Discussiones Mathematicae
General Algebra and Applications 43 (2023) 241–247
https://doi.org/10.7151/dmgaa.1438

A NOTE ON THE ABUNDANCE OF PARTIAL TRANSFORMATION SEMIGROUPS WITH FIXED POINT SETS

Rattiya Wijarajak and Yanisa Chaiya

Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University (Rangsit Campus)
Pathum Thani, 12120, Thailand

e-mail: rattiya.wija@dome.tu.ac.th
yanisa@mathstat.sci.tu.ac.th

Abstract

Given a non-empty set X and let $P(X)$ be the partial transformation semigroup on X. For a fixed non-empty subset Y of X, let

$$PFix(X,Y) = \{\alpha \in P(X) : y\alpha = y \text{ for all } y \in \mathrm{dom}(\alpha) \cap Y\}.$$

Then $PFix(X,Y)$ is a subsemigroup of $P(X)$. In this paper, we show that $PFix(X,Y)$ is always abundant, even though it is not regular. Moreover, unit regular and coregular elements of such semigroup are all completely characterized.

Keywords: partial transformation semigroup, abundance, unit regularity, coregularity.

2020 Mathematics Subject Classification: 20M20.

1. Introduction and preliminaries

Let X be a non-empty set and $P(X)$ the partial transformation semigroup on X. Fix a non-empty subset Y of X and consider the subsemigroup $PFix(X,Y)$ of $P(X)$ defined by

$$PFix(X,Y) = \{\alpha \in P(X) : y\alpha = y \text{ for all } y \in \mathrm{dom}(\alpha) \cap Y\},$$

which was first introduced and named *partial transformation semigroup with a fixed point set* Y in [3]. The authors showed that $PFix(X,Y)$ need not be regular and proved that an element $\alpha \in PFix(X,Y)$ is regular if and only if $\mathrm{dom}(\alpha) \cap Y = \mathrm{ran}(\alpha) \cap Y$, where $\mathrm{dom}(\alpha)$ and $\mathrm{ran}(\alpha)$ mean the domain of α and the range of α, respectively. Later in [10], the authors provided a complete description of Green's relations on $PFix(X,Y)$ and applied the results to obtain characterizations of left regular, right regular, intra-regular, and complete regular elements in such a semigroup.

Although $PFix(X,Y)$ is not regular, it contains a regular subsemigroup

$$Fix(X,Y) = \{\alpha \in PFix(X,Y) : \mathrm{dom}(\alpha) = X\}$$

which has been discovered before in [5] and its significant properties were described in [1, 2, 8, 9].

In this paper, we describe more regular properties of $PFix(X,Y)$ and show that $PFix(X,Y)$ is always abundant.

Throughout this paper, we write the functions on the right; in particular, this means that for a composition $\alpha\beta$, the transformation α is applied first. To simplify the notation, we often write the singleton set $\{a\}$ as a. For element $\alpha \in PFix(X,Y)$, we write

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

and take as understood that the script i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $\mathrm{ran}(\alpha) = \{a_i\}$ and $a_i\alpha^{-1} = A_i \subseteq \mathrm{dom}(\alpha)$.

2. ABUNDANCE OF $PFix(X,Y)$

On a semigroup S, $a, b \in S$ are $\mathcal{L}^*$-related in S if and only if a and b are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined in the dual way. The semigroup S is said to be *left abundant* if each $\mathcal{L}^*$-class contains an idempotent. *Right abundant semigroup* is defined dually. A semigroup which is both left and right abundant will be called an *abundant semigroup*.

Of course, regular semigroups are abundant and in this case we have $\mathcal{L} = \mathcal{L}^*$, $\mathcal{R} = \mathcal{R}^*$. The aim of this section is to show that $PFix(X,Y)$ is an abundant semigroup which is not regular. Note that we write id_A to mean the identity map on the set A.

Recall the well-known characterizations of the relations $\mathcal{L}$ and $\mathcal{R}$ on $P(X)$; and $\mathcal{L}^*$ and $\mathcal{R}^*$ on any semigroup S in Lemmas 1 and 2, respectively.

