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A NOTE ON THE ABUNDANCE OF PARTIAL TRANSFORMATION SEMIGROUPS WITH FIXED POINT SETS

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Abstract

Given a non-empty set X and let $P(X)$ be the partial transformation semigroup on X. For a fixed non-empty subset Y of X, let

$$PFix(X,Y) = \{\alpha \in P(X) : y\alpha = y \text{ for all } y \in \mathrm{dom}(\alpha) \cap Y\}.$$

Then $PFix(X,Y)$ is a subsemigroup of $P(X)$. In this paper, we show that $PFix(X,Y)$ is always abundant, even though it is not regular. Moreover, unit regular and coregular elements of such semigroup are all completely characterized.

Keywords: partial transformation semigroup, abundance, unit regularity, coregularity.

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1. Introduction and preliminaries

Let X be a non-empty set and $P(X)$ the partial transformation semigroup on X. Fix a non-empty subset Y of X and consider the subsemigroup $PFix(X,Y)$ of $P(X)$ defined by

$$PFix(X,Y) = \{\alpha \in P(X) : y\alpha = y \text{ for all } y \in \mathrm{dom}(\alpha) \cap Y\},$$

which was first introduced and named *partial transformation semigroup with a fixed point set* Y in [3]. The authors showed that $PFix(X,Y)$ need not be regular and proved that an element $\alpha \in PFix(X,Y)$ is regular if and only if $\mathrm{dom}(\alpha) \cap Y = \mathrm{ran}(\alpha) \cap Y$, where $\mathrm{dom}(\alpha)$ and $\mathrm{ran}(\alpha)$ mean the domain of α and the range of α, respectively. Later in [10], the authors provided a complete description of Green's relations on $PFix(X,Y)$ and applied the results to obtain characterizations of left regular, right regular, intra-regular, and complete regular elements in such a semigroup.

Although $PFix(X,Y)$ is not regular, it contains a regular subsemigroup

$$Fix(X,Y) = \{\alpha \in PFix(X,Y) : \mathrm{dom}(\alpha) = X\}$$

which has been discovered before in [5] and its significant properties were described in [1, 2, 8, 9].

In this paper, we describe more regular properties of $PFix(X,Y)$ and show that $PFix(X,Y)$ is always abundant.

Throughout this paper, we write the functions on the right; in particular, this means that for a composition $\alpha\beta$, the transformation α is applied first. To simplify the notation, we often write the singleton set $\{a\}$ as a. For element $\alpha \in PFix(X,Y)$, we write

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

and take as understood that the script i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $\mathrm{ran}(\alpha) = \{a_i\}$ and $a_i\alpha^{-1} = A_i \subseteq \mathrm{dom}(\alpha)$.

2. Abundance of $PFix(X,Y)$

On a semigroup S, $a, b \in S$ are $\mathcal{L}^*$-related in S if and only if a and b are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined in the dual way. The semigroup S is said to be *left abundant* if each $\mathcal{L}^*$-class contains an idempotent. *Right abundant semigroup* is defined dually. A semigroup which is both left and right abundant will be called an *abundant semigroup.*

Of course, regular semigroups are abundant and in this case we have $\mathcal{L} = \mathcal{L}^*$, $\mathcal{R} = \mathcal{R}^*$. The aim of this section is to show that $PFix(X,Y)$ is an abundant semigroup which is not regular. Note that we write id_A to mean the identity map on the set A.

Recall the well-known characterizations of the relations $\mathcal{L}$ and $\mathcal{R}$ on $P(X)$; and $\mathcal{L}^*$ and $\mathcal{R}^*$ on any semigroup S in Lemmas 1 and 2, respectively.

Lemma 1 [6]**.** *Let $\alpha, \beta \in P(X)$. Then*

1. $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\mathrm{ran}(\alpha) = \mathrm{ran}(\beta)$*;*

2. $(\alpha,\beta)\in\mathcal{R}$ *if and only if* $\ker(\alpha)=\ker(\beta)$,
where $\ker(\gamma)=\{(x_1,x_2)\in \mathrm{dom}(\gamma)\times\mathrm{dom}(\gamma) : x_1\gamma=x_2\gamma\}$ *for any* $\gamma\in P(X)$.

