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NOTES ON RINGS AND NEAR-RINGS WITH HOMO MULTIPLIERS

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Abstract

In this study, we provide new types of mappings on the rings and near-rings, which are called homoleft multipliers, homoright multipliers, and homo multipliers. We also investigated how these mappings affected the structure of near-rings. Moreover, we present examples that show the existence of such mappings.

Keywords: 3-prime near-rings, Lie ideals, homoleft multipliers, homoright multipliers, homomultipliers.

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1. Introduction

All properties of a near-ring remain verified in a ring. It might generally be described as near-ring where $(\mathcal{N},+)$ is a group (not necessarily abelian) while $(\mathcal{N},\cdot)$ is a semigroup and $a\cdot(b+c)=a\cdot b+a\cdot c$ for all $a,b,c\in\mathcal{N}$ (left distributive law). This near-ring will be termed a left near-ring. The near-ring $\mathcal{N}$ is referred to as a right near-ring if $(a+b)\cdot c=a\cdot c+b\cdot c$.

In this text, we will only refer to a "near-ring" as a left near-ring with the multiplicative center $Z(\mathcal{N})$. This convention will be strictly followed unless otherwise stated explicitly. Further, we will write ab for just $a.b$ for simplicity of notation. Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$ and usually $\mathcal{N}$ will be 3-prime, if for $x, y \in \mathcal{N}, x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$, for all $x \in \mathcal{N}$ (left distributivity yields $x0 = 0$). For any pair of elements $x, y \in \mathcal{N}$, $x \circ y = xy + yx$ and $[x, y] = xy - yx$ will denote the well-known Jordan product and Lie product respectively. An additive subgroup U of $\mathcal{N}$ is said to be a Lie ideal if $[\mathrm{u}, \mathrm{n}] \in U$, for all $\mathrm{u} \in \mathrm{U}, \mathrm{n} \in \mathcal{N}$.

Near-rings, in which the nonzero elements form a multiplicative group, are called near-fields. They were introduced at the beginning of this century and soon proved useful in coordinating certain important classes of geometric planes. Later on, they also turned out to be an essential tool in studying doubly transitive groups and incidence structures.

The study of "proper" near-rings, on the other hand, culminated in theorems that considerably generalize Wedderburn-Artin's Theorem and Jabobson's Density Theorem in ring theory. For near-rings, density is equivalent to an interesting interpolation property since one is dealing with non-linear functions (in contrast to the "linear situation" in ring theory). Also, when every function on an omega group is (or can be) interpolated by a polynomial function, these structure theorems provide answers. Polynomial near-rings turn out to be very useful in determining generalized ideals in omega groups. Finally, a class of near-rings, the "planar" ones, give rise to balanced incomplete block designs, structures that are needed for experimental designs. From planar near-rings, block designs with extraordinarily high efficiencies can be constructed.

In dealing with literature, many authors have studied commutativity theorems for prime or semiprime rings and near-rings admitting derivations, semiderivations, generalized derivations, generalized semiderivations, multipliers, homoderivations, and generalized homoderivations satisfying certain conditions. For more details, see the references [1],...,[9], and [10]. In [10] El Sofy (2000) defined a new concept which is called "homoderivation" on a ring R to be an additive mapping h from R into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. Thus, he combines two important concepts: derivation and homomorphism on rings; he also proves the commutativity of prime rings by admitting a homoderivation that satisfies some algebraic conditions. Later, several authors studied homoderivations acting on appropriate subsets of the prime ring and $*$-prime rings (see [1] and [8]). Following this line of investigation, in [6], [7] and [9], Boua et al studied the structure of near-rings and also Jordan right ideals equipped with homoderivations and generalized homoderivations, which satisfy some algebraic identities.

Similar to the approach taken by El Sofi and Boua regarding homoderivations, in this article we thought of combining both homomorphism and multiplier into one concept we called "homomultiplier" on rings and near-rings, which is defined as follows:

An additive mapping L from $\mathcal{N}$ into itself is said to be homoright multiplier (or, homoleft multiplier) if $L(xy) = L(x)L(y) + xL(y)(or, L(xy) = L(x)L(y) + L(x)y)$ for all $x, y \in \mathcal{N}$. If L is both homoright multiplier as well as homoleft multiplier, then L will be called homomultiplier.

The example below demonstrates the existence of such ring mappings.

Example 1. Let $\mathcal{S}$ be a ring such that Char $\mathcal{S} \neq 2$. Define

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z, 0 \in \mathcal{S} \right\},$$

and $L_1, L_2, L_3 : \mathcal{R} \longrightarrow \mathcal{R}$ such that:

$$L_1 \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & -\mathrm{z} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_2 \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & -\mathrm{z} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_3 \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & \mathrm{z} \\ 0 & 0 & 0 \end{pmatrix}$$

Then, $\mathcal{R}$ is a ring, we can easily see that L_1 is a homo multiplier (is both a homoleft multiplier and a homoright multiplier) which is not a homomorphism, nor a multiplier, L_2 is a homoleft multiplier which is not a homoright multiplier, nor a homomorphism, nor a left multiplier and L_3 is a homoright multiplier, which is neither a homoleft multiplier, nor homomorphism or right multiplier.

We begin to study this type of mapping on rings, especially semiprime rings, and arrive at a very important result that was proven in Lemma 3, we found that the only homoleft multiplier or homoright multiplier on a semiprime ring is the zero mapping. It is considered a semiprime condition that is very necessary to obtain this result, and example 1 confirms this.

