implies that $sa^2 \in I$ and it follows that $a^2 \in I : s$. We get $a \in I : s$ and hence $sa \in I$. Thus, I is an S-semiprime ideal which implies I is an S-k-semiprime ideal of R since I is a k-ideal of R. ■

Proposition 4.9. *Let R be a commutative semiring and S a multiplicatively closed subset of R disjoint with a k-ideal I of R. If $R \subseteq T$ be an extension of commutative semirings, I be an S-k-semiprime ideal of T then $I \cap R$ is an S-k-semiprime ideal of R.*

Proof. Let I be an S-k-semiprime ideal of T. For every $a \in T$ with $a^2 \in I$ implies that $sa \in I$. Now let $x^2 \in I \cap R$ for $x \in R \subseteq T$. Then $x^2 \in I$ which implies that $sx \in I$.
So $sx \in I \cap R$ which implies that $I \cap R$ is an S-semiprime ideal of R.

We have $I \cap R \subseteq \overline{I \cap R}$. Let $x \in \overline{I \cap R}$ then $x \in R$ and $x + y \in I \cap R$ for some $y \in I \cap R$. This implies that $x \in T$ and $x + y \in I$ for some $y \in I$. Since I

is a k-ideal of R so $x \in I$ and hence $x \in I \cap R$.
Therefore $I \cap R = \overline{I \cap R}$. Hence $I \cap R$ is an S-k-semiprime ideal of R. ■

Proposition 4.10. *Let R be a commutative semiring, $S \subseteq R$ a multiplicatively closed set and J a k-ideal of R disjoint with S. Let I be a proper k-ideal of R containing J such that $I/J \cap \hat{S} = \emptyset$. Then I is an S-k-semiprime ideal of R if and only if I/J is an $\hat{S}$-k-semiprime ideal of R/J.*

Proof. Let I be an S-k-semiprime ideal of R, then there exists an $s \in S$ such that for all $a \in R$ with $a^2 \in I$ implies $sa \in I$ and $I = \overline{I}$. Let $\hat{a} \in R/J$ such that $\hat{a}^2 \in I/J$, then $\widehat{a^2} \in I/J$. Since I is a k-ideal so $a^2 \in I$ and thus $sa \in I$ and therefore $\widehat{sa} \in I/J$. Since $I/J \subseteq \overline{I/J}$ so consider that $\hat{x} \in \overline{I/J}$ which implies that $\hat{x} \in R/J$ and $\hat{x} + \hat{y} \in I/J$ for some $\hat{y} \in I/J$. Then $x \in R$ and $x + y \in I$ for some $y \in I$ and so $x \in I$. Thus we get $\hat{x} \in I/J$. Therefore I/J is an $\hat{S}$-k-semiprime ideal of R/J.

Conversely, if $I/J \cap \hat{S} = \emptyset$ then I must be disjoint with S. Let I/J be an $\hat{S} - k$-semiprime ideal of R/J, then there exists $\hat{s} \in \hat{S}$ such that for all $\hat{a} \in R/J$ with $\widehat{a^2} \in I/J$ implies $\widehat{sa} \in I/J$. Let $a \in I$ with $a^2 \in I$ then we have $\widehat{a^2} \in I/J$. Thus $\hat{s}\hat{a} \in I/J$ and hence $sa \in I$. Since $I \subseteq \overline{I}$ so consider that $x \in \overline{I}$ which implies that $x \in R$ and $x + y \in I$ for some $y \in I$. Then $\hat{x} \in R/J$ and $x \hat{+} y \in I/J$ for some $\hat{y} \in I/J$ and so $\hat{x} \in I/J$. Thus we get $x \in I$. Therefore I is an S-k-semiprime ideal of R. ■

Now similar to definitions of S-m-system and S-k-m-system we can define S-p-system and S-k-p-system respectively and further discuss the characterization theorem for S-semiprime ideal and S-k-semiprime ideal of a semiring.

Definition 4.11. Let R be a semiring. A nonempty subset N of R containing a multiplicative closed set S is called an S-p-system if for any $x \in R$, there exists an $s \in S$ and $r \in R$ such that $sx \in N$ implies that $xrx \in N$.