Lemma 1 [6]. *Let $\alpha, \beta \in P(X)$. Then*

1. $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\mathrm{ran}(\alpha) = \mathrm{ran}(\beta)$*;*

2. $(\alpha,\beta) \in \mathcal{R}$ *if and only if* $\ker(\alpha) = \ker(\beta)$,
   where $\ker(\gamma) = \{(x_1,x_2) \in \operatorname{dom}(\gamma) \times \operatorname{dom}(\gamma) : x_1\gamma = x_2\gamma\}$ *for any* $\gamma \in P(X)$.

Lemma 2 [7]**.** *Let S be a semigroup. Then*

$$\mathcal{L}^* = \big\{(a,b) \in S \times S : (\forall s,t \in S^1)\ as = at \Leftrightarrow bs = bt\big\},$$

$$\mathcal{R}^* = \big\{(a,b) \in S \times S : (\forall s,t \in S^1)\ sa = ta \Leftrightarrow sb = tb\big\}.$$

For the semigroup $PFix(X,Y)$, we have the characterization of the relation $\mathcal{L}^*$ as shown in the following lemma.

Lemma 3. *Let $\alpha,\beta \in PFix(X,Y)$. Then $(\alpha,\beta) \in \mathcal{L}^*$ if and only if* $\operatorname{ran}(\alpha) = \operatorname{ran}(\beta)$.

Proof. Assume $\operatorname{ran}(\alpha) = \operatorname{ran}(\beta)$. Then α and β are known to be $\mathcal{L}$-related in $P(X)$. Hence α and β are $\mathcal{L}^*$-related in $PFix(X,Y)$.

Conversely, assume that $(\alpha,\beta) \in \mathcal{L}^*$ and define $\gamma = id_{\operatorname{ran}(\alpha)}$. Clearly, $\operatorname{ran}(\gamma) = \operatorname{ran}(\alpha)$ and $\alpha\gamma = \alpha$. Applying the characterization of the relation $\mathcal{L}^*$ from Lemma 2 (with α,β in the roles of a,b and γ and the identity in the roles of s and t, respectively), we conclude that $\beta\gamma = \beta$ and $\operatorname{ran}(\beta) = \operatorname{ran}(\beta\gamma) = (\operatorname{ran}(\beta) \cap \operatorname{dom}(\gamma))\gamma \subseteq \operatorname{ran}(\gamma) = \operatorname{ran}(\alpha)$. Similary, $\operatorname{ran}(\alpha) \subseteq \operatorname{ran}(\beta)$ whence $\operatorname{ran}(\alpha) = \operatorname{ran}(\beta)$. ∎

Lemma 4. *The semigroup $PFix(X,Y)$ is left abundant.*

Proof. For each $\alpha \in PFix(X,Y)$, we have $id_{\operatorname{ran}(\alpha)}$ is an idempotent in the $\mathcal{L}^*$-class of α. Hence, an arbitrary $\mathcal{L}^*$-class of $PFix(X,Y)$ contains an idempotent. Therefore, $PFix(X,Y)$ is left abundant. ∎

Next, we give the characterization of the relation $\mathcal{R}^*$ on $PFix(X,Y)$ as in the following lemma.

Lemma 5. *Let $\alpha,\beta \in PFix(X,Y)$. Then $(\alpha,\beta) \in \mathcal{R}^*$ if and only if* $\ker(\alpha) = \ker(\beta)$.

Proof. Assume $\ker(\alpha) = \ker(\beta)$. Then α and β are known to be $\mathcal{R}$-related in $P(X)$. Hence α and β are $\mathcal{R}^*$-related in $PFix(X,Y)$.