This result made us think about studying these mappings on the near-rings, and we found some important preliminary results there. Furthermore, we proved the most important following result: when $\mathcal{N}$ be a semiprime near-ring and L is a homoleft multiplier (or, homoright multiplier) on such that $L(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $L = 0$. Refer to Theorem 1. That suggests a link between the commutativity of the multiplication in near-rings and homoright multipliers being zero maps.

From all of this, we made an important point that is to change the course of the study and work on studying the commutative of addition. Hence the importance of Lie ideals emerged as we studied this type of mapping on near-rings with Lie ideals, which enriched the article with many distinctive results.

2. Some preliminaries

We begin with several lemmas that are useful throughout the paper, most of which are known for zero-symmetric 3-prime left near-rings and can be found in [2] and [5].

Lemma 1. Let $\mathcal{N}$ be a 3-prime near-ring.

(a) If there is an element $z \neq 0$ of $Z(\mathcal{N})$ such that $z + z \in Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.

(b) If $z \in Z(\mathcal{N})$ and x is an element of $\mathcal{N}$ such that xz or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.

(c) If $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal, then $\mathcal{N}$ is a commutative ring.

Lemma 2. Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $U \subseteq Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.

The following lemma proves an extremely important result in ring theory: if $\mathcal{R}$ is a semiprime ring, then the only additive mapping, which is a homoleft multiplier or a homoright multiplier on $\mathcal{R}$ is the zero mapping.

Lemma 3. Let $\mathcal{R}$ be a semiprime ring. If $\mathcal{R}$ admits a homoleft multiplier or homoright multiplier L, then $L = 0$.

Proof. For all $x, y, z \in \mathcal{R}$, we have

$$\begin{aligned} L(x(yz)) &= L(x)L(yz) + L(x)yz \\ &= L(x)L(y)L(z) + L(x)L(y)z + L(x)yz \end{aligned} \tag{1}$$

and

$$\begin{aligned} L((xy)z) &= L(xy)L(z) + L(xy)z \\ &= L(x)L(y)L(z) + L(x)yL(z) + L(x)L(y)z + L(x)yz \end{aligned} \tag{2}$$

From (1) and (2), we obtain $L(x)yL(z) = 0$ for all $x, y, z \in \mathcal{R}$. Therefore, $L = 0$ by the semiprimeness of $\mathcal{R}$. By the same way we can prove that there is no nonzero homoright multipliers on $\mathcal{R}$. ■

In the rest of this paper, we will study the homomultipliers on near-rings and the Lie ideals of near-rings.

Lemma 4. Let $\mathcal{N}$ be a near-ring.

(i) If L is a homoleft multiplier on $\mathcal{N}$, then $L(xy) = L(x)y + L(x)L(y)$ for all $x, y \in \mathcal{N}$.

(ii) If L is a homoright multiplier on $\mathcal{N}$, then $L(xy) = xL(y) + L(x)L(y)$ for all $x, y \in \mathcal{N}$.

Proof. (i) Let L be a homoleft multiplier on $\mathcal{N}$. Then, for all $x, y \in \mathcal{N}$ we have

$$\begin{aligned} L(x(y+y)) &= L(x)L(y+y) + L(x)(y+y) \\ &= L(x)L(y) + L(x)L(y) + L(x)y + L(x)y. \end{aligned} \tag{3}$$

Also, for all $x, y \in \mathcal{N}$ we get

$$\begin{aligned} L(x(y+y)) &= L(xy+xy) \\ &= L(xy) + L(xy) \\ &= L(x)L(y) + L(x)y + L(x)L(y) + L(x)y \end{aligned} \tag{4}$$

Comparing (3) and (4), we conclude that

$$L(x)L(y) + L(x)y = L(x)y + L(x)L(y) \text{ for all } x, y \in \mathcal{N}.$$

(ii) By the same way as used in (i), we can prove that $L(xy) = xL(y) + L(x)L(y)$ for all $x, y \in \mathcal{N}$, when L is a homoright multiplier. ■

Lemma 5. Let $\mathcal{N}$ be a near-ring.

(i) If $\mathcal{N}$ admits a homoright multiplier L, then

$$L(xy)L(z) = L(x)L(yz) + xL(y)L(z) \text{ for all } x, y, z \in \mathcal{N}.$$

(ii) If $\mathcal{N}$ admits a homoleft multiplier L, then

$$L(xy)(L(z) + z) = L(x)L(y)L(z) + L(x)L(y)z + L(x)yz \text{ for all } x, y, z \in \mathcal{N}.$$

and

$$L(xy)(z + L(z)) = L(x)yz + L(x)L(y)z + L(x)L(y)L(z) \text{ for all } x, y, z \in \mathcal{N}.$$

Proof. (i) Let L be a homoright multiplier of $\mathcal{N}$, then, for all $x, y, z \in \mathcal{N}$ we have

$$\begin{aligned} L(x(yz)) &= L(x)L(yz) + xL(yz) \\ &= L(x)L(yz) + xL(y)L(z) + xyL(z) \end{aligned} \tag{5}$$

and also we have

$$L((xy)z) = L(xy)L(z) + xyL(z) \text{ for all } x, y, z \in \mathcal{N}. \tag{6}$$

Using (5) and (6), we can easily arrive at $L(xy)L(z) = L(x)L(yz) + xL(y)L(z)$ for all $x, y, z \in \mathcal{N}$.

(ii) Let L be a homoleft multiplier of $\mathcal{N}$. Then, for all $x, y \in \mathcal{N}$ we get

$$\begin{aligned} L(x(yz)) &= L(x)L(yz) + L(x)yz \\ &= L(x)L(y)L(z) + L(x)L(y)z + L(x)yz \end{aligned} \tag{7}$$

and

$$\begin{aligned} L((xy)z) &= L(xy)L(z) + L(xy)z \\ &= L(xy)(L(z) + z) \end{aligned} \tag{8}$$

According to (7) and (8), we get

$$L(xy)(L(z) + z) = L(x)L(y)L(z) + L(x)L(y)z + L(x)yz \text{ for all } x, y, z \in \mathcal{N}.$$

Although for all $x, y, z \in \mathcal{N}$, we have

$$\begin{aligned} L(x(yz)) &= L(x)yz + L(x)L(yz) \\ &= L(x)yz + L(x)L(y)z + L(x)L(y)L(z) \end{aligned} \tag{9}$$

and

$$\begin{aligned} L((xy)z) &= L(xy)z + L(xy)L(z) \\ &= L(xy)(z + L(z)) \end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$L(xy)(z + L(z)) = L(x)yz + L(x)L(y)z + L(x)L(y)L(z) \text{ for all } x, y, z \in \mathcal{N}.$$

■

Lemma 6. Let $\mathcal{N}$ be a semiprime near-ring. Then there is no nonzero homo-multipliers on $\mathcal{N}$.

Proof. Let L be a homomultiplier of $\mathcal{N}$, then

$$L(xy) = L(x)L(y) + L(x)y \text{ for all } \ x, y \in \mathcal{N}$$

and

$$L(xy) = L(x)L(y) + xL(y) \text{ for all } \ x, y \in \mathcal{N}.$$

From the above expressions, we obtain, $L(x)y = xL(y)$ for all $x, y \in \mathcal{N}$, putting $y = yz$ in the last relation and use it to obtain $xL(y)L(z) = 0$ for all $x, y, z \in \mathcal{N}$, in view of semiprimeness of $\mathcal{N}$, we get $L(y)L(z) = 0$ for all $y, z \in \mathcal{N}$. Therefore, $0 = L(y)L(zx) = L(y)L(z)L(x) + L(y)zL(x) = L(y)zL(x)$ for all $x, y, z \in \mathcal{N}$. Thus, $L = 0$ as $\mathcal{N}$ is semiprime. ■

Lemma 7. Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If L is a homoleft multiplier (or, homoright multiplier) on $\mathcal{N}$ such that $L(U) = \{0\}$, then $L = 0$.

Proof. Let L be a homoleft multiplier of $\mathcal{N}$. Thus $0 = L([u, n]) = L(n)u$ for all $u \in U, n \in \mathcal{N}$, replace u by $[u, x]$ to get $0 = L(n)[u, x] = L(n)xu$ for all $u \in U, n, x \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $U \neq \{0\}$, we conclude that $L = 0$.

If L is a homoright multiplier of $\mathcal{N}$ and $L(U) = \{0\}$, then $0 = L([u, n]) = uL(n)$ for all $u \in U, n \in \mathcal{N}$, replacing n by nx, we obtain $0 = unL(x)$ for all $u \in U, n, x \in \mathcal{N}$. Using the fact that $\mathcal{N}$ is 3-prime and $U \neq \{0\}$, we find that $L = 0$. ■

3. Main Results

Theorem 1. Let $\mathcal{N}$ be a semiprime near-ring. If L is a homoleft multiplier (or, homoright multiplier) on $\mathcal{N}$ such that $L(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $L = 0$.

Proof. Let L be a homoleft multiplier of $\mathcal{N}$. We have

$$L(xy)(L(z) + z) = (L(z) + z)L(xy) \text{ for all } \ x, y, z \in \mathcal{N}.$$

Therefore, by Lemma 5 (ii), we get

$$\begin{aligned}
L(x)L(y)L(z) + L(x)L(y)z + L(x)yz &= L(xy)(L(z) + z) \\
&= (L(z) + z)L(x)L(y) + (L(z) + z)L(x)y \\
&= L(x)L(y)(L(z) + z) + L(x)(L(z) + z)y \\
&= L(x)L(y)L(z) + L(x)L(y)z + (L(x)L(z) \\
&\quad + L(x)z)y \\
&= L(x)L(y)L(z) + L(x)L(y)z + L(xz)y
\end{aligned}$$

$$\begin{aligned} &= L(x)L(y)L(z) + L(x)L(y)z + yL(xz) \\ &= L(x)L(y)L(z) + L(x)L(y)z + yL(x)L(z) \\ &\quad +yL(x)z \end{aligned}$$

it follows that $yL(x)L(z) = L(x)yL(z) = 0$ for all $x, y, z \in \mathcal{N}$. Since $\mathcal{N}$ is a semiprime, we conclude that $L = 0$.
Now, if L is a homoright multiplier of $\mathcal{N}$, then

$$L(xy)L(z) = L(z)L(xy) \text{ for all } \ x, y, z \in \mathcal{N},$$

by Lemma 5 (i), we get

$$L(x)L(y)L(z) + L(x)yL(z) + xL(y)L(z) = L(z)L(x)L(y) + L(z)xL(y)$$

and that easily implies $L(x)yL(z) = 0$ for all $x, y, z \in \mathcal{N}$, which assures that $L = 0$ by semiprimeness of $\mathcal{N}$. ■

Theorem 2. Let $\mathcal{N}$ be a 3-prime near-ring and U be a Lie ideal of $\mathcal{N}$. If L is a homoleft multiplier (or, homoright multiplier) on $\mathcal{N}$ such that $L(U) \subseteq Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian or $L = 0$.

Proof. Let L be a homoleft multiplier of $\mathcal{N}$, and $Z(\mathcal{N}) \supseteq L(U) \neq \{0\}$, then there exists $u \in U$ such that $0 \neq L(u) \in Z(\mathcal{N})$, but $L(u) + L(u) = L(u + u) \in Z(\mathcal{N})$. Thus $(\mathcal{N}, +)$ is abelian by Lemma 1 (a).
Now suppose that $L(U) = \{0\}$, by using Lemma 7, we arrive at $L = 0$. The second part of theorem can be proved by the same way. ■

Theorem 3. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, U a nonzero Lie ideal of $\mathcal{N}$. If L is a homoleft multiplier of $\mathcal{N}$ which verifies one of the following assertions:

(i) $L(u \circ n) = 0$ for all $u \in U, n \in \mathcal{N}$,

(ii) $L([u, n]) = 0$ for all $u \in U, n \in \mathcal{N}$,

then $(\mathcal{N}, +)$ is abelian.

Proof. (i) Suppose that

$$L(u \circ n) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{11}$$

It follows that

$$\begin{aligned} 0 &= L(u \circ un) \\ &= L(u(u \circ n)) \\ &= L(u)L(u \circ n) + L(u)(u \circ n) \\ &= L(u)(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}, \end{aligned}$$

which implies $L(u)un = -L(u)nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the previous equation and use it to get $L(u)n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, and the 3-primeness of $\mathcal{N}$ gives

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{12}$$

Now, if there exists $u_0 \in U \cap Z(\mathcal{N})$, then replace n by $u_0 n$ in (11) and invoking it, we obtain

$$\begin{aligned} 0 &= L\left(u_0(u \circ n)\right) \\ &= L\left(u_0\right) L(u \circ n) + L\left(u_0\right)(u \circ n) \\ &= L\left(u_0\right)(u \circ n) \\ &= L\left(u_0\right)(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L\left(u_0\right) un = -L\left(u_0\right) nu$ for all $u \in U, n \in \mathcal{N}$. Taking nt in place of n in the above equation and use it to get $L\left(u_0\right) n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either $L\left(u_0\right) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$ by applying the 3-primeness of $\mathcal{N}$.

Returning to (12), we obtain either $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$, it follows that $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

(ii) Suppose that

$$L([u,n]) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{13}$$

It follows that

$$\begin{aligned} 0 &= L([u,un]) \\ &= L(u[u,n]) \\ &= L(u)L([u,n]) + L(u)[u,n] \\ &= L(u)[u,n] \text{ for all } u \in U, n \in \mathcal{N}, \end{aligned}$$

which implies $L(u)un = L(u)nu$ for all $u \in U, n \in \mathcal{N}$, by choosing nt in place of n in the previous equation and use it to get $L(u)n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, and the 3-primeness of $\mathcal{N}$ gives

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{14}$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, replace n by $u_0 n$ in (13) and using it, we find that

$$\begin{aligned} 0 &= L\left(u_0[u,n]\right) \\ &= L\left(u_0\right) L([u,n]) + L\left(u_0\right)[u,n] \\ &= L\left(u_0\right)[u,n] \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L(u_0)un = L(u_0)nu$ for all $u \in U, n \in \mathcal{N}$. For $n = nt$, the last relation gives easily $L(u_0)n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either $L(u_0) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$ by the 3-primeness of $\mathcal{N}$.

Returning to (14), we obtain either $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$ which forces that $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2. ■

Theorem 4. Let $\mathcal{N}$ be a 3-prime near-ring, U a nonzero Lie ideal of $\mathcal{N}$ and L be a homoright multiplier of $\mathcal{N}$ such that

(i) $L(u \circ n) = (u \circ n)$ for all $u \in U, n \in \mathcal{N}$, or

(ii) $L([u,n]) = [u,n]$ for all $u \in U, n \in \mathcal{N}$,

then $(\mathcal{N}, +)$ is abelian.

Proof. (i) By assumption, we have

$$L(u \circ n) = (u \circ n) \text{ for all } u \in U, n \in \mathcal{N} \tag{15}$$

Replacing n by un in (15) and using it again, we get

$$\begin{aligned} u(u \circ n) &= L(u(u \circ n)) \\ &= L(u)L(u \circ n) + uL(u \circ n) \\ &= L(u)(u \circ n) + u(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L(u)un = -L(u)nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the last equation and use it to get $L(u)n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{16}$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, replace n by $u_0 n$ in (15) and invoking it to get

$$\begin{aligned} u_0(u \circ n) &= L(u_0[u,n]) \\ &= L(u_0) L(u \circ n) + u_0 L(u \circ n) \\ &= L(u_0)(u \circ n) + u_0(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L(u_0)(u \circ n) = 0$ for all $u \in U, n \in \mathcal{N}$, equivalently $L(u_0)un = -L(u_0)nu$ for all $u \in U, n \in \mathcal{N}$. Replace n by nt in the latter equation, we can easily arrive at $L(u_0)n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either $L(u_0) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$ by the 3-primeness of $\mathcal{N}$.
Returning to (16), we obtain either $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$, it follows that $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

(ii) Assuming that

$$L([u,n]) = [u,n] \text{ for all } u \in U, n \in \mathcal{N}. \tag{17}$$

Replacing n by un in (17) and using it again, we get

$$\begin{aligned} u([u,n]) &= L(u[u,n]) \\ &= L(u)L([u,n]) + uL([u,n]) \\ &= L(u)([u,n]) + u[u,n] \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L(u)un = L(u)nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the last equation, we easily find that $L(u)n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, by the 3-primeness of $\mathcal{N}$, we obviously have

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{18}$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, replace n by $u_0 n$ in (17) and use it to get

$$\begin{aligned} u_0[u,n] &= L(u_0[u,n]) \\ &= L(u_0) L([u,n]) + u_0 L([u,n]) \\ &= L(u_0)[u,n] + u_0[u,n] \text{ for all } u \in U, n \in \mathcal{N}, \end{aligned}$$

which implies that $L(u_0) un = L(u_0) nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the previous equation, we find that $L(u_0) n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so, by the 3-primeness of $\mathcal{N}$, we arrive at $L(u_0) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$. Return to (17), we obtain either $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$, it follows that $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

■

Theorem 5. Let $\mathcal{N}$ be a 3-prime near-ring and U a nonzero Lie ideal of $\mathcal{N}$ and L is a homoright multiplier of $\mathcal{N}$. If $\mathcal{N}$ has one of the following properties:

(i) $L([u,n]) = u \circ n$ for all $u \in U, n \in \mathcal{N}$,

(ii) $L(u \circ n) = [u,n]$ for all $u \in U, n \in \mathcal{N}$,

then $(\mathcal{N}, +)$ is abelian.

Proof. (i) Suppose that

$$L([u,n]) = u \circ n \text{ for all } u \in U, n \in \mathcal{N}. \tag{19}$$

Replace n by un in (19) and use it to get

$$\begin{aligned} u(u \circ n) &= L(u[u,n]) \\ &= L(u)L([u,n]) + uL([u,n]) \\ &= L(u)(u \circ n) + u(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Therefore, $L(u)un = -L(u)nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the last equation and use it to get $L(u)n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{20}$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, replace n by $u_0 n$ in (19), we find that

$$\begin{aligned} u_0(u \circ n) &= L(u_0[u,n]) \\ &= L(u_0)L([u,n]) + u_0 L([u,n]) \\ &= L(u_0)(u \circ n) + u_0(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

It follows that $L(u_0)un = -L(u_0)nu$ for all $u \in U$, $n \in \mathcal{N}$, replace n by nt in the last equation, we easily obtain $L(u_0)n[-u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either $L(u_0) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$. Return to (20), we find that $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$ so, $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

(ii) Let L be a homoright multiplier of $\mathcal{N}$ and

$$L(u \circ n) = [u,n] \text{ for all } u \in U, n \in \mathcal{N}. \tag{21}$$

Replace n by un in (21) and using it again, we have

$$\begin{aligned} u[u,n] &= L(u(u \circ n)) \\ &= L(u)L(u \circ n) + uL(u \circ n) \\ &= L(u)[u,n] + u[u,n] \text{ for all } u \in U, n \in \mathcal{N}, \end{aligned}$$

which implies that $L(u)un = L(u)nu$ for all $u \in U, n \in \mathcal{N}$. For $n = nt$, the preceding relation gives $L(u)n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so by the 3-primeness of $\mathcal{N}$, we obtain

$$L(u) = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{22}$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, replace n by $u_0 n$ in (21) and invoking it, we get

$$\begin{aligned} u_0[u,n] &= L(u_0(u \circ n)) \\ &= L(u_0)L(u \circ n) + u_0 L(u \circ n) \\ &= L(u_0)[u,n] + u_0[u,n] \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

Thus, $L(u_0)un = L(u_0)nu$ for all $u \in U, n \in \mathcal{N}$, replace n by nt in the last equation and use it to get $L(u_0)n[u,t] = 0$ for all $u \in U, n, t \in \mathcal{N}$, so either $L(u_0) = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$. Return to (22), we obtain $L(U) = \{0\}$ or $U \subseteq Z(\mathcal{N})$, it follows that $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2. ■

Theorem 6. Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, U a nonzero Lie ideal of $\mathcal{N}$ and L is a homoleft multiplier (or, homoright multiplier) of $\mathcal{N}$. If $L([u,n]) \pm u \circ n \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, then $L = 0$.

Proof. Suppose that

$$L([u,n]) + u \circ n \in Z(\mathcal{N}) \text{ for all } u \in U, n \in \mathcal{N}. \tag{23}$$

For $n = u$, we get $2u^2 \in Z(\mathcal{N})$ for all $u \in U$. Also, for $n = u^2$, we obtain $u\left(2u^2\right) = u \circ u^2 \in Z(\mathcal{N})$ for all $u \in U$. By Lemma 1 (b), we get $2u^2 = 0$ or $u \in Z(\mathcal{N})$ for all $u \in U$, using the 2-torsion freeness of $\mathcal{N}$, we arrive at

$$u^2 = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U \tag{24}$$

Suppose there exists $u_0 \in U$ such that $u_0 \in Z(\mathcal{N})$, putting u_0 instead of u in our hypothesis to get $u_0 \circ n = 2u_0 n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$. By Lemma 1 (b), we obtain either $u_0 = 0$ or $2n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, then (24) becomes

$$u^2 = 0 \text{ for all } u \in U \text{ or } 2n \in Z(\mathcal{N}) \text{ for all } n \in \mathcal{N}.$$

If $u^2 = 0$ for all $u \in U$, then $0 = u(u+v)^2 = uvu$ for all $u, v \in U$, it follows that $0 = vu[-v,n]u = vunvu$ for all $u, v \in U, n \in \mathcal{N}$ and the 3-primeness of $\mathcal{N}$ implies $vu = 0$ for all $u, v \in U$. Thus $0 = v[u,n] = -vnu = vn(-u)$ for all $u, v \in U, n \in \mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we find that $U = \{0\}$; a contradiction.

If $2n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, then $2n^2 = n(2n) \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, by Lemma 1(b), we obtain $2n = 0$ or $n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$. By the 2-torsion freeness of $\mathcal{N}$, we arrive at $n = 0$ or $n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, that is $n \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, therefore, $\mathcal{N}$ is a commutative ring by Lemma 1 (c), and Lemma 3 forces that $L = 0$.

By the same way we can prove the second part of our theorem. ■

Theorem 7. Let $\mathcal{N}$ be a 3-prime near-ring and U be a Lie ideal of $\mathcal{N}$.

(i) If L is a homoright multiplier (or, homoleft multiplier) on $\mathcal{N}$ such that $L([u,n]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, then $(\mathcal{N},+)$ is abelian or $L = 0$.

(ii) If L is a homoright multiplier (or, homoleft multiplier) on $\mathcal{N}$ such that $L(u \circ n) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, then $(\mathcal{N},+)$ is abelian or $L = 0$.

Proof. (i) If L is a homoright multiplier and $L([u,n]) \in Z(\mathcal{N})$, by our assumption we have $L(u[u,n]) = L([u,un]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, so we obtain $L(u[u,n])L(z) = L(z)L(u[u,n])$ for all $u \in U, n, z \in \mathcal{N}$, using Lemma 5 (i) lastly forces

$$L(u)L[u,n]L(z)+L(u)[u,n]L(z)+uL[u,n]L(z) = L(z)L(u)L([u,n])+L(z)uL([u,n]) \tag{25}$$

Let $z = [v,m]$, where $v \in U, m \in \mathcal{N}$, in (25) to get $L(u)[u,n]L([v,m]) = 0$ which can be written as $L(u)[u,n]\mathcal{N}L([v,m]) = \{0\}$ for all $u, v \in \mathrm{U}, m, n \in \mathcal{N}$, using the 3-primeness of $\mathcal{N}$, we conclude that

$$L(u)[u,n] = 0 \text{ or } L([v,m]) = 0 \text{ for all } u, v \in U, m, n \in \mathcal{N}. \tag{26}$$

If $L(u)[u,n]=0$ for all $u\in U, n\in \mathcal{N}$, put $n=nt$ in the last equation and using it again we can easily find that $L(u)\mathcal{N}[u,t]=\{0\}$ for all $u\in U, t\in\mathcal{N}$, so

$$L(u)=0 \ \text{ or } \ [u,t]=0 \text{ for all } u\in U, t\in\mathcal{N}. \tag{27}$$

If there exists $u_0\in U$ such that $L(u_0)=0$, then from our hypothesis, we obtain $L([u_0,m])=u_0L(m)\in Z(\mathcal{N})$ for all $m\in\mathcal{N}$, thus $u_0L([v,m])\in Z(\mathcal{N})$ for all $v\in U, m\in\mathcal{N}$, and using Lemma 1(b), we get $u_0\in Z(\mathcal{N})$ or $L([v,m])=0$ for all $v\in U, m\in\mathcal{N}$, the both cases of (27) implies $U\subseteq Z(\mathcal{N})$ or $L([v,m])=0$ for all $v\in U, m\in\mathcal{N}$ and by Lemma 2, we conclude that $(\mathcal{N},+)$ is abelian or $L([v,m])=0$ for all $v\in U, m\in\mathcal{N}$.
Now, if $L([v,m])=0$ for all $v\in U$, $m\in\mathcal{N}$, then $L(vm)=L(mv)$ for all $v\in U$, $m\in\mathcal{N}$. Taking $[w,x]$ in place of v, where $w\in U$, $x\in\mathcal{N}$, in the last equation and use the fact that $L([w,x])=0$ for all $w\in U$, $x\in\mathcal{N}$, we can arrive at $[w,x]L(m)=0$, where $w\in U$, $x,m\in\mathcal{N}$, it follows that $0=[w,x]L(mn)=[w,x]mL(n)$ which gives the desired result by using Lemma 2.

Now, suppose that L is a homoleft multiplier of $\mathcal{N}$, and $L([u,n])\in Z(\mathcal{N})$ for all $u\in U, n\in\mathcal{N}$. If $Z(\mathcal{N})=\{0\}$, then

$$\begin{aligned} 0 &= L([u,un]) \\ &= L(u[u,n]) \\ &= L(u)[u,n] \text{ for all } u\in U, n\in\mathcal{N}. \end{aligned}$$

$L(u)un=L(u)nu$ for all $u\in U, n\in\mathcal{N}$, put $n=mt$, we easily find $L(u)\mathcal{N}[u,t]=\{0\}$ for all $u\in U$, $t\in\mathcal{N}$, by the 3-primeness of $\mathcal{N}$, we arrive at $L(u)=0$ or $u\in Z(\mathcal{N})$ for all $u\in U$, $n\in\mathcal{N}$, in all cases, we conclude that $L(u)=0$ for all $u\in U$, therefore, $L=0$ by Lemma 7.

Now, assume that $Z(\mathcal{N})\neq\{0\}$, replacing n by zn, where $z\in Z(\mathcal{N})-\{0\}$ in our assumption, we find that $L([u,zn])=L([u,n]z)=L([u,n])(L(z)+z)\in Z(\mathcal{N})$ for all $u\in U, n\in\mathcal{N}$, using Lemma 1 (b) lastly gives

$$L([u,n])=0 \ \text{ or } \ L(z)+z\in Z(\mathcal{N}) \ \text{ for all } \ u\in U, n\in\mathcal{N}. \tag{28}$$

If $L([u,n])=0$ for all $u\in U, n\in\mathcal{N}$, using the same proof as used above in the first case we arrive at

$$L(u)=0 \ \text{ or } \ u\in Z(\mathcal{N}) \ \text{ for all } \ u\in U \tag{29}$$

If there is $u_0\in U$ such that $u_0\in Z(\mathcal{N})$, then the fact that $L([u,n])=0$ for all u $\in U, n\in\mathcal{N}$ implies $0=L([u,u_0n])=L(u_0[u,n])=L(u_0)[u,n]$ for all $u\in U, n\in\mathcal{N}$, so $L(u_0)un=L(u_0)nu$ for all $u\in U, n\in\mathcal{N}$, put $n=n^{'}t$ in the last equation and use it to get $L(u_0)\mathcal{N}[u,t]=\{0\}$ for all $u\in U, t\in\mathcal{N}$, by the 3-primeness we arrive at $L(u_0)=0$ or $u\in Z(\mathcal{N})$ for all $u\in U$, then (29) becomes

$L(U) = \{0\}$ or $u \subseteq Z(\mathcal{N})$ for all $u \in U$, it follows that either $L = 0$ or $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

Supposing that $L(z) + z \in Z(\mathcal{N})$, then $L(xy)(L(z) + z) = (L(z) + z)L(xy)$ for all $x, y \in \mathcal{N}$. By Lemma 5(ii) we get

$$(30) L(xy)(L(z) + z) = L(x)L(y)L(z) + L(x)L(y)z + L(x)yz \text{ for all } x, y \in \mathcal{N},$$

also, for all $x, y \in \mathcal{N}$, we have

$$\begin{aligned} (L(z) + z)L(xy) &= (L(z) + z)L(x)L(y) + (L(z) + z)L(x)y \\ &= L(x)L(y)L(z) + L(x)L(y)z + L(x)yL(z) + L(x)yz \end{aligned} \tag{31}$$

From (30) and (31) we obtain $L(x)yL(z) = 0$ for all $x, y \in \mathcal{N}$, using the 3-primeness of $\mathcal{N}$ we obtain either $L = 0$ or $L(z) = 0$. Suppose that $L(z) = 0$, since $L(zx) = L(xz)$ for all $x \in \mathcal{N}$, we get $L(x)z = 0$ for all $x \in \mathcal{N}$, and the 3-primeness of $\mathcal{N}$ with $z \neq 0$ forces that $L = 0$.

(ii) If L is a homoright multiplier of $\mathcal{N}$ and $L(u \circ n) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, by assumption we have $L(u(u \circ n)) = L((u \circ un)) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, so we can get $L(u(u \circ n))L(z) = L(z)L(u(u \circ n)\ )$ for all $u \in U, n, z \in \mathcal{N}$, using Lemma 5(i) lastly, we obtain

$$L(u)L(u \circ n)L(z) + L(u)(u \circ n)L(z) + uL(u \circ n)L(z) = L(z)L(u)L(u \circ n) + L(z)uL(u \circ n) \tag{32}$$

Let $z = (v \circ m)$ where $v \in U$, $m \in \mathcal{N}$, in (32) to get $L(u)(u \circ n)L(v \circ m) = 0$ for all $u, v \in \mathcal{N}$, which can be written as $L(u)(u \circ n)\mathcal{N}L(v \circ m) = \{0\}$ for all $u, v \in U, m, n \in \mathcal{N}$, using the 3-primeness of $\mathcal{N}$ we conclude that

$$(33) \qquad L(u)(u \circ n) = 0 \text{ or } L(v \circ m) = 0 \text{ for all } u, v \in U, m, n \in \mathcal{N}.$$

If $L(u)(u \circ n) = 0$ for all $u \in \mathrm{U}, n \in \mathcal{N}$, then $L(u)un = -L(u)nu$ for all $u \in U, n \in \mathcal{N}$, putting $n = nt$ in the last equation and use it to implies $L(u)\mathcal{N}[-u, t] = \{0\}$ for all $u \in U, t \in \mathcal{N}$, by the 3-primeness of $\mathcal{N}$, we arrive at $L(u) = 0$ or $[-u, n] = 0$ for all $u \in U, n \in \mathcal{N}$. If there is $u_0 \in U$ such that $L(u_0) = 0$, then from our hypothesis we obtain $L\,(u_0 \circ m) = u_0 L(m) \in Z(\mathcal{N})$ for all $m \in \mathcal{N}$, thus $u_0 L(v \circ m) \in Z(\mathcal{N})$ for all $v \in U, m \in \mathcal{N}$, and using Lemma 1(b) lastly implies that

$$(34) \qquad u_0 \in Z(\mathcal{N}) \text{ or } L(v \circ m) = 0 \text{ for all } v \in U, m \in \mathcal{N},$$

in all cases, we can conclude

$$U \subseteq Z(\mathcal{N}) \text{ or } L(v \circ m) = 0 \text{ for all } v \in U, m \in \mathcal{N}$$

If $L(v \circ m) = 0$ for all $v \in U, m \in \mathcal{N}$, then

$$(35) \qquad L(vm) = -L(mv) \text{ for all } v \in U, m \in \mathcal{N}.$$

Now, if we replace m by $(w \circ x)$, where $w \in U$, $x \in \mathcal{N}$, in (35) and use the fact $L(w \circ x) = 0$ for all $w \in U, x \in \mathcal{N}$, we can arrive at

$$(w \circ x)L(v) = 0 \text{ for all } v \in U, x, m \in \mathcal{N}. \tag{36}$$

It follows that $(w \circ x)L([v, n]) = 0$ for all $w, v \in U$, $x, n \in \mathcal{N}$, so $(w \circ x)L(vn) = (w \circ x)L(nv)$ for all $w, v \in U$, $x, n \in \mathcal{N}$. Using (35) in the latter expression implies $2(w \circ x)L(vn) = 0$ for all $w, v \in U, x, n \in \mathcal{N}$, by the 2-torsion freeness of $\mathcal{N}$, and (36) lastly implies $(w \circ x)vL(n) = 0$ for all $w, v \in U, x, n \in \mathcal{N}$ and this result leads to $0 = (w \circ x)vL(mn) = (w \circ x)vmL(n)$ for all $w, v \in U, x, n, m \in \mathcal{N}$, by the 3-primeness of $\mathcal{N}$, we conclude that either $L = 0$ or $(w \circ x)v = 0$ for all $w, v \in U, x \in \mathcal{N}$, so we can easily verify $(w \circ x) = 0$ for all $w \in U, x \in \mathcal{N}$. i.e. $wx = -xw$ for all $w \in U, x \in \mathcal{N}$, if we replace x by nx in the last expression and use it, we obtain $x[-w, n] = 0$ for all $w \in U, x, n \in \mathcal{N}$, we conclude that $U \subseteq Z(\mathcal{N})$ which gives the desired result by Lemma 2.

Now, Suppose that L is a homoleft multiplier of $\mathcal{N}$ and $L(u \circ n) \in Z(\mathcal{N})$ for all $u \in U$, $n \in \mathcal{N}$. If $Z(\mathcal{N}) = \{0\}$, then

$$\begin{aligned} 0 &= L(u \circ un) \\ &= L(u(u \circ n)) \\ &= L(u)(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}. \end{aligned}$$

That is, $L(u)un = -L(u)nu$ for all $u \in U$, $n \in \mathcal{N}$. Putting $n = mt$ in the above equation, we can easily arrive at $L(u)\mathcal{N}[-u, t] = \{0\}$ for all $u \in U$, $t \in \mathcal{N}$, then by the 3-primeness of $\mathcal{N}$ we arrive at $L(u) = 0$ or $-u \in Z(\mathcal{N})$ for all $u \in U$. Since $Z(\mathcal{N}) = \{0\}$, the both cases force that $L(u) = 0$ for all $u \in U$, therefore, $L = 0$ by Lemma 7. Now, we suppose that $Z(\mathcal{N}) \neq \{0\}$, then there exists a nonzero element $z \in Z(\mathcal{N})$, from our assumption we find that $L(u \circ zn) = L((u \circ n)z) = L(u \circ n)(L(z) + z) \in Z(\mathcal{N})$ for all $u \in U$, $n \in \mathcal{N}$, using Lemma 1(b) lastly gives

$$L(u \circ n) = 0 \text{ or } L(z) + z \in Z(\mathcal{N}) \text{ for all } u \in U, n \in \mathcal{N}.$$

If $L(u \circ n) = 0$ for all $u \in U$, $n \in \mathcal{N}$, using the same proof as used above in the first case we arrive at

$$L(u) = 0 \text{ or } -u \in Z(\mathcal{N}) \text{ for all } u \in U.$$

If there is $u_0 \in U \cap Z(\mathcal{N})$, then using the fact $L(u \circ n) = 0$ for all $u \in U$, $n \in \mathcal{N}$ implies that

$$\begin{aligned} 0 &= L(u \circ (-u_0)n) \\ &= L((-u_0)(u \circ n)) \\ &= L(-u_0)(u \circ n) \text{ for all } u \in U, n \in \mathcal{N}, \end{aligned}$$

which implies that $L(-u_0)un = L(-u_0)un$ for all $u \in U$, $n \in \mathcal{N}$, put $n = mt$ in the last equation and use it to get $L(-u)_0\mathcal{N}[-u, t] = \{0\}$ for all $u \in U$, $t \in \mathcal{N}$, then by the 3-primeness of $\mathcal{N}$ we arrive at $L(u_0) = 0$ or $-u \in Z(\mathcal{N})$ for all $u \in U$, then $L(U) = \{0\}$ or $-u \subseteq Z(\mathcal{N})$, it follows that either $L = 0$ or $(\mathcal{N}, +)$ is abelian by Lemma 7 and Lemma 2.

When $L(z) + z \in Z(\mathcal{N})$, using the same proof as above in (ii) after equation (29), we get the required result.

■

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