Discussiones Mathematicae
General Algebra and Applications 44 (2024) 261–276
https://doi.org/10.7151/dmgaa.1436

σ-FILTERS OF DISTRIBUTIVE LATTICES

M. Sambasiva Rao

Department of Mathematics
MVGR College of Engineering, Chintalavalasa
Vizianagaram, Andhra Pradesh, India-535005
e-mail: mssraomaths35@rediffmail.com

Abstract

The concept of σ-filters is introduced in distributive lattices and studied some properties of these classes of filters. Two sets of equivalent conditions are derived one for every μ-filter to become a σ-filter and the other for every filter to become a σ-filter of a distributive lattice. A one-to-one correspondence is established between the set of all prime σ-filters of a distributive lattice and the set of all prime σ-filters of its quotient lattice with respect to a congruence.

Keywords: prime filter, co-annihilator, μ-filter; O-filter, σ-filter, *pm*-lattice.

2020 Mathematics Subject Classification: 06D99.

Introduction

In 1970, the theory of relative annihilators was introduced in lattices by Mark Mandelker [14] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. Speed [13] and Cornish [4] made an extensive study of annihilators in distributive lattices. The class of annulets played a vital role in characterizing many a algebraic structures like normal lattices [3], quasi-complemented lattices [4]. In [7], Pawar and Thakare introduced the class of *pm*-lattices and characterized the *pm*-latices in topological terms. In [11], the author investigated thoroughly the properties co-annihilator filters and μ-filters of distributive lattices. An extensive investigation of co-annihilators was made in residuated lattices by Rasouli in [8]. In [5], the authors studied certain properties co-annihilator filters of residuated lattice in the name of α-filters. In [10], the author studied the properties of O-filters of distributive lattices and characterized the O-filters with the help of minimal prime filters. The main aim of this paper is to study some further properties of co-annihilators in the form of σ-filters of distributive lattices.

In this note, the concept of σ-filters is introduced in distributive lattices and their properties are studied with the help of prime filters, co-annihilator filters, μ-filters, and O-filters. It is observed that every σ-filter of a distributive lattice is a μ-filter but the converse is not true in general. However, some equivalent conditions are derived for every μ-filter of a distributive lattice to become a σ-filter. It is also observed that every O-filter of a distributive lattice is a σ-filter but not the converse in general. Some necessary and sufficient conditions are derived for every σ-filter of a distributive lattice to become an O-filter. Some equivalent conditions are derived to prove that the class of all filters of the form $\sigma(F)$ of a distributive lattice to become a sublattice to the lattice of all filters of the distributive lattice. A set of equivalent conditions is derived for every filter of a distributive lattice to become a σ-filter. For any ideal I of a distributive lattice L, a one-to-one correspondence is obtained between the set of all prime σ-filters of a distributive lattice L and the set of all prime σ-filters of the quotient lattice $L_{/\psi_I}$ where ψ_I is an ideal congruence.

1. Preliminaries

The reader is referred to [1, 2, 9, 10] and [11] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results are presented for the ready reference of the reader.

Definition 1 [1]**.** A lattice $(L, \wedge, \vee)$ is called *distributive* if for all $x, y, z \in L$, it satisfies either of the following properties:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of a lattice L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice. A proper ideal P of a lattice L is called *prime* if for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. A proper ideal M of a lattice is called *maximal* if there exists no proper ideal N such that $M \subset N$. Two prime filters P and Q of a lattice L are called *co-maximal* if $P \vee Q = L$. The *annihilator* [13] of a non-empty set A of a lattice is the set $A^* = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$. The *pseudo-complement* [6] b^* of an element b of a lattice L is the element satisfying

$$a \wedge b = 0 \text{ if and only if } a \leq b^*$$

where $\leq$ is the induced order in L. A lattice in which every element has a pseudo-complement is called a pseudo-complemented lattice. The annihilator of the principal ideal is given by $(x]^* = (x^*]$ where * is the pseudo-complementation on the lattice L.