Theorem 4.12. *Let R be a semiring and S a multiplicatively closed subset of R. A proper ideal I of a semiring R is an S-semiprime ideal of R if and only if I^c is an S-p-system.*

Proof. Let I be an S-semiprime ideal of R if and only if for any $x \in R$ if $xRx \subseteq I$ then there exists an $s \in S$ such that $sx \in I$ if and only if $sx \in I^c$ then there exists $r \in R$ such that $xrx \notin I$ and so $xrx \in I^c$ if and only if I^c is an S-p-system. ■

Definition 4.13. Let R be a semiring. A nonempty subset N of R containing a multiplicative closed set S is called an S-k-p-system if (i)for any $x \in R$, there exists an $s \in S$ and $r \in R$ such that $sx \in N$ implies that $xrx \in N$ and (ii) $x \in N$ implies that $x \notin \overline{N^c}$.

Example 4.14. Let us consider the commutative semiring $R = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in \mathbb{Z}_{12}^{+}\}$ and $S = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\}$ be the multiplicatively closed subset of R. We consider the ideal $I = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}\}$. Then I^c is an S-k-p-system.

Theorem 4.15. *Let R be a semiring and S a multiplicatively closed subset of R. A proper ideal I of a semiring R is an S-k-semiprime ideal of R if and only if I^c is an S-k-p-system.*

Proof. Let I be a proper ideal of R. Suppose I^c is an S-k-p-system. Let $x \in R$ such that $xRx \subseteq \overline{I}$. If possible let $sx \notin I$ for any $s \in S$ which implies that $sx \in I^c$ for some $s \in S$. But I^c is S-k-p-system so there exists $r \in R$ such that $xrx \in I^c$ and $xrx \notin \overline{(I^c)^c} = \overline{I}$. Which is a contradiction. Hence $sx \in I$ and so I is S-k-semiprime ideal of R.

Conversely, suppose I is an S-k-semiprime ideal of R. So I^c is S-p-system. Let $x \in I^c$ which implies that $x \notin I = \overline{I}$ and thus $x \notin \overline{(I^c)^c}$. Hence I^c is S-k-p-system. ■

Definition 4.16. Let R be a semiring, S a multiplicatively closed subset of R not containing 0. The semiring R is said to be an S-semiprime semiring if and only if $< 0 >$ is an S-semiprime ideal of R.

Theorem 4.17. *Let R be a semiring, S a multiplicatively closed subset of R not containing 0. The semiring R is an S-semiprime semiring if and only if there exists $s \in S$ such that for all $a \in R$, if $aRa = 0$ then that $sa = 0$.*

Proof. Suppose R be as S-semiprime semiring. Then $< 0 >$ is an S-semiprime ideal of R. Let $a \in R$ with $aRa = 0 \in < 0 >$. This implies that $sa \in < 0 >$ and it follows that $sa = 0$.

Conversely, let there exists $s \in S$ for all $a \in R$ with $aRb = 0$ implies that $sa = 0$. Let $x \in R$, we have $xRx \in < 0 >$. It implies that $xRx = 0$ and thus $sx = 0$. Hence we get that $sx \in < 0 >$. Therefore $< 0 >$ is S-semiprime ideal and so R is an S-semiprime semiring. ■

Now we give an analogous result to Proposition 3.31. The proof is similar.

Proposition 4.18. *Let R be a semiring with identity and S a multiplicatively closed subset of R. A proper k-ideal J of R is an S-k-semiprime ideal of R if and only if $M_n(J)$ is an $M_n^d(S)$-k-semiprime ideal of $M_n(R)$.*

We now present an analogous result to one of the most exciting ring theory results. An ideal of a ring is a semiprime ideal if and only if it is the intersection of some prime ideals of that ring. T.Y.Lam showed this in [9, Theorem 10.11] in the case of a noncommutative ring. S.Kar et. al. [11] has given a similar result for the case of k-semiprime ideal. Here we attempt to discuss the case of the S-semiprime ideal and S-k-semiprime ideal of a semiring.

Proposition 4.19. *Let R be a semiring and S a multiplicatively closed subset of R. Let M be an S-m-system of a semiring R and P be a maximal ideal, maximal with respect to the condition that M is disjoint with P. Then P is an S-prime ideal of R.*

Proof. Suppose $sx, sy \notin P$ for all $s \in S$ but $< x >< y > \subseteq P$. Since P is maximal with respect to $M \cap P = \emptyset$ so we can write there exists $m, m' \in M \subseteq R$ such that $m \in P+ < x >, m' \in P+ < y >$. There exists $s' \in S$ and $r \in R$ such that $s'm, s'm' \in M$ implies that $mrm' \in M$ because M is an S-m-system.

Moreover, $mrm' \in (P+ < x >)R(P+ < y >) \subseteq P+ < x >< y > \subseteq P$. Which is a contradiction. Hence P is an S-prime ideal. ■

Definition 4.20. Let R be a semiring and S be any multiplicatively closed subset of R. For any ideal I of R, we define $\Gamma(I) = \{r \in R \mid M \cap I \neq \emptyset$ for any S-m-system M containing r$\}$.