Lemma 2 [7]**.** *Let S be a semigroup. Then*

$$\mathcal{L}^*=\big\{(a,b)\in S\times S : (\forall s,t\in S^1)\ as=at \Leftrightarrow bs=bt\big\},$$

$$\mathcal{R}^*=\big\{(a,b)\in S\times S : (\forall s,t\in S^1)\ sa=ta \Leftrightarrow sb=tb\big\}.$$

For the semigroup $PFix(X,Y)$, we have the characterization of the relation $\mathcal{L}^*$ as shown in the following lemma.

Lemma 3. *Let $\alpha,\beta\in PFix(X,Y)$. Then $(\alpha,\beta)\in\mathcal{L}^*$ if and only if* $\mathrm{ran}(\alpha)=\mathrm{ran}(\beta)$.

Proof. Assume $\mathrm{ran}(\alpha)=\mathrm{ran}(\beta)$. Then α and β are known to be $\mathcal{L}$-related in $P(X)$. Hence α and β are $\mathcal{L}^*$-related in $PFix(X,Y)$.

Conversely, assume that $(\alpha,\beta)\in\mathcal{L}^*$ and define $\gamma=id_{\mathrm{ran}(\alpha)}$. Clearly, $\mathrm{ran}(\gamma)=\mathrm{ran}(\alpha)$ and $\alpha\gamma=\alpha$. Applying the characterization of the relation $\mathcal{L}^*$ from Lemma 2 (with α,β in the roles of a,b and γ and the identity in the roles of s and t, respectively), we conclude that $\beta\gamma=\beta$ and $\mathrm{ran}(\beta)=\mathrm{ran}(\beta\gamma)=(\mathrm{ran}(\beta)\cap\mathrm{dom}(\gamma))\gamma\subseteq\mathrm{ran}(\gamma)=\mathrm{ran}(\alpha)$. Similary, $\mathrm{ran}(\alpha)\subseteq\mathrm{ran}(\beta)$ whence $\mathrm{ran}(\alpha)=\mathrm{ran}(\beta)$. ■

Lemma 4. *The semigroup $PFix(X,Y)$ is left abundant.*

Proof. For each $\alpha\in PFix(X,Y)$, we have $id_{\mathrm{ran}(\alpha)}$ is an idempotent in the $\mathcal{L}^*$-class of α. Hence, an arbitrary $\mathcal{L}^*$-class of $PFix(X,Y)$ contains an idempotent. Therefore, $PFix(X,Y)$ is left abundant. ■

Next, we give the characterization of the relation $\mathcal{R}^*$ on $PFix(X,Y)$ as in the following lemma.

Lemma 5. *Let $\alpha,\beta\in PFix(X,Y)$. Then $(\alpha,\beta)\in\mathcal{R}^*$ if and only if* $\ker(\alpha)=\ker(\beta)$.

Proof. Assume $\ker(\alpha)=\ker(\beta)$. Then α and β are known to be $\mathcal{R}$-related in $P(X)$. Hence α and β are $\mathcal{R}^*$-related in $PFix(X,Y)$.

Conversely, assume that $(\alpha,\beta)\in\mathcal{R}^*$. To prove that $\ker(\alpha)=\ker(\beta)$, we first establish that $\mathrm{dom}(\alpha)=\mathrm{dom}(\beta)$. Since $id_{\mathrm{dom}(\alpha)}\alpha=\alpha$, using Lemma 2, we deduce that $id_{\mathrm{dom}(\alpha)}\beta=\beta$. Consequently, $\mathrm{dom}(\beta)=\mathrm{dom}(id_{\mathrm{dom}(\alpha)}\beta)\subseteq\mathrm{dom}(id_{\mathrm{dom}(\alpha)})=\mathrm{dom}(\alpha)$. Similarly, we have $\mathrm{dom}(\alpha)\subseteq\mathrm{dom}(\beta)$, and thus $\mathrm{dom}(\alpha)=\mathrm{dom}(\beta)$. Now, let $(a,b)\in\ker(\alpha)$. This implies that $a\alpha=b\alpha$, and two cases arise.