Conversely, assume that $(\alpha,\beta) \in \mathcal{R}^*$. To prove that $\ker(\alpha) = \ker(\beta)$, we first establish that $\operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$. Since $id_{\operatorname{dom}(\alpha)}\alpha = \alpha$, using Lemma 2, we deduce that $id_{\operatorname{dom}(\alpha)}\beta = \beta$. Consequently, $\operatorname{dom}(\beta) = \operatorname{dom}(id_{\operatorname{dom}(\alpha)}\beta) \subseteq \operatorname{dom}(id_{\operatorname{dom}(\alpha)}) = \operatorname{dom}(\alpha)$. Similarly, we have $\operatorname{dom}(\alpha) \subseteq \operatorname{dom}(\beta)$, and thus $\operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$. Now, let $(a,b) \in \ker(\alpha)$. This implies that $a\alpha = b\alpha$, and two cases arise.

Case 1. $a \in Y$ and $b \in X \setminus Y$. Let $Y \setminus \{a\} = \{y_i\}$, $X \setminus (Y \cup \{b\}) = \{x_j\}$, and define $\gamma \in PFix(X, Y)$ as follows:

$$\gamma = \begin{pmatrix} \{a, b\} & y_i & x_j \\ a & y_i & x_j \end{pmatrix}.$$

We can observe that $\gamma\alpha = \alpha$, and then, by Lemma 2, $\gamma\beta = \beta$. Hence, $b\beta = b\gamma\beta = a\beta$, which implies $(a, b) \in \ker(\beta)$.

Case 2. $a, b \in X \setminus Y$. Let $Y = \{y_i\}$, $X \setminus (Y \cup \{a, b\}) = \{x_j\}$, and define γ as described in Case 1. Using the same proof as presented in Case 1, we can conclude that $(a, b) \in \ker(\beta)$.

Similarly, we have $\ker(\beta) \subseteq \ker(\alpha)$, which implies that $\ker(\alpha) = \ker(\beta)$, as required. ■

Lemma 6. *The semigroup $PFix(X, Y)$ is right abundant.*

Proof. For any $\alpha \in PFix(X, Y)$, write

$$\alpha = \begin{pmatrix} A_i & C_j \\ y_i & c_j \end{pmatrix},$$

where $y_i \in A_i \cap Y$ for all i and $C_j \subseteq X \setminus Y$. For each j, choose $c_j' \in C_j$ and let

$$\gamma = \begin{pmatrix} A_i & C_j \\ y_i & c_j' \end{pmatrix}.$$

Then γ is an idempotent in $PFix(X, Y)$ with $\ker(\alpha) = \ker(\gamma)$, that is, γ is in $\mathcal{R}^*$-class of α. Therefore, $PFix(X, Y)$ is right abundant. ■

Using Lemmas 4 and 6, we obtain

Theorem 7. *The semigroup $PFix(X, Y)$ is an abundant semigroup.*

3. UNIT REGULAR AND COREGULAR ELEMENTS OF $PFix(X, Y)$

Let S be a monoid with identity 1. An element $u \in S$ is a *unit* if there exists $u' \in S$ such that $uu' = 1 = u'u$. Moreover, an element $a \in S$ is said to be *unit regular* if there exists a unit $u \in S$ such that $a = aua$. In particular, if all elements of S are unit regular, then S is called a *unit regular semigroup.*

Notice that $PFix(X, Y)$ is a monoid having id_X as an identity. It is clear that $\alpha \in PFix(X, Y)$ is a unit if and only if α is bijective with $\text{dom}(\alpha) = X$, that is, $\alpha|_Y = id_Y$ and $\alpha|_{X\setminus Y} : X \setminus Y \to X \setminus Y$ is a bijection.

For each $\alpha \in PFix(X, Y)$, let

$$\pi_\alpha = \{x\alpha^{-1} : x \in \mathrm{ran}(\alpha)\} \text{ and } \pi_\alpha(X\backslash Y) = \{x\alpha^{-1} : x \in \mathrm{ran}(\alpha)\backslash Y\}.$$

A subset P of X is said to be a *cross section of* π_α if $P \subseteq \mathrm{dom}(\alpha)$ and $|P \cap x\alpha^{-1}| = 1$ for all $x\alpha^{-1} \in \pi_\alpha$. In particular, P is said to be a cross section of $\pi_\alpha(X \setminus Y)$ if $P \subseteq \mathrm{dom}(\alpha)$ such that $P\alpha \subseteq \mathrm{ran}(\alpha)\backslash Y$ and $|P \cap x\alpha^{-1}| = 1$ for all $x\alpha \in \pi_\alpha(X\backslash Y)$.