A bounded distributive lattice L is called a *pm-lattice* if every prime ideal of L is contained in a unique maximal ideal of L. Pawar and Thakare [7] have proved that if L is a *pm*-lattice then the space $\max(L)$ of all maximal ideals of the lattice L is a T_2-space (and hence it is normal). A proper filter P of L is said to be *prime* if for any $x, y \in L$, $x \vee y \in P$ implies $x \in P$ or $y \in P$. A prime filter P of a lattice L is called *minimal* if it is the minimal element in the class of all prime filters. The *dual pseudo-complement* [12] b^+ of an element b of a lattice L is the element satisfying

$$a \vee b = 1 \text{ if and only if } b^+ \leq a$$

where $\leq$ is the induced order in L. A lattice in which every element has a dual pseudo-complement is called a dual pseudo-complemented lattice.

Theorem 2 [9]**.** *A prime filter P of a distributive lattice L with* 1 *is minimal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \vee y = 1$.*

For any non-empty subset A of a distributive lattice L with 1, the co-annihilator of A is define as the set $A^+ = \{x \in L \mid x \vee a = 1 \text{ for all } a \in A\}$. For any non-empty subset A of L, A^+ is a filter of L with $A \cap A^+ = \{1\}$.

Lemma 3 [11]**.** *Let L be a distributive lattice with* 1*. For any subsets A and B of L,*

(1) *$A \subseteq B$ implies $B^+ \subseteq A^+$,*

(2) *$A \subseteq A^{++}$,*

(3) *$A^{+++} = A^+$,*

(4) *$A^+ = L$ if and only if $A = \{1\}$.*

From the above lemma, it can be pointed out that the correspondence $A \mapsto A^+$ is a Galois connection between the subsets of the lattice L. In case of filters of lattices, the following properties of co-annihilators hold.

Proposition 1.1 [11]**.** *Let L be a distributive lattice with* 1*. For any filters F and G of L, we have*

(1) $F^+ \cap F^{++} = \{1\}$,
(2) $F \cap G = \{1\}$ *implies* $F \subseteq G^+$,
(3) $(F \vee G)^+ = F^+ \cap G^+$,
(4) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

It is clear that $([x))^+ = \{x\}^+$ and is simply denoted by $(x)^+$ which is called the *co-annulet*. Then clearly $(0)^+ = \{1\}$. It is clear that $[x)^+ = [x^+)$ where $^+$ is the dual pseudo-complementation on the lattice L. An element x of a lattice L is called *co-dense* if $(x)^+ = \{1\}$. The following corollary is a direct consequence of the above results.

Corollary 4 [11]**.** *Let L be a distributive lattice with 1. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$,
(2) $(a \wedge b)^+ = (a)^+ \cap (b)^+$,
(3) $(a \vee b)^{++} = (a)^{++} \cap (b)^{++}$,
(4) $(a)^+ \cap (b)^+ = \{1\}$ *if and only if* $(a)^+ \subseteq (b)^{++}$,
(5) $(a)^+ = L$ *if and only if* $a = 1$.

A filter F of a distributive lattice L with 1 is called a *co-annihilator filter* [11] if $F = F^{++}$. A filter F of a distributive lattice L with 1 is called a μ-filter [11] of L if $x \in F$ implies $(x)^{++} \subseteq F$ for all $x \in L$. Every co-annihilator filter of a distributive lattice is a μ-filter. A filter F of a distributive lattice L is called an O-filter [10] if $F = O(I)$ for some ideal I of L, where $O(I) = \{x \in L \mid x \vee a = 1 \text{ for some } a \in I\}$. Throughout this note, all lattices are bounded distributive lattice unless otherwise mentioned.

2. Main results

In this section, the concept of σ-filters is introduced in lattices. A set of equivalent conditions is derived for every filter of a lattice to become a σ-filter. Interconnections among σ-filters, μ-filters, O-filters, and minimal prime filters are investigated.

Definition 5. For any filter F of a lattice L, define the set $\sigma(F)$ as follows

$$\sigma(F) = \{x \in X \mid (x)^+ \vee F = L\}.$$

Clearly $\sigma(L) = L$. For $F = \{1\}$, obviously we get $\sigma(\{1\}) = \{1\}$.