Case 1. $a \in Y$ and $b \in X \setminus Y$. Let $Y \setminus \{a\} = \{y_i\}$, $X \setminus (Y \cup \{b\}) = \{x_j\}$, and define $\gamma \in PFix(X, Y)$ as follows:

$$\gamma = \begin{pmatrix} \{a,b\} & y_i & x_j \\ a & y_i & x_j \end{pmatrix}.$$

We can observe that $\gamma\alpha = \alpha$, and then, by Lemma 2, $\gamma\beta = \beta$. Hence, $b\beta = b\gamma\beta = a\beta$, which implies $(a, b) \in \ker(\beta)$.

Case 2. $a, b \in X \setminus Y$. Let $Y = \{y_i\}$, $X \setminus (Y \cup \{a, b\}) = \{x_j\}$, and define γ as described in Case 1. Using the same proof as presented in Case 1, we can conclude that $(a, b) \in \ker(\beta)$.

Similarly, we have $\ker(\beta) \subseteq \ker(\alpha)$, which implies that $\ker(\alpha) = \ker(\beta)$, as required. ■

Lemma 6. *The semigroup $PFix(X, Y)$ is right abundant.*

Proof. For any $\alpha \in PFix(X, Y)$, write

$$\alpha = \begin{pmatrix} A_i & C_j \\ y_i & c_j \end{pmatrix},$$

where $y_i \in A_i \cap Y$ for all i and $C_j \subseteq X \setminus Y$. For each j, choose $c'_j \in C_j$ and let

$$\gamma = \begin{pmatrix} A_i & C_j \\ y_i & c'_j \end{pmatrix}.$$

Then γ is an idempotent in $PFix(X, Y)$ with $\ker(\alpha) = \ker(\gamma)$, that is, γ is in $\mathcal{R}^*$-class of α. Therefore, $PFix(X, Y)$ is right abundant. ■

Using Lemmas 4 and 6, we obtain

Theorem 7. *The semigroup $PFix(X, Y)$ is an abundant semigroup.*

3. UNIT REGULAR AND COREGULAR ELEMENTS OF $PFix(X, Y)$

Let S be a monoid with identity 1. An element $u \in S$ is a *unit* if there exists $u' \in S$ such that $uu' = 1 = u'u$. Moreover, an element $a \in S$ is said to be *unit regular* if there exists a unit $u \in S$ such that $a = aua$. In particular, if all elements of S are unit regular, then S is called a *unit regular semigroup.*

Notice that $PFix(X, Y)$ is a monoid having id_X as an identity. It is clear that $\alpha \in PFix(X, Y)$ is a unit if and only if α is bijective with $\operatorname{dom}(\alpha) = X$, that is, $\alpha|_Y = id_Y$ and $\alpha|_{X\setminus Y} : X \setminus Y \to X \setminus Y$ is a bijection.

For each $\alpha \in PFix(X, Y)$, let

$$\pi_\alpha = \{x\alpha^{-1} : x \in \operatorname{ran}(\alpha)\} \text{ and } \pi_\alpha(X\backslash Y) = \{x\alpha^{-1} : x \in \operatorname{ran}(\alpha)\backslash Y\}.$$

A subset P of X is said to be a *cross section of* π_α if $P \subseteq \operatorname{dom}(\alpha)$ and $|P \cap x\alpha^{-1}| = 1$ for all $x\alpha^{-1} \in \pi_\alpha$. In particular, P is said to be a cross section of $\pi_\alpha(X \setminus Y)$ if $P \subseteq \operatorname{dom}(\alpha)$ such that $P\alpha \subseteq \operatorname{ran}(\alpha)\backslash Y$ and $|P \cap x\alpha^{-1}| = 1$ for all $x\alpha \in \pi_\alpha(X\backslash Y)$.

We now characterize all unit regular elements of $PFix(X,Y)$.

Theorem 8. *Let $\alpha \in PFix(X,Y)$. Then α is unit regular if and only if the following conditions hold:*

1. $\operatorname{dom}(\alpha) \cap Y = \operatorname{ran}(\alpha) \cap Y$*;*
2. *if* $\operatorname{ran}(\alpha)\backslash Y \neq \emptyset$*, then there exists a cross section P of $\pi_\alpha(X\backslash Y)$ such that* $|X\backslash(Y \cup C)| = |X\backslash(Y \cup P)|$*, where $C = \operatorname{ran}(\alpha)\backslash Y$.*

Proof. Assume that α is unit regular. Then $\alpha = \alpha\beta\alpha$ for some a unit β in $PFix(X,Y)$, that is, α is regular. So $\operatorname{dom}(\alpha) \cap Y = \operatorname{ran}(\alpha) \cap Y$ and (1) holds. Let $C = \operatorname{ran}(\alpha)\backslash Y = \{c_j\}$ and choose $P = \{c_j\beta\}$. In order to show that P is a cross section of $\pi_\alpha(X \setminus Y) = \{c_j\alpha^{-1}\}$, for each j, we let $x_j \in \operatorname{dom}(\alpha)$ in which $x_j\alpha = c_j$. If there is $c_{j_0}\beta \in P \setminus \operatorname{dom}(\alpha)$, then $c_{j_0} = x_{j_0}\alpha = x_{j_0}(\alpha\beta\alpha) = (c_{j_0}\beta)\alpha \notin \operatorname{ran}(\alpha)\backslash Y$, a contradiction. This implies $P \subseteq \operatorname{dom}(\alpha)$. In addition, if there exists $(c_{j_0}\beta)\alpha \in P\alpha \cap Y$, then we choose $x \in c_{j_0}\alpha^{-1}$. Consequently, $x\alpha \in \operatorname{ran}(\alpha) \setminus Y$. However, $x\alpha = x\alpha\beta\alpha = (c_{j_0}\beta)\alpha \in Y$, which leads to a contradiction. Thus, $P\alpha \subseteq \operatorname{ran}(\alpha) \setminus Y$. To show $|P \cap c_j\alpha^{-1}| = 1$ for all j, we first assume to contrary that there is j_0 such that $P \cap c_{j_0}\alpha^{-1} = \emptyset$. Then $c_{j_0} = x_{j_0}\alpha = x_{j_0}(\alpha\beta\alpha) = (c_{j_0}\beta)\alpha \neq c_{j_0}$ since $c_{j_0}\beta \in P$, a contradiction. Thus $P \cap c_j\alpha^{-1} \neq \emptyset$ for all j. Now, assume that $(c_{j_1}\beta)\alpha = (c_{j_2}\beta)\alpha$ for some $c_{j_1}\beta, c_{j_2}\beta \in P$. Then $c_{j_1} = x_{j_1}\alpha = x_{j_1}(\alpha\beta\alpha) = (c_{j_1}\beta)\alpha = (c_{j_2}\beta)\alpha = x_{j_2}(\alpha\beta\alpha) = x_{j_2}\alpha = c_{j_2}$. We can conclude that $|P \cap c_j\alpha^{-1}| = 1$ for all j. Therefore, P is a cross section of $\pi_\alpha(X \setminus Y)$. Since $\operatorname{dom}(\beta) = X = (Y \cup \{c_j\}) \cup (X\backslash(Y \cup \{c_j\}))$; $\operatorname{ran}(\beta) = X = (Y \cup \{c_j\beta\}) \cup (X\backslash(Y \cup \{c_j\beta\}))$ and β is bijective, we get $\beta|_{X\backslash(Y\cup\{c_j\})} : X\backslash(Y\cup\{c_j\}) \to X\backslash(Y\cup\{c_j\beta\})$ is also bijective. Hence $|X\backslash(Y\cup C)| = |X\backslash(Y\cup P)|$.

Conversely, assume the conditions hold. By (1), we can write α as

$$\alpha = \begin{pmatrix} A_i & C_j \\ y_i & c_j \end{pmatrix},$$

where $y_i \in A_i \cap Y$ for all i; $C_j \subseteq X\backslash Y$ and $c_j \in X\backslash Y$ for all j. If $\operatorname{ran}(\alpha)\backslash Y = \emptyset$, then $J = \emptyset$ and $\alpha = \alpha id_X\alpha$, that is, α is unit regular. If $\operatorname{ran}(\alpha)\backslash Y \neq \emptyset$, then we let P be a cross section of $\pi_\alpha(X\backslash Y)$ satisfying (2). So $|P \cap C_j| = 1$ for all j. Let $c'_j \in P \cap C_j$. Hence $|X\backslash(Y \cup \{c_j\})| = |X\backslash(Y \cup \{c'_j\})|$. So, there exists a bijection $\sigma : X\backslash(Y \cup \{c_j\}) \to X\backslash(Y \cup \{c'_j\})$. Let $Y = \{y_k\}$, $X \setminus (Y \cup \{c_j\}) = \{z_t\}$ and define $\beta : X \to X$ by

$$\beta = \begin{pmatrix} y_k & c_j & z_t \\ y_k & c'_j & z_t\sigma \end{pmatrix}.$$

So β is a unit of $PFix(X,Y)$ and $\alpha = \alpha\beta\alpha$. Therefore, α is unit regular. ∎

Corollary 9. *$PFix(X,Y)$ is a unit regular semigroup if and only if $Y = X$.*

Proof. Assume $Y \neq X$. Let $y \in Y$ and $x \in X \backslash Y$. Define $\alpha : \{x\} \to X$ by $x\alpha = y$. Then $\alpha \in PFix(X,Y)$ and $\mathrm{dom}(\alpha) \cap Y \neq \mathrm{ran}(\alpha) \cap Y$. Thus α is not regular which is absolutely not unit regular.

Conversely, if $Y = X$, then each element of $PFix(X,Y)$ is of the form id_A, where $A \subseteq Y$ which is unit regular by Theorem 8. Therefore, $PFix(X,Y)$ is a unit regular semigroup. ∎

We finish that note with the characterization of the coregular semigroups $PFix(X,Y)$. The first study of coregular semigroups of (full) transformations, one can find in [4].

An element a in a semigroup S is said to be *coregular*, if $a = aba = bab$ for some $b \in S$ and S is a *coregular semigroup* if all of its elements are coregular.

The following theorem is the characterization of the coregular elements of $PFix(X,Y)$.

Theorem 10. *Let $\alpha \in PFix(X,Y)$. Then α is coregular if and only if the following conditions hold:*

1. $\mathrm{ran}(\alpha) \subseteq \mathrm{dom}(\alpha)$;
2. $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$.

Proof. Assume α is coregular. Then there exists $\beta \in PFix(X,Y)$ such that $\alpha = \alpha\beta\alpha = \beta\alpha\beta$. Hence $\alpha = \beta\alpha\beta = \beta(\alpha\beta\alpha)\beta = (\beta\alpha\beta)(\alpha\beta\alpha)\beta = (\beta\alpha\beta)\alpha(\beta\alpha\beta) = \alpha^3$. Since $\mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^3) \subseteq \mathrm{dom}(\alpha^2) \subseteq \mathrm{dom}(\alpha)$, we obtain $\mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^2)$. Hence $\mathrm{ran}(\alpha) = \mathrm{dom}(\alpha)\alpha = \mathrm{dom}(\alpha^2)\alpha = [(\mathrm{ran}(\alpha) \cap \mathrm{dom}(\alpha))\alpha^{-1}]\alpha \subseteq \mathrm{ran}(\alpha) \cap \mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\alpha)$. Let $x \in \mathrm{ran}(\alpha)$. Then $x \in \mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^2)$ and $x = z\alpha$ for some $z \in \mathrm{dom}(\alpha)$. So, $x\alpha^2 = (z\alpha)\alpha^2 = z\alpha^3 = z\alpha = x = xid_{\mathrm{ran}(\alpha)}$. Hence $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$.

Conversely, assume that the conditions hold. Since $\mathrm{ran}(\alpha) \subseteq \mathrm{dom}(\alpha)$, we obtain $\mathrm{dom}(\alpha^3) = \mathrm{dom}(\alpha)$. For each $x \in \mathrm{dom}(\alpha^3)$, we get $x\alpha^3 = (x\alpha)\alpha^2 = x\alpha$ since $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$. Thus $\alpha^3 = \alpha$ whence α is coregular. ∎

Corollary 11. *$PFix(X,Y)$ is a coregular semigroup if and only if $Y = X$.*

Proof. Since coregularity implies regularity, we immediately get $Y = X$.

Conversely, if $Y = X$, then each element of $PFix(X,Y)$ is of the form id_A, where $A \subseteq Y$ which obviously satisfies all sufficient conditions in Theorem 10. So, it is coregular and $PFix(X,Y)$ is a coregular semigroup, as required. ∎

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