Theorem 4.21. *Let R be a semiring and S be a multiplicatively closed subset of R. For any ideal I of R, $\Gamma(I) = \bigcap_{I \subseteq P,\ P \text{ is an } S\text{-prime ideal}} P$.*

Proof. Let $x \in \Gamma(I)$. Let P be an S-prime ideal of R such that $I \subseteq P$. Let us consider that $x \notin P$ then $x \in P^c$. By Theorem 3.18 we have P^c is an S-m-system. So $P^c \cap I \neq \emptyset$. This is a contradiction as $I \subseteq P$. Hence $x \in P$ for all S-prime ideals P such that $I \subseteq P$. Hence $x \in \bigcap_{\text{I} \subseteq \text{P, P is an S-prime ideal}} \text{P}$.

Conversely, let $x \in \bigcap_{\text{I} \subseteq \text{P, P is an S-prime ideal}} \text{P}$. Let us assume that $x \notin \Gamma(I)$. So by definition, there exists an S-m-system M such that $x \in M$ and $M \cap I = \emptyset$. By Zorn's lemma there exists a maximal ideal J of R such that $M \cap J = \emptyset$. By Proposition 4.19, J is an S-prime ideal. Since $x \in M$ so $x \notin J$ and thus $x \notin I$. Therefore $x \notin \bigcap_{\text{I} \subseteq \text{P, P is an S-prime ideal}} \text{P}$. Which is a contradiction. Therefore $x \in \Gamma(I)$. ■

Now we propose the equivalent result of Proposition 4.21 in the S-k-prime ideal version with the following definition.

Definition 4.22. Let R be a semiring and S be any multiplicatively closed subset of R. For any ideal I of R, we define $\overline{\Gamma}(I) = \{r \in R \mid M \cap I \neq \emptyset$ for any S-k-m-system containing r$\}$.

Proposition 4.23. *Let R be a semiring and S be a multiplicatively closed subset of R. For any ideal I of R, $\overline{\Gamma}(I) = \bigcap_{I \subseteq P,\ P \text{ is an } S\text{-}k\text{-prime ideal}} P$.*

Proof. Let $x \in \overline{\Gamma}(I)$. Let P be an S-k-prime ideal of R such that $I \subseteq P$. Let us consider that $x \notin P$ then $x \in P^c$. By Theorem 3.21 we have P^c is an S-k-m-system. So $P^c \cap I \neq \emptyset$. This is a contradiction as $I \subseteq P$. Hence $x \in P$ for all S-k-prime ideals P such that $I \subseteq P$. Hence $x \in \bigcap_{\text{I} \subseteq \text{P, P is an S-k-prime ideal}} \text{P}$.

Conversely, let $x \in \bigcap_{\text{I} \subseteq \text{P, P is an S-k-prime ideal}} \text{P}$. Let us assume that $x \notin \overline{\Gamma}(I)$. So by definition, there exists an S-k-m-system M such that $x \in M$ and $M \cap I = \emptyset$. By Zorn's lemma there exists a maximal ideal J of R such that $M \cap J = \emptyset$. By Proposition 4.19, J is an S-k-prime ideal. Since $x \in M$ so $x \notin J$ and thus $x \notin I$. Therefore $x \notin \bigcap_{\text{I} \subseteq \text{P, P is an S-k-prime ideal}} \text{P}$. Which is a contradiction. Therefore $x \in \overline{\Gamma}(I)$. ∎

Proposition 4.24. *Let R be a semiring and S a multiplicatively closed subset of R. If I and J be two ideals of R such that $I \subseteq J$ then $\Gamma(I) \subseteq \Gamma(J)$ and $\overline{\Gamma}(I) \subseteq \overline{\Gamma}(J)$.*

Proof. Let $r \in \Gamma(I)$ then for any S-m-system M containing r we have $M \cap I \neq \emptyset$. This implies that for any S-m-system M containing r we have $M \cap J \neq \emptyset$. Thus $r \in \Gamma(J)$ and hence $\Gamma(I) \subseteq \Gamma(J)$.

Also, Let $r \in \overline{\Gamma}(I)$ then for any S-k-m-system M containing r we have $M \cap I \neq \emptyset$. This implies that for any S-k-m-system M containing r we have $M \cap J \neq \emptyset$. Thus $r \in \overline{\Gamma}(J)$ and hence $\overline{\Gamma}(I) \subseteq \overline{\Gamma}(J)$. ∎

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