We now characterize all unit regular elements of $PFix(X,Y)$.

Theorem 8. *Let $\alpha \in PFix(X,Y)$. Then α is unit regular if and only if the following conditions hold:*

1. $\mathrm{dom}(\alpha) \cap Y = \mathrm{ran}(\alpha) \cap Y$*;*
2. *if* $\mathrm{ran}(\alpha)\backslash Y \neq \emptyset$*, then there exists a cross section P of $\pi_\alpha(X\backslash Y)$ such that* $|X\backslash(Y \cup C)| = |X\backslash(Y \cup P)|$*, where $C = \mathrm{ran}(\alpha)\backslash Y$.*

Proof. Assume that α is unit regular. Then $\alpha = \alpha\beta\alpha$ for some a unit β in $PFix(X,Y)$, that is, α is regular. So $\mathrm{dom}(\alpha) \cap Y = \mathrm{ran}(\alpha) \cap Y$ and (1) holds. Let $C = \mathrm{ran}(\alpha)\backslash Y = \{c_j\}$ and choose $P = \{c_j\beta\}$. In order to show that P is a cross section of $\pi_\alpha(X \setminus Y) = \{c_j\alpha^{-1}\}$, for each j, we let $x_j \in \mathrm{dom}(\alpha)$ in which $x_j\alpha = c_j$. If there is $c_{j_0}\beta \in P \setminus \mathrm{dom}(\alpha)$, then $c_{j_0} = x_{j_0}\alpha = x_{j_0}(\alpha\beta\alpha) = (c_{j_0}\beta)\alpha \notin \mathrm{ran}(\alpha)\backslash Y$, a contradiction. This implies $P \subseteq \mathrm{dom}(\alpha)$. In addition, if there exists $(c_{j_0}\beta)\alpha \in P\alpha \cap Y$, then we choose $x \in c_{j_0}\alpha^{-1}$. Consequently, $x\alpha \in \mathrm{ran}(\alpha) \setminus Y$. However, $x\alpha = x\alpha\beta\alpha = (c_{j_0}\beta)\alpha \in Y$, which leads to a contradiction. Thus, $P\alpha \subseteq \mathrm{ran}(\alpha) \setminus Y$. To show $|P \cap c_j\alpha^{-1}| = 1$ for all j, we first assume to contrary that there is j_0 such that $P \cap c_{j_0}\alpha^{-1} = \emptyset$. Then $c_{j_0} = x_{j_0}\alpha = x_{j_0}(\alpha\beta\alpha) = (c_{j_0}\beta)\alpha \neq c_{j_0}$ since $c_{j_0}\beta \in P$, a contradiction. Thus $P \cap c_j\alpha^{-1} \neq \emptyset$ for all j. Now, assume that $(c_{j_1}\beta)\alpha = (c_{j_2}\beta)\alpha$ for some $c_{j_1}\beta, c_{j_2}\beta \in P$. Then $c_{j_1} = x_{j_1}\alpha = x_{j_1}(\alpha\beta\alpha) = (c_{j_1}\beta)\alpha = (c_{j_2}\beta)\alpha = x_{j_2}(\alpha\beta\alpha) = x_{j_2}\alpha = c_{j_2}$. We can conclude that $|P \cap c_j\alpha^{-1}| = 1$ for all j. Therefore, P is a cross section of $\pi_\alpha(X \setminus Y)$. Since $\mathrm{dom}(\beta) = X = (Y \cup \{c_j\}) \cup (X\backslash(Y \cup \{c_j\}))$; $\mathrm{ran}(\beta) = X = (Y \cup \{c_j\beta\}) \cup (X\backslash(Y \cup \{c_j\beta\}))$ and β is bijective, we get $\beta|_{X\backslash(Y\cup\{c_j\})} : X\backslash(Y\cup\{c_j\}) \to X\backslash(Y\cup\{c_j\beta\})$ is also bijective. Hence $|X\backslash(Y\cup C)| = |X\backslash(Y\cup P)|$.