Lemma 6. *For any filter F of a lattice L, $\sigma(F)$ is a filter of L.*

Proof. Clearly $1 \in \sigma(F)$. Let $x, y \in \sigma(F)$. Then $(x)^+ \vee F = L$ and $(y)^+ \vee F = L$. Hence

$$\begin{aligned}(x \wedge y)^+ \vee F &= \{(x)^+ \cap (y)^+\} \vee F \\ &= \{(x)^+ \vee F\} \cap \{(y)^+ \vee F\} \\ &= L \cap L \\ &= L\end{aligned}$$

which gives that $x \wedge y \in \sigma(F)$. Let $x \in \sigma(F)$ and $x \leq y$. Then $(x)^+ \subseteq (y)^+$ and thus $L = (x)^+ \vee F \subseteq (y)^+ \vee F$. Hence $y \in \sigma(F)$. Thus $\sigma(F)$ is a filter of L. ■

In the following result, some elementary properties of $\sigma(F)$ are derived.

Lemma 7. *For any two filters F, G of a lattice L, we have*

(1) *$\sigma(F) \subseteq F$,*

(2) *$F \subseteq G$ implies $\sigma(F) \subseteq \sigma(G)$,*

(3) *$\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$.*

Proof. (1) Let $x \in \sigma(F)$. Then $(x)^+ \vee F = L$. Hence $x \in (x)^+ \vee F$. Thus $x = a \wedge b$ for some $a \in (x)^+$ and $b \in F$. Since $a \in (x)^+$, we get $a \vee x = 1$. Thus $x = x \vee x = (a \wedge b) \vee x = (a \vee x) \wedge (b \vee x) = 1 \wedge (b \vee x) = b \vee x \in F$. Therefore $\sigma(F) \subseteq F$.

(2) Suppose $F \subseteq G$. Let $x \in \sigma(F)$. Then $L = (x)^+ \vee F \subseteq (x)^+ \vee G$. Therefore $x \in \sigma(G)$.

(3) Clearly $\sigma(F \cap G) \subseteq \sigma(F) \cap \sigma(G)$. Conversely, let $x \in \sigma(F) \cap \sigma(G)$. Then $(x)^+ \vee F = (x)^+ \vee G = L$. Now $(x)^+ \vee (F \cap G) = \{(x)^+ \vee F\} \cap \{(x)^+ \vee G\} = L \cap L = L$. Hence $x \in \sigma(F \cap G)$. Thus $\sigma(F) \cap \sigma(G) \subseteq \sigma(F \cap G)$. Therefore $\sigma(F \cap G) = \sigma(F) \cap \sigma(G)$. ■

Definition 8. A filter F of a lattice L is called a σ-filter if $F = \sigma(F)$.

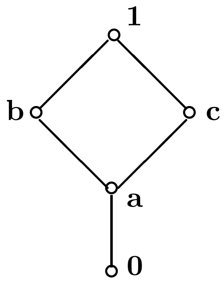
Clearly the improper filters $\{1\}$ and L are trivial σ-filters of L. It is obvious that a proper σ-filter of a lattice contains no co-dense elements. In [11], the class of all μ-filters of a lattice L is characterized in terms of co-annihilators of the lattice. In the following theorem, it is proved that the class of all μ-filters of a lattice L contains properly the class of all σ-filters of L.

Proposition 2.1. *Every σ-filter of a lattice is a μ-filter.*

Proof. Let F be a σ-filter of a lattice L. Then $\sigma(F) = F$. Let $x \in F$. Then $(x)^+ \vee F = L$. Now, let $t \in (x)^{++}$. Then $(x)^+ \subseteq (t)^+$. Hence $L = (x)^+ \vee F \subseteq (t)^+ \vee F$. Thus $t \in \sigma(F) = F$, which proves that $(x)^{++} \subseteq F$. Therefore F is a μ-filter of L. ■

The converse of the above proposition is not true. i.e. every μ-filter of a lattice need not be a σ-filter. For consider the following example.

Example 9. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure.



Consider the filter $F = \{b, 1\}$. It can be easily observed that $(b)^{++} \subseteq F$. Hence F is a μ-filter of L. Observe that $(b)^{+} \vee F = \{a, b, c, 1\} \neq L$. Therefore F is not a σ-filter of L.

However, in the following theorem, some equivalent conditions are given for every μ-filter of a lattice to become a σ-filter.

Theorem 10. *Let L be a lattice. Then the following assertions are equivalent:*

(1) *every μ-filter is a σ-filter,*

(2) *every co-annihilator filter is a σ-filter,*

(3) *for each $x \in L$, $(x)^{++}$ is a σ-filter,*

(4) *for each $x \in L$, $(x)^{+} \vee (x)^{++} = L$.*

Proof. (1)⇒(2) Since every co-annihilator filter is a μ-filter, it is clear.

(2)⇒(3) Since each $(x)^{++}$ is a co-annihilator filter, it is clear.

(3)⇒(4) Assume statement (3). Let $x \in L$. Since $(x)^{++}$ is a σ-filter of L, we get $(x)^{++} = \sigma\big((x)^{++}\big)$. Clearly $x \in (x)^{++} = \sigma\big((x)^{++}\big)$. Hence $(x)^{+} \vee (x)^{++} = L$.

(4)⇒(1) Assume that $(x)^{+} \vee (x)^{++} = L$ for each $x \in L$. Let F be a μ-filter of L. Clearly $\sigma(F) \subseteq F$. Conversely, let $x \in F$. Since F is a μ-filter, we get $(x)^{++} \subseteq F$. Hence $L = (x)^{+} \vee (x)^{++} \subseteq (x)^{+} \vee F$. Thus $x \in \sigma(F)$. Therefore F is a σ-filter of L. ∎

In [10], authors studied the properties of O-filters and proved that every O-filter of a lattice is the intersection of all minimal prime filters containing it. In the following result, it is proved that the class of all σ-filters is properly contained in the class of all O-filters.

Theorem 11. *Every σ-filter of a lattice is an O-filter.*

Proof. Let F be a σ-filter of a lattice L. Then $\sigma(F) = F$. Consider $S = \{x \in L \mid (x)^{++} \vee F = L\}$. We first show that S is an ideal of L. Clearly $0 \in S$. Let $x, y \in S$. Then $(x \vee y)^{++} \vee F = \big\{(x)^{++} \cap (y)^{++}\big\} \vee F = \big\{(x)^{++} \vee F\big\} \cap \big\{(y)^{++} \vee F\big\} = L \cap L = L$. Hence $x \vee y \in S$. Let $x \in S$ and $y \leq x$. Then $L = (x)^{++} \vee F \subseteq (y)^{++} \vee F$. Hence $y \in S$. Thus S is an ideal of L. We now show that $F = O(S)$. Let $x \in O(S)$. Then $x \vee y = 1$ for some $y \in S$. Now

$$\begin{aligned} x \vee y = 1 &\Rightarrow y \in (x)^{+} \\ &\Rightarrow (y)^{++} \subseteq (x)^{+} \\ &\Rightarrow L = (y)^{++} \vee F \subseteq (x)^{+} \vee F \qquad \text{since } y \in S \\ &\Rightarrow x \in \sigma(F) = F \qquad \text{since } F \text{ is a } \sigma\text{-filter} \end{aligned}$$

which yields that $O(S) \subseteq F$. Conversely, let $x \in F = \sigma(F)$. Then $(x)^{+} \vee \sigma(F) = L$. Therefore $0 \in (x)^{+} \vee \sigma(F)$. Hence $0 = a \wedge b$ for some $a \in (x)^{+}$ and $b \in \sigma(F)$. Thus $a \vee x = 1$ and $(b)^{+} \vee F = L$. Now

$$\begin{aligned} a \wedge b = 0 &\Rightarrow (a \wedge b)^{+} = (0)^{+} = \{1\} \\ &\Rightarrow (a)^{+} \cap (b)^{+} = \{1\} \\ &\Rightarrow (b)^{+} \subseteq (a)^{++} \\ &\Rightarrow L = (b)^{+} \vee F \subseteq (a)^{++} \vee F \qquad \text{since } b \in \sigma(F) \\ &\Rightarrow a \in S \ \text{ and } \ a \vee x = 1 \\ &\Rightarrow x \in O(S) \end{aligned}$$

which gives $F = \sigma(F) \subseteq O(S)$. Hence $F = O(S)$. Therefore F is an O-filter of L. ■

The converse of the above theorem is not true, i.e., every O-filter of a lattice need not be a σ-filter. For, consider the distributive lattice given in Example 9. Consider $F = \{1, b\}$ and $I = \{0, a, c\}$. Clearly F is a filter and I is an ideal of L such that $F = O(I)$. Hence F is an O-filter of L. Now, observe that $\sigma(F) = \{1\}$, because $(b)^{+} \vee F = \{1, a, b, c\} \neq L$. Therefore F is not a σ-filter of L.

Proposition 2.2. *Each co-annulet of a lattice is an O-filter.*

Proof. Let L be a lattice and $a \in L$. Then $(a)^{+}$ is a co-annulet of L. It is easy to check that $(a)^{+} = O((a])$. ■

Theorem 12. *Let L be a lattice. Then the following assertions are equivalent:*

(1) *L is a pm-lattice,*

(2) *every O-filter is a σ-filter,*

(3) *for any* $a, b \in L$, $a \vee b = 1$ *implies* $(a)^+ \vee (b)^+ = L$,

(4) *each co-annulet is a* σ*-filter,*

(5) *for any two distinct maximal ideals* M, N *of* L, *there exists* $a \notin M$ *and* $b \notin N$ *such that* $a \wedge b = 0$,

(6) *any two distinct minimal prime filters are co-maximal.*

Proof. (1)$\Rightarrow$(2) Assume that L is a *pm*-lattice. Then every prime ideal of L is contained in a unique maximal ideal of L. Let F be an O-filter of L. Then there exists an ideal I of L such that $F = O(I)$. Clearly $\sigma(F) \subseteq F$. Conversely, let $x \in F = O(I)$. Then there exists $s \in I$ such that $x \vee s = 1$. Suppose $(x)^+ \vee F \neq L$. Then there exists a prime ideal P such that $\{(x)^+ \vee F\} \cap P = \emptyset$. Then $P \vee (x]$ is an ideal of L such that $P \subseteq P \vee (x]$. Suppose $s \in P \vee (x]$. Then $s = t \vee x$ for some $t \in P$. Hence $1 = x \vee s = x \vee (t \vee x) = t \vee x$, which implies $t \in (x)^+ \subseteq (x)^+ \vee F$. Thus $t \in \{(x)^+ \vee F\} \cap P$, which is a contradiction. Therefore $s \notin P \vee (x]$, which means that $P \vee (x]$ is a proper ideal of L. Then there exists a maximal ideal M_1 such that $P \vee (x] \subseteq M_1$. Again, we have $P \vee (s]$ is an ideal such that $P \subseteq P \vee (s]$. Suppose $x \in P \vee (s]$. Then $x = t \vee s$ for some $t \in P$. Hence $1 = x \vee s = (t \vee s) \vee s = t \vee s$. Thus $t \in (s)^+ \subseteq (x)^+ \vee O(I)$ because of $s \in I$. Hence $t \in \{(x)^+ \vee O(I)\} \cap P = \{(x)^+ \vee F\} \cap P$, which is a contradiction. Therefore $x \notin P \vee (s]$, which means that $P \vee (s]$ is a proper ideal of L. Then there exists a maximal ideal M_2 such that $P \vee (s] \subseteq M_2$. Since $x \vee s = 1$, we get $s \notin M_1$ and $x \notin M_2$. Therefore $M_1 \neq M_2$. Thus the prime ideal P is contained in two distinct maximal ideals, which is a contradiction to the hypothesis. Hence $(x)^+ \vee F = L$. Therefore $x \in \sigma(F)$, which means $\sigma(F) = F$.

(2)$\Rightarrow$(3) Let $a, b \in L$ be such that $a \vee b = 1$. By the above proposition, $(a)^+$ is an O-filter of L. Hence $b \in (a)^+ = \sigma((a)^+)$. Therefore $(a)^+ \vee (b)^+ = L$.

(3)$\Rightarrow$(4) Assume the condition (3). Let $(a)^+, a \in L$ be a co-annulet in L. Clearly $\sigma((a)^+) \subseteq (a)^+$. Conversely, let $x \in (a)^+$. Then $x \vee a = 1$. By (3), we get $(x)^+ \vee (a)^+ = L$. Hence $x \in \sigma((a)^+)$. Thus $(a)^+ = \sigma((a)^+)$. Therefore $(a)^+$ is a σ-filter.

(4)$\Rightarrow$(5) Assume condition (4) holds. Let M and N be two distinct maximal ideals of L. Choose $x \in M - N$. Since $x \notin N$, we get $N \vee (x] = L$. Hence, $a \vee x = 1$ for some $a \in N$. Thus $x \in (a)^+$. By condition (4), we get $\sigma((a)^+) = (a)^+$. Since $x \in (a)^+ = \sigma((a)^+)$, we get $(x)^+ \vee (a)^+ = L$. Then $0 \in (a)^+ \vee (x)^+$. Then there exist two elements $s \in (a)^+$ and $t \in (x)^+$ such that $s \wedge t = 0$. If $s \in N$, then $1 = s \vee a \in N$, which is a contradiction. If $t \in M$, then $1 = t \vee x \in M$, which is also a contradiction. Therefore there exist $t \notin M$ and $s \notin M$ such that $s \wedge t = 0$.

(5)$\Rightarrow$(6) Assume condition (5). Let P and Q be two distinct minimal prime filters of L. Then $L - P$ and $L - Q$ are distinct maximal ideals of L. By (5), there exist $a \notin L - P$ and $b \notin L - Q$ such that $a \wedge b = 0$. Hence $a \in P$ and $b \in Q$ such that $a \wedge b = 0$. Hence $0 = a \wedge b \in P \vee Q$. Therefore $P \vee Q = L$.

(6)$\Rightarrow$(1) Assume the condition (6). Let P be a prime ideal of L. Let M_1 and M_2 be two maximal ideals of L such that $P \subseteq M_1$ and $P \subseteq M_2$. Suppose $M_1 \neq M_2$. Then $L - M_1$ and $L - M_2$ are distinct minimal prime filters of L such that $L - M_1 \subseteq L - P$ and $L - M_2 \subseteq L - P$. By condition (6), we get $(L - M_1) \vee (L - M_2) = L$. Hence $L = (L - M_1) \vee (L - M_2) \subseteq L - P$, which is a contradiction. Thus P should be contained in a unique maximal ideal. Therefore L is a *pm*-lattice. ∎

Definition 13. For any proper filter F of a lattice L, define the set $\omega(F)$ as $\omega(F) = \{x \in L \mid (x)^+ \nsubseteq F\}$.

Proposition 2.3. *Let L be a lattice and M be a maximal filter of L. Then the set $\omega(M)$ is a filter of L such that $\omega(M) \subseteq M$.*

Proof. Since M is proper, we get $(1)^+ \nsubseteq M$. Hence $1 \in \omega(M)$. Suppose $x, y \in \omega(M)$. Then $(x)^+ \nsubseteq M$ and $(y)^+ \nsubseteq M$. Hence $M \subset M \vee (x)^+$ and $M \subset M \vee (y)^+$. Since M is maximal, we get $M \vee (x)^+ = L$ and $M \vee (y)^+ = L$. Thus, we get

$$M \vee (x \wedge y)^+ = M \vee \left\{(x)^+ \cap (y)^+\right\} = \left\{M \vee (x)^+\right\} \cap \left\{M \vee (y)^+\right\} = L \cap L = L.$$

If $(x \wedge y)^+ \subseteq M$, then $M = L$ which is a contradiction. Hence $(x \wedge y)^+ \nsubseteq M$. Thus $x \wedge y \in \omega(M)$. Again, let $x \in \omega(M)$ and $x \leq y$. Then $(x)^+ \nsubseteq M$ and $x \leq y$. Since $x \leq y$, we get $(x)^+ \subseteq (y)^+$. Hence $(y)^+ \nsubseteq M$. Hence $y \in \omega(M)$. Therefore $\omega(M)$ is a filter of L. Now, let $x \in \omega(M)$. Then $(x)^+ \nsubseteq M$. Hence, there exists $a \in (x)^+$ such that $a \notin M$. Since $a \in (x)^+$, we get $a \vee x = 1$. Suppose $x \notin M$. Then $M \vee [x) = L$. Since $a \notin M$, we get $M \vee [a) = L$. Hence $L = