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REVERSE DERIVATIONS AND GENERALIZED REVERSE DERIVATIONS IN SEMIRINGS

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Abstract

In this article we extend the results on reverse derivation in rings to semirings. First we dispose of reverse derivations in prime semirings analogous to Herstein's result [7]. Then, we prove that reverse derivation is just an ordinary derivation in semiprime semirings if and only if it is a central derivation. We also define generalized reverse derivations and obtain some commutativity results which extend the results in [11]. The primary technique we use in these results is the use of derivations and reverse derivations in ring of differences R^{Δ} corresponding to the semiring R and the fact that R is embedded in R^{Δ}. This fact allows us to travel back and forth between R and R^{Δ} and serve as a key tool in obtaining the desired results.

Keywords: Reverse derivations, derivations, semirings, l-generalized reverse derivations, r-generalized reverse derivations, generalized reverse derivations and semiprime semirings.

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1. Introduction

Herstein introduced the concept of reverse derivations in rings [7]. In the same paper, he proved that in prime rings reverse derivation is just an ordinary derivation. This made the researchers turn their focus to semiprime rings wherein the concept of reverse derivation and derivation are not the one and the same as shown by Samman and Alyamani in [9] (Example 2.3). Hence, from thereon several researchers extended the idea of reverse derivations in many ways. In one direction Sugantha Meena and Chandramouleeswaran [10] introduced reverse derivations in semirings and presented some of its basic properties. In our paper, we prove that in an additively cancellative semiprime semirings, a reverse derivation is just an ordinary derivation if and only if it is a central derivation, analogous to the one in rings. In the other direction Aboubakr and González [1] introduced the concept of generalized reverse derivations in semiprime rings. These generalizations served as motivation for several authors and plenty of research papers were produced in the last decade. Recently Ahmed and Dudek [2] introduced generalized reverse derivations in semirings and presented conditions that lead to the commutativity of additively inverse semirings. For more information on derivation in semirings one can refer to the survey paper by Dimitrov [3] . Inspired by these, we present some commutativity results in additively cancellative semiprime semirings using the concept of generalized reverse derivation.

2. Definitions and Examples

In this section we define the concept of reverse derivation and generalized reverse derivation in semirings. We refer to Golan [6] for the basic definitions presented below in semiring theory and we refer to Ganesh and Selvan [4] for the definition of the derivation d^{Δ} in ring of differences (R^{Δ}) of a semiring (R) corresponding to the derivation d in R. This serves as a platform to travel between R and R^{Δ} and help immensely to use the results in rings and to be able to extend it to semirings.

Definition 2.1. [6] Let R be a semiring and $a, b, c \in R$. R is called additively cancellative if and only if $a + b = a + c$ implies $b = c$.

Definition 2.2. [6] Let R be a semiring and $a, b \in R$. R is called a yoked semiring if there exists a $r \in R$ such that either $a = b + r$ or $b = a + r$.

Definition 2.3. [6] Let R be an additively cancellative semiring and then the corresponding ring of differences, denoted by R^{Δ} is defined as follows:

$$R^{\Delta} = \{a - b : a, b \in R\}$$

In R^Δ, we have $a - b = c - d$ if and only if there exists $r, r' \in R$ such that $a + r = c + r'$ and $b + r = d + r'$. The zero element and multiplicative identity of R^Δ are $r - r$ and $1 - 0$ respectively. For $a - b, c - d \in R^\Delta$, addition and multiplication is given by

$$(a - b) + (c - d) = (a + c) - (b + d)$$
$$(a - b)(c - d) = (ac + bd) - (ad + bc)$$

We also note that the embedding of R to the ring of differences R^Δ is due to the map $r \mapsto r - 0$, for each $r \in R$.

Remark 2.4. In the next two sections, we refer the semiring R to be additively cancellative (unless stated otherwise) so that the corresponding R^Δ can be defined as above.

Definition 2.5. [6] A function d of R into R is called a derivation of a semiring R if it satisfies the following conditions.

(i) $d(r + s) = d(r) + d(s), \quad \forall\, r, s \in R$ and

(ii) $d(rs) = d(r)s + rd(s), \quad \forall\, r, s \in R.$

Definition 2.6. [4] Let R be a semiring with derivation d. Let R^Δ be the ring of differences of the semiring R. Then, d^Δ is a function in R^Δ induced by d, defined as follows.

$$d^\Delta \ : \ R^\Delta \to R^\Delta$$
$$d^\Delta(a - b) = d(a) - d(b), \quad \forall\, a, b \in R.$$

We refer the reader to [4] to see that d^Δ is indeed a derivation in R^Δ, if $d.$ is a derivation in R.

Definition 2.7. [6] Let R be a semiring and I be an ideal of R. We call I a k-ideal if $r \in I$ and $r + s \in I$ implies $s \in I$, for $r, s \in R$. k-ideals are also referred as subtractive ideals.

Definition 2.8. Let R be a semiring and f be a mapping of R to itself. We call f a central map if $f(r) \in Z(R)$ for all $r \in R$, where $Z(R)$ is the centre of R. Further, we call f a central derivation (central reverse derivation) if f is a derivation of R (if f is a reverse derivation of R).

The following definitions in semirings is analogous to the one in rings [1, 7].

Definition 2.9. [7] Let R be a semiring and let ∂ be an additive mapping of R into R such that $\partial(rs) = \partial(s)r + s\partial(r)$, for all $r, s \in R$. We call this map ∂ to be a reverse derivation of R.

We present some examples below to motivate the study of reverse derivations in semirings.

Example 2.10. Let R be a semiring and $S = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in R \right\}$. Then, it is easy to see that S is a semiring. Let us define a map $\partial : S \to S$ by $\partial \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Clearly, $\partial(AB) = \partial(B)A + B\partial(A)$ for all $A, B \in S$, proving that ∂ is a reverse derivation. In addition, one can easily verify that ∂ is also a derivation.

Example 2.11. Let R be a semiring and $S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b, c \in R \right\}$. Then, we can clearly see that S is a semiring. Let d be a map from S to S given by $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+b \\ 0 & 0 \end{pmatrix}$. It is easy to check that $d(AB) = d(A)B + Ad(B)$ for all $A, B \in S$, but $d(B)A + Bd(A) \neq d(AB)$. Hence, d is not a reverse derivation but an ordinary derivation.

Example 2.12. Let R be a semiring and $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R \right\}$. It is clear that S is a semiring. Let d be a map from S to S given by $d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. If we set $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$, then it is easy to verify that $d(AB) = \begin{pmatrix} 0 & ae+bf \\ 0 & 0 \end{pmatrix} = d(A)B + Ad(B)$ for all $A, B \in S$. Thus, d is a derivation. On the other hand, we see that $d(B)A + Bd(A) = \begin{pmatrix} 0 & ec+db \\ 0 & 0 \end{pmatrix} \neq d(AB)$. Hence, d is not a reverse derivation.

The examples presented above shows that the set of derivations and the set of reverse derivations are not the same. In addition, the following example illustrate the fact that there exists reverse derivations that are not derivations.

Example 2.13. [10] Let R be a semiring and let $S = R \oplus R$. If we define addition and multiplication componentwise, then S becomes a semiring. Let S_1 and S_2 be semirings defined in the same way as S and let ∂_1 and ∂_2 be the reverse derivations respectively. Let us define a map $\partial : S_1 \to S_2$ by $\partial(r_1, r_2) = (\partial_2(r_1), \partial_1(r_2))$. Then ∂ is indeed a reverse derivation but not a derivation.

Remark 2.14. Based on the above Definition 2.9, one can define the corresponding map ∂^Δ in ring of differences R^Δ of a semiring R, as an additive map such that $\partial^\Delta(a - b) = \partial(a) - \partial(b)$, where $a, b \in R$. We prove that this ∂^Δ is indeed a reverse derivation in R^Δ in Lemma 3.6.

Definition 2.15. [1] Let R be a semiring and ∂ be a reverse derivation. We define l-generalized reverse derivation (r-generalized reverse derivation) as an additive map $F : R \to R$ such that it satistifes $F(rs) = F(s)r + s\partial(r)$ for all $r, s \in R$ (resp. $F(rs) = \partial(s)r + sF(r)$).

Remark 2.16. Based on the above Definition 2.15, one can define the corresponding map F^{Δ} in ring of differences R^{Δ} of a semiring R, as an additive map such that $F^{\Delta}(a - b) = F(a) - F(b)$, where $a, b \in R$. We prove that this F^{Δ} is indeed a l-generalized reverse derivation in R^{Δ} in Lemma 4.1 (resp. r-generalized reverse derivation in Remark 4.2).

Example 2.17. Let R be a semiring and $S = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in R \right\}$. Then, it is easy to see that S is a semiring. Let us define the mappings $F : S \to S$ and $\partial : S \to S$ as follows:

$$F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \partial \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From Example 2.10 we know that ∂ is a reverse derivation. For $A, B \in S$ it is easy to verify that $F(AB) = F(B)A + B\partial(A)$ and $F(AB) = \partial(B)A + BF(A)$. Hence, F is both l-generalized reverse derivation and r-generalized reverse derivation.

3. Reverse derivations in rings and semirings

After Herstein bid adieu to reverse derivation in prime rings, it is natural for the researchers to turn their attention to reverse derivations in semiprime rings. Samman and Alyamani [9] proved that in a semiprime rings, a reverse derivation is just an ordinary derivation if and only if it is a central derivation. This served as a motivation for the author to investigate the same in the context of semiprime semirings. We present two different proofs of the main result. In the first proof (Theorem 3.8) we prove it directly in semirings, whereas in the second proof (Theorem 3.11) we travel to R^{Δ} and come back to R to prove the result. We also present a shorter proof of Herstein's Theorem 2.1 in [7]. First, we shall state the following results from [8].

Lemma 3.1. [8] Let R be an ring and for any $r \in R$, let $T(r) = \{a \in R : a(rx - xr) = 0, \forall x \in R\}$. Then $T(r)$ is an ideal of R.

Lemma 3.2. [8] Let R be a prime ring and for $r \neq 0$ such that $r(ax - xa) = 0$, for all $x \in R$, then $a \in Z(R)$, where $Z(R)$ is the centre of R.

Now, we are ready to give a shorter proof of a weaker version of Theorem 2.1 in [7].

Theorem 3.3. If R is prime ring and ∂ is a non-zero reverse derivation of R, then ∂ is just an ordinary derivation of R.

Proof. Since ∂ is non-zero, there exists $y \in R$ such that $\partial(y) \neq 0$. So, we have

$$\begin{aligned}
\partial(xy^2) &= \partial((xy)y), \quad \forall x \in R \\
\partial(y^2)x + y^2\partial(x) &= \partial(y)xy + y\partial(xy), \quad \forall x \in R \\
\partial(y)yx + y\partial(y)x + y^2\partial(x) &= \partial(y)xy + y\partial(y)x + y^2\partial(x), \quad \forall x \in R \\
\partial(y)yx &= \partial(y)xy, \quad \forall x \in R \\
\partial(y)(yx - xy) &= 0, \quad \forall x \in R
\end{aligned}$$

Then, by Lemma 3.2, we have $y \in Z(R)$ so that $\partial(xy) = \partial(yx) = \partial(x)y + x\partial(y)$. Thus, ∂ is just an ordinary derivation. ■

Motivated by Lemma 3.1, we investigate the analogue in semiring and it turns out that $T(r)$ is infact a k-ideal.

Lemma 3.4. Let R be an additively cancellative semiring and for any $r \in R$, let $T(r) = \{a \in R : arx = axr, \forall x \in R\}$. Then $T(r)$ is a k-ideal of R.

Proof. It is easy to see that $T(r)$ is a left ideal of R. Now let us prove that it is also a right ideal. That is, for $u \in T(r)$ and $x \in R$, then we need to prove $ux \in R$. Note that if $x \in R$, then $xy \in R$ for $y \in R$ and since $u \in T(r)$ we get $urxy = uxyr$. Thus, $urxy + uxry = uxyr + uxry$. Since $u \in T(r)$ and R is additively cancellative, $uxry = uxyr$ for all $x \in R$ so that $ux \in T(r)$. Thus, $T(r)$ is a two-sided ideal of R.
If $a \in T(r)$ and $a + b \in T(r)$ then

$$\begin{aligned}
(a+b)rx &= (a+b)xr \\
arx + brx &= axr + bxr \\
arx + brx &= arx + bxr && \text{(since } a \in T(r)\text{)} \\
brx &= bxr && \text{(since } R \text{ is additivelly cancellative)}
\end{aligned}$$

Thus $b \in T(r)$ and hence $T(r)$ is a k-ideal of R. ■

Lemma 3.5. Let R be a semiring, r be an element of R and $Z(R)$ be the center of R. If $r(uv) = r(vu), \forall u, v \in R$, then $r \in Z(R)$.

Proof. Let R^Δ be the corresponding ring of differences of R. Let $x, y \in R^\Delta$ and $x = a - b, y = c - d$ where $a, b, c, d \in R$. Then for any $r \in R$, we have

$$\begin{aligned} r(xy) &= r((a-b)(c-d)) \\ &= r((ac+bd)-(ad+bc)) \\ &= rac + rbd - rad - rbc \\ &= rca + rdb - rda - rcb \qquad \text{(by hypothesis)} \\ &= r((ca+db)-(cb+da)) \\ &= r((c-d)(a-b)) \\ &= r(yx) \end{aligned}$$

Thus, $r \in Z(R^\Delta)$ and hence we have $r \in Z(r)$. ■

Lemma 3.6. Let R be a semiring and ∂ be a reverse derivation of R, then the mapping ∂^Δ (as given in Remark 2.14) corresponding to ∂ is a reverse derivation of R^Δ.

Proof. Let $x, y \in R^\Delta$ and let $x = a - b, y = c - d$ for $a, b, c, d \in R$. Then, we have

$$\begin{aligned} \partial^\Delta(xy) &= \partial^\Delta((a-b)(c-d)) \\ &= \partial^\Delta((ac+bd)-(ad+bc)) \\ &= \partial(ac+bd) - \partial(ad+bc) \\ &= \partial(ac) + \partial(bd) - \partial(ad) - \partial(bc) \\ &= \partial(c)a + c\partial(a) + \partial(d)b + d\partial(b) \\ &\qquad - \partial(d)a - d\partial(a) - \partial(c)b - c\partial(b) \\ &= \partial(c)a - \partial(c)b - \partial(d)a + \partial(d)b \\ &\qquad + c\partial(a) - c\partial(b) - d\partial(a) + d\partial(b) \\ &= (\partial(c) - \partial(d))(a-b) + (c-d)(\partial(a) - \partial(b)) \\ &= (\partial^\Delta(c-d))(a-b) + (c-d)(\partial^\Delta(a-b)) \\ &= \partial^\Delta(y)x + y\partial^\Delta(x) \end{aligned}$$

Thus, ∂^Δ is a reverse derivation in R^Δ. ■

Theorem 3.7. Let R be a prime semiring and ∂ be a non-zero reverse derivation of R. If R^Δ is the corresponding ring of differences of R and ∂^Δ the corresponding mapping induced by ∂, then R is a commutative semi-integral domain (a semiring in which product of any two non-zero element is again a non-zero element) and ∂ is just an ordinary derivation.

Proof. We know that of R is prime, then R^{Δ} is prime [by Lemma 3.1 in [4]). If ∂ is a non-zero derivation of R, then by Lemma 3.6, the corresponding map ∂^{Δ} is a non-zero reverse derivation in R^{Δ}. Now apply Theorem 2.1 in [7] to see that R^{Δ} is a commutative integral domain and ∂^{Δ} is just an ordinary derivation. The result follows immediately since R is embedded in R^{Δ} and the restriction of ∂^{Δ} to R is just ∂. ■

The above theorem disposes of reverse derivation in prime semirings. Now, we turn our focus to semiprime semirings.

Theorem 3.8. If R is an additively cancellative semiprime semiring then ∂ is a non-zero reverse derivation of R if and only if it is a central derivation. Furthermore, ∂ is just an ordinary derivation of R.

Proof. Note that if ∂ is a central derivation then it is obviously a reverse derivation. We now prove that ∂ is just a central derivation. If ∂ is a reverse derivation, then for $r, s \in R$, we have the following.

$$\begin{aligned}
\partial(rs^2) &= \partial((rs)s) \\
\partial(s^2)r + s^2\partial(r) &= \partial(s)rs + s\partial(rs) \\
\partial(s)sr + s\partial(s)r + s^2\partial(r) &= \partial(s)rs + s\partial(s)r + s^2\partial(r) \\
\partial(s)sr &= \partial(s)rs
\end{aligned}$$

where the last equation is due to the fact that R is additively cancellative. Then, by Lemma 3.5, we have $\partial(s) \in Z(R)$, for all $s \in R$; so that ∂ is central. Also note that, $\partial(rs) = \partial(s)r + s\partial(r) = \partial(r)s + r\partial(s)$. Thus, ∂ is just an ordinary derivation. ■

Lemma 3.9. Let R be a semiring and ∂ be a reverse derivation of R. Let R^{Δ} be the corresponding ring of differences of R and ∂^{Δ} be the corresponding reverse derivation in R^{Δ} induced by ∂. If $\partial(r) \in Z(R), \forall r \in R$, then $\partial^{\Delta}(x) \in Z(R^{\Delta}), \forall x \in R^{\Delta}$, where $Z(R)$ is the centre of R and $Z(R^{\Delta})$ is the centre of R^{Δ}.

Proof. Let $x = a - b$ and $y = c - d$, where $a, b, c, d \in R$. Then,

$$\begin{aligned}
\partial^{\Delta}(x)y &= (\partial^{\Delta}(a-b))(c-d) \\
&= (\partial(a) - \partial(b))(c-d) \\
&= \partial(a)c - \partial(a)d - \partial(b)c + \partial(b)d \\
&= c\partial(a) - d\partial(a) - c\partial(b) + d\partial(b), \qquad \text{(since } \partial \text{ is a central map)} \\
&= (c-d)\partial(a) - (c-d)\partial(b) \\
&= (c-d)(\partial(a) - \partial(b)) \\
&= (c-d)\partial^{\Delta}(a-b)
\end{aligned}$$

$$= y\partial^{\Delta}(x)$$

Thus, ∂^{Δ} is a central map in R^{Δ}. ■

Remark 3.10. The converse of the above lemma is also true.

Theorem 3.11. Let R be an additively cancellative semiprime semiring and R^{Δ} be its corresponding ring of differences. Let ∂ be a reverse derivation of R, then ∂ is just an ordinary derivation if and only if $\partial(r) \in Z(R)$ for all $r \in R$, where $Z(R)$ is the centre of R.

Proof. If ∂ is a central derivation in R, then it is clearly an ordinary derivation of R. Now we prove that if ∂ is a reverse derivation, then it is indeed a central derivation.

Note that if R is semiprime, then so is R^{Δ} (by Lemma 3.6 in [5]). Also if ∂ is a reverse derivation, then the map ∂^{Δ} corresponding to ∂ is a reverse derivation of R^{Δ} by Lemma 3.6. Now if we apply Proposition 3.1 in [9], we note that ∂^{Δ} is a central derivation of R^{Δ}. The result then follows if we apply Remark 3.10. ■

Lemma 3.12. Let R be a semiprime semiring and $a, b \in R$. If ∂ is reverse derivation from R to R defined $\partial(r) = ar + rb$, then $\partial = 0$.

Proof. Let R^{Δ} be the corresponding ring of differences of the semiring R. We define the induced map $\partial^{\Delta} : R^{\Delta} \to R^{\Delta}$ defined by $\partial^{\Delta}(u) = au + ub$ for $u \in R^{\Delta}$ and $a, b \in R$. First we prove that ∂^{Δ} is a reverse derivation in R^{Δ}.
Let $u, v \in R^{\Delta}$ such that $u = x - y, v = z - w$ for $x, y, z, w \in R$.

$$\begin{aligned}
\partial^{\Delta}(uv) &= \partial^{\Delta}((x-y)(z-w)) \\
&= \partial^{\Delta}((xz+yw)-(xw+yz)) \\
&= \partial(xz)+\partial(yw)-\partial(xw)-\partial(yz) \\
&= \partial(z)x + z\partial(x) + \partial(w)y + w\partial(y) \\
&\qquad - \partial(w)x - w\partial(x) - \partial(z)y - z\partial(y) \\
&= (az+zb)x + z(ax+xb) + (aw+wb)y + w(ay+yb) \\
&\qquad - (aw+wb)x - w(ax+xb) - (az+zb)y - z(ay+yb) \\
&= azx + zbx + zax + zxb + awy + wby + way + wyb \\
&\qquad - awx - wbx - wax - wxb - azy - zby - zay - zyb \\
&= azx - azy - awx + awy + zbx - zby - wbx + wby \\
&\qquad + zax - zay + zxb - zyb - wax + way - wxb + wyb \\
&= (az - aw + zb - wb)(x-y) + (z-w)(ax - ay - xb - yb) \\
&= [a(z-w) + (z-w)b](x-y) + (z-w)[a(x-y) + (x-y)b] \\
&= \partial^{\Delta}(z-w)(x-y) + (z-w)\partial^{\Delta}(x-y)
\end{aligned}$$

$$= \partial^{\Delta}(v)u + v\partial^{\Delta}(u)$$

Thus, ∂^{Δ} is a reverse derivation in R^{Δ}. Now, applying Proposition 3.4 in [9] yields $\partial^{\Delta} = 0$ which implies $\partial = 0$ since ∂ is the restriction of ∂^{Δ} to R. ■

4. Generalized reverse derivations and Commutativity of semiprime semirings

Aboubakr and González in [1] introduced the concept of generalized reverse derivations in semiprime rings. In an attempt to generalize this concept to semirings, Ahmad and Dudek [2] introduced the generalised reverse derivations in semirings and established some commutativity results in the class of prime MA-semirings. For example, whenever generalized reverse derivation vanishes on a commutator, then the semiring is commutative. They also established commutativity conditions on semiprime MA-semirings.

In this section, we generalize the results in [1] to additively cancellative semiprime semirings. We prove that the generalized reverse derivation is just an ordinary derivation if and only if it is contained in the centralizer of a non zero ideal. We also investigate and extend the commutativity results obtained in [11] to additively cancellative semiprime semirings using the concept of generalized reverse derivations.

Lemma 4.1. Let R be a semiprime semiring and F be a l-generalized reverse derivation associated with a reverse derivation ∂. If R^{Δ} is the corresponding ring of differences of the semiring R and ∂^{Δ} is the induced reverse derivation in R^{Δ}, then F^{Δ} (as defined in Remark 2.16) is an induced l-generalized reverse derivation of R^{Δ} corresponding to F, associated with ∂^{Δ}.

Proof. Let $x = a - b, y = c - d \in R^{\Delta}$, where $a, b, c, d \in R$. We need to prove the following.

$$F^{\Delta}(xy) = F^{\Delta}(y)x + y\partial^{\Delta}(x) \tag{1}$$

Consider the LHS of (1). We have

$$\begin{aligned}
F^{\Delta}(xy) &= F^{\Delta}((a-b)(c-d)) \\
&= F^{\Delta}((ac+bd)-(ad+bc)) \\
&= F(ac+bd) - F(ad+bc) \\
&= F(c)a + c\partial(a) + F(d)b + d\partial(b) - F(d)a - d\partial(a) \\
&\qquad - F(c)b - c\partial(b) \\
&= F(c)a - F(c)b - F(d)a + F(d)b + c\partial(a) - c\partial(b) \\
&\qquad - d\partial(a) + d\partial(b)
\end{aligned}$$

$$
\begin{aligned}
&= ((F(c) - F(d))(a - b) + (c - d)(\partial(a) - \partial(b)) \\
&= (F^{\Delta}(c - d))(a - b) + (c - d)(\partial^{\Delta}(a - b)) \\
&= F^{\Delta}(y)x + y\partial^{\Delta}(x)
\end{aligned}
$$

Thus, equation (1) holds and hence the lemma is true. ■

Remark 4.2. We can prove the above result for r-generalized reverse derivation in the same way.

Theorem 4.3. Let R be a semiring, I be a nonzero ideal of R and $\partial : I \to R$ be a non-zero reverse derivation. Then there exists an l-generalized reverse derivation $F : R \to R$ corresponding to ∂ if and only if $\partial(I), F(I) \subseteq C_R(I)$. Furthermore, if $\partial(I) \subseteq C_R(I)$ then, ∂ is a derivation on I and if $F(I) \subseteq C_R(I)$, then F is an r-generalized derivation corresponding to ∂ on I.

Proof. Let R be a semiring and ∂ be a reverse derivation in I. Let R^{Δ} be the corresponding ring of differences of R and ∂^{Δ} be the reverse derivation in R^{Δ} induced by ∂. Let F be a l-generalized reverse derivation of I, then by Lemma 4.1 we have F^{Δ} be the corresponding l-generalized reverse derivation of R^{Δ}. Let I^{Δ} be the corresponding ideal of I in R^{Δ}. Hence by Theorem 3.1 in [1] we have $\partial^{\Delta}(I^{\Delta}), F^{\Delta}(I^{\Delta}) \subseteq C_{R^{\Delta}}(I^{\Delta})$. Now we prove that $C_{R^{\Delta}}(I^{\Delta}) \cap R = C_R(I)$.

$C_{R^{\Delta}}(I^{\Delta}) \cap R \subseteq C_R(I)$ is clear. On the other hand, let $x \in C_R(I)$ and $y \in I^{\Delta}$ and let $y = a - b$ for $a, b \in I$. Then we have, $xy = x(a - b) = xa - xb = ax - bx = (a - b)x = yx$ which implies that $x \in C_{R^{\Delta}}(I^{\Delta})$ proving $C_{R^{\Delta}}(I^{\Delta}) \cap R \supseteq C_R(I)$. Now we note that since R is embedded in R^{Δ} the restriction of ∂^{Δ} and F^{Δ} to R is just ∂ and F respectively and hence we are done. ■

Theorem 4.4. Let R be a semiring, I be a nonzero ideal of R and $\partial : I \to R$ be a non-zero reverse derivation. Then there exists an r-generalized reverse derivation $F : R \to R$ corresponding to ∂ if and only if $\partial(I), F(I) \subseteq C_R(I)$. Furthermore, if $\partial(I) \subseteq C_R(I)$ then, ∂ is a derivation on I and if $F(I) \subseteq C_R(I)$, then F is an l-generalized derivation corresponding to ∂ on I.

Proof. Similar to Theorem 4.3. ■

Motivated by the commutativity results obtained in [11], we study the following situations in the context of semirings. One should note that the results in [11] holds true for a multiplicative (generalized) reverse derivation in semiprime rings. For a ring R, it is defined as a map F from R to R associated with any map g (not necessarily a reverse derivation) such that $F(rs) = F(s)r + sg(r)$. We define F to be a multiplicative (generalized) reverse derivation in semirings analogously. In the remainder of this section the notation $a \circ b$ denotes the Jordan product given by $ab + ba$ for $a, b \in R$ and similarly $x \circ y = xy + yx$ for $x, y \in R^{\Delta}$.

Lemma 4.5. Let R be a semiprime semiring and F be a multiplicative (generalized) reverse derivation and g be any map from R to R. Let R^{Δ} be the corresponding ring of differences of R and F^{Δ} be the multiplicative (generalized) reverse derivation corresponding to F and g^{Δ} be the map corresponding to g. Let I be an ideal of R and I^{Δ} be the corresponding ideal in R^{Δ}. For $r, s \in I$, we have the following.

(i) If $F(r)F(s) + rs = 0$, then $F^{\Delta}(x)F^{\Delta}(y) + xy = 0$ for all $x, y \in I^{\Delta}$;

(ii) If $F(r)F(s) = rs$, then $F^{\Delta}(x)F^{\Delta}(y) = xy$ for all $x, y \in I^{\Delta}$;

(iii) If $F(r)F(s) + sr = 0$, then $F^{\Delta}(x)F^{\Delta}(y) + yx = 0$ for all $x, y \in I^{\Delta}$;

(iv) If $F(r)F(s) = sr$, then $F^{\Delta}(x)F^{\Delta}(y) = yx$ for all $x, y \in I^{\Delta}$;

(v) If $F(r)F(s) + g(s)F(r) = 0$, then $F^{\Delta}(x)F^{\Delta}(y) + g^{\Delta}(y)F^{\Delta}(x) = 0$ for all $x, y \in I^{\Delta}$;

(vi) If $F(r)F(s) = g(s)F(r)$, then $F^{\Delta}(x)F^{\Delta}(y) = g^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

(vii) If $F(rs) + r \circ s = 0$, then $F^{\Delta}(xy) + x \circ y = 0$ for all $x, y \in I^{\Delta}$;

(viii) If $F(rs) = r \circ s$, then $F^{\Delta}(xy) = x \circ y$ for all $x, y \in I^{\Delta}$;

(ix) If $F(rs) + rs = sr$, then $F^{\Delta}(xy) + xy = yx$ for all $x, y \in I^{\Delta}$;

(x) If $F(rs) + sr = rs$, then $F^{\Delta}(xy) + yx = xy$ for all $x, y \in I^{\Delta}$;

(xi) If $F(r)g(s) + sr = g(s)F(r)$, then $F^{\Delta}(x)g^{\Delta}(y) + yx = g^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

(xii) If $F(r)g(s) = sr + g(s)F(r)$, then $F^{\Delta}(x)g^{\Delta}(y) = yx + g^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

(xiii) If $F(r)g(s) + rs = g(s)F(r)$, then $F^{\Delta}(x)g^{\Delta}(y) + xy = g^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

(xiv) If $F(r)g(s) = rs + g(s)F(r)$, then $F^{\Delta}(x)g^{\Delta}(y) = xy + g^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

(xv) If $F(r)g(s) + rs = g(s)F(r) + sr$, then $F^{\Delta}(x)g^{\Delta}(y) + xy = g^{\Delta}(y)F^{\Delta}(x) + yx$ for all $x, y \in I^{\Delta}$;

(xvi) If $F(r)g(s) + sr = g(s)F(r) + rs$, then $F^{\Delta}(x)g^{\Delta}(y) + yx = g^{\Delta}(y)F^{\Delta}(x) + xy$ for all $x, y \in I^{\Delta}$;

(xvii) If $F(rs) = F(r)F(s)$, then $F^{\Delta}(xy) = F^{\Delta}(x)F^{\Delta}(y)$ for all $x, y \in I^{\Delta}$;

(xviii) If $F(rs) = F(s)F(r)$, then $F^{\Delta}(xy) = F^{\Delta}(y)F^{\Delta}(x)$ for all $x, y \in I^{\Delta}$;

Proof. (i) Let $x, y \in I^{\Delta}$ where $x = a - b, y = c - d$ for $a, b, c, d \in I$. Then, we have

$$\begin{aligned} F^{\Delta}(x)F^{\Delta}(y) + xy &= F^{\Delta}(a-b)F^{\Delta}(c-d) + (a-b)(c-d) \\ &= (F(a) - F(b))(F(c) - F(d)) + ac + bd - ad - bc \\ &= F(a)F(c) - F(a)F(d) - F(b)F(c) + F(b)F(d) \\ &\qquad + ac + bd - ad - bc \\ &= (F(a)F(c) + ac) - (F(a)F(d) + ad) \\ &\qquad - (F(b)F(c) + bc) + (F(b)F(d) + bd) \\ &= 0 \end{aligned}$$

since each term in the penultimate step is zero by hypothesis.
Proof of the results (ii) to (xviii) are fairly straightforward if we follow along the lines of proof of result (i). ■

In the following theorems we denote R be a semiprime semiring and R^{Δ} be the corresponding ring of differences (semirpime ring); and F be a multiplicative (generalized) reverse derivation of R associated with reverse derivation ∂ and F^{Δ} be the corresponding multiplicative (generalized) reverse derivation associated with reverse derivation ∂^{Δ}; g be any map in R and g^{Δ} be the corresponding map in R^{Δ}; and I be an ideal of R and I^{Δ} be the corresponding ideal of R^{Δ}. We use these corresponding mappings and ideals in R^{Δ} to prove the results in a ring and then we use the fact that F^{Δ} and g^{Δ} restricted to R is just F and g respectively.

Theorem 4.6. For $r, s \in I$, if $F(r)F(s) + rs = 0$, then R is commutative.

Proof. Apply above Lemma 4.5(i) and Theorem 2.1 in [11], then the result is immediate. ■

Theorem 4.7. For $r, s \in I$, if $F(r)F(s) = rs$, then R is commutative.

Proof. Apply above Lemma 4.5(ii) and Theorem 2.1 in [11], then the result follows. ■

Theorem 4.8. For $r, s \in I$, if $F(r)F(s) + sr = 0$, then R is commutative.

Proof. Apply above Lemma 4.5(iii) and Theorem 2.2 in [11], then the result is immediate. ■

Theorem 4.9. For $r, s \in I$, if $F(r)F(s) = sr$, then R is commutative.

Proof. Apply above Lemma 4.5(iv) and Theorem 2.2 in [11], then the result follows. ■

Theorem 4.10. For $r, s \in I$, if $F(r)F(s)+g(s)F(r) = 0$, then R is commutative.

Proof. Apply above Lemma 4.5(v) and Theorem 2.3 in [11] and Corollary 2.4 in [11], then the result holds. ■

Theorem 4.11. For $r, s \in I$, if $F(r)F(s) = g(s)F(r)$, then R is commutative.

Proof. Apply above Lemma 4.5(vi) and Theorem 2.3 in [11] and Corollary 2.4 in [11], then the result holds. ■

Theorem 4.12. For $r, s \in I$, if $F(rs) + r \circ s = 0$, then R is commutative.

Proof. Apply above Lemma 4.5(vii) and Theorem 2.8 in [11], then the result follows. ■

Theorem 4.13. For $r, s \in I$, if $F(rs) = r \circ s$, then R is commutative.

Proof. Apply above Lemma 4.5(viii) and Theorem 2.8 in [11], then the result follows. ■

Theorem 4.14. For $r, s \in I$, if $F(rs) + rs = sr$, then R is commutative.

Proof. Apply above Lemma 4.5(ix) and Theorem 2.9 in [11], then the result is immediate. ■

Theorem 4.15. For $r, s \in I$, if $F(rs) + sr = rs$, then R is commutative.

Proof. Apply above Lemma 4.5(x) and Theorem 2.9 in [11], then the result is immediate. ■

Theorem 4.16. For $r, s \in I$, if $F(r)g(s) + sr = g(s)F(r)$, then g is commuting.

Proof. Apply above Lemma 4.5(xi) and Theorem 2.10 in [11], then the result holds true. ■

Theorem 4.17. For $r, s \in I$, if $F(r)g(s) = sr + g(s)F(r)$, then g is commuting.

Proof. Apply above Lemma 4.5(xii) and Theorem 2.10 in [11], then the result holds true. ■

Theorem 4.18. For $r, s \in I$, if $F(r)g(s) + rs = g(s)F(r)$, then g is commuting.

Proof. Apply above Lemma 4.5(xiii) and Theorem 2.11 in [11], then the result follows. ■

Theorem 4.19. For $r, s \in I$, if $F(r)g(s) = rs + g(s)F(r)$, then g is commuting.

Proof. Apply above Lemma 4.5(xiv) and Theorem 2.11 in [11], then the result follows. ■

Theorem 4.20. For $r, s \in I$, if $F(r)g(s) + rs = g(s)F(r) + sr$, then g is commuting.

Proof. Apply above Lemma 4.5(xv) and Theorem 2.12 in [11], then the result is immediate. ■

Theorem 4.21. For $r, s \in I$, if $F(r)g(s) + sr = g(s)F(r) + rs$, then g is commuting.

Proof. Apply above Lemma 4.5(xvi) and Theorem 2.12 in [11], then the result is immediate. ■

Theorem 4.22. For $r, s \in I$, if one of the following condition holds, then g is commuting.

(i) $F(rs) = F(r)F(s)$;

(ii) $F(rs) = F(s)F(r)$

Proof. Apply above Lemma 4.5(xvii),(xviii) and Theorem 2.13 in [11], then the result is true. ■

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