Conversely, assume the conditions hold. By (1), we can write α as

$$\alpha = \begin{pmatrix} A_i & C_j \\ y_i & c_j \end{pmatrix},$$

where $y_i \in A_i \cap Y$ for all i; $C_j \subseteq X\backslash Y$ and $c_j \in X\backslash Y$ for all j. If $\mathrm{ran}(\alpha)\backslash Y = \emptyset$, then $J = \emptyset$ and $\alpha = \alpha id_X\alpha$, that is, α is unit regular. If $\mathrm{ran}(\alpha)\backslash Y \neq \emptyset$, then we let P be a cross section of $\pi_\alpha(X\backslash Y)$ satisfying (2). So $|P \cap C_j| = 1$ for all j. Let $c'_j \in P \cap C_j$. Hence $|X\backslash(Y \cup \{c_j\})| = |X\backslash(Y \cup \{c'_j\})|$. So, there exists a bijection $\sigma : X\backslash(Y \cup \{c_j\}) \to X\backslash(Y \cup \{c'_j\})$. Let $Y = \{y_k\}$, $X \setminus (Y \cup \{c_j\}) = \{z_t\}$ and define $\beta : X \to X$ by

$$\beta = \begin{pmatrix} y_k & c_j & z_t \\ y_k & c'_j & z_t\sigma \end{pmatrix}.$$

So β is a unit of $PFix(X,Y)$ and $\alpha = \alpha\beta\alpha$. Therefore, α is unit regular. ∎

Corollary 9. *$PFix(X,Y)$ is a unit regular semigroup if and only if $Y = X$.*

Proof. Assume $Y \neq X$. Let $y \in Y$ and $x \in X \backslash Y$. Define $\alpha : \{x\} \to X$ by $x\alpha = y$. Then $\alpha \in PFix(X,Y)$ and $\mathrm{dom}(\alpha) \cap Y \neq \mathrm{ran}(\alpha) \cap Y$. Thus α is not regular which is absolutely not unit regular.

Conversely, if $Y = X$, then each element of $PFix(X,Y)$ is of the form id_A, where $A \subseteq Y$ which is unit regular by Theorem 8. Therefore, $PFix(X,Y)$ is a unit regular semigroup. ∎

We finish that note with the characterization of the coregular semigroups $PFix(X,Y)$. The first study of coregular semigroups of (full) transformations, one can find in [4].

An element a in a semigroup S is said to be *coregular*, if $a = aba = bab$ for some $b \in S$ and S is a *coregular semigroup* if all of its elements are coregular.

The following theorem is the characterization of the coregular elements of $PFix(X,Y)$.

Theorem 10. *Let $\alpha \in PFix(X,Y)$. Then α is coregular if and only if the following conditions hold:*

1. $\mathrm{ran}(\alpha) \subseteq \mathrm{dom}(\alpha)$*;*
2. $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$.

Proof. Assume α is coregular. Then there exists $\beta \in PFix(X,Y)$ such that $\alpha = \alpha\beta\alpha = \beta\alpha\beta$. Hence $\alpha = \beta\alpha\beta = \beta(\alpha\beta\alpha)\beta = (\beta\alpha\beta)(\alpha\beta\alpha)\beta = (\beta\alpha\beta)\alpha(\beta\alpha\beta) = \alpha^3$. Since $\mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^3) \subseteq \mathrm{dom}(\alpha^2) \subseteq \mathrm{dom}(\alpha)$, we obtain $\mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^2)$. Hence $\mathrm{ran}(\alpha) = \mathrm{dom}(\alpha)\alpha = \mathrm{dom}(\alpha^2)\alpha = [(\mathrm{ran}(\alpha) \cap \mathrm{dom}(\alpha))\alpha^{-1}]\alpha \subseteq \mathrm{ran}(\alpha) \cap \mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\alpha)$. Let $x \in \mathrm{ran}(\alpha)$. Then $x \in \mathrm{dom}(\alpha) = \mathrm{dom}(\alpha^2)$ and $x = z\alpha$ for some $z \in \mathrm{dom}(\alpha)$. So, $x\alpha^2 = (z\alpha)\alpha^2 = z\alpha^3 = z\alpha = x = xid_{\mathrm{ran}(\alpha)}$. Hence $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$.

Conversely, assume that the conditions hold. Since $\mathrm{ran}(\alpha) \subseteq \mathrm{dom}(\alpha)$, we obtain $\mathrm{dom}(\alpha^3) = \mathrm{dom}(\alpha)$. For each $x \in \mathrm{dom}(\alpha^3)$, we get $x\alpha^3 = (x\alpha)\alpha^2 = x\alpha$ since $\alpha^2|_{\mathrm{ran}(\alpha)} = id_{\mathrm{ran}(\alpha)}$. Thus $\alpha^3 = \alpha$ whence α is coregular. ∎

Corollary 11. *$PFix(X,Y)$ is a coregular semigroup if and only if $Y = X$.*

Proof. Since coregularity implies regularity, we immediately get $Y = X$.

Conversely, if $Y = X$, then each element of $PFix(X,Y)$ is of the form id_A, where $A \subseteq Y$ which obviously satisfies all sufficient conditions in Theorem 10. So, it is coregular and $PFix(X,Y)$ is a coregular semigroup, as required. ∎

Acknowledgement

The authors gratefully acknowledge the financial support provided by Faculty of Science and Technology, Contract No. SciGR 8/2565.

References

[1] Y. Chaiya, P. Honyam and J. Sanwong, *Maximal subsemigroups and finiteness conditions on transformation semigroups with fixed sets*, Turkish J. Math. **41** (2017) 43–54.
https://doi.org/10.3906/mat-1507-7

[2] Y. Chaiya, P. Honyam and J. Sanwong, *Natural partial orders on transformation semigroups with fixed sets*, Int. J. Math. Math. Sci. **Article ID 2759090** (2016) 1–7.
https://doi.org/10.1155/2016/2759090

[3] R. Chinram and W. Yonthanthum, *Regularity of the semigroups of transformations with a fixed point set*, Thai J. Math. **18** (2020) 1261–1268.

[4] I. Dimitrova and J. Koppitz, *Coregular semigroups of full transformations*, Demonstr. Math. **XLIV (4)** (2011) 739–753.
https://doi.org/10.1515/dema-2013-0342

[5] P. Honyam and J. Sanwong, *Semigroups of transformations with fixed sets*, Quaest. Math. **36** (2013) 79–92.
https://doi.org/10.2989/16073606.2013.779958

[6] J.M. Howie, Fundamentals of Semigroup Theory, London Mathematics Society Monographs, New Series **12** (Clarendon Press, Oxford, 1995).

[7] E.S. Lyapin, Semigroups (Am. Math. Soc, Providence, 1963).

[8] N. Nupo and C. Pookpienlert, *On connectedness and completeness of Cayley digraphs of transformation semigroups with fixed sets*, Int. Electron. J. Algebra **28** (2020) 110–126.
https://doi.org/10.24330/ieja.768190

[9] N. Nupo and C. Pookpienlert, *Domination parameters on Cayley digraphs of transformation semigroups with fixed sets*, Turkish J. Math. **50 (9)** (2021) 1775–1788.
https://doi.org/10.3906/mat-2104-18

[10] R. Wijarajak and Y. Chaiya, *Green's relations and regularity on semigroups of partial transformations with fixed sets*, Commun. Algebra **45 (4)** 3827–3839.
https://doi.org/10.1080/00927872.2022.2045606

Received 22 October 2021
Revised 7 March 2022
Accepted 7 March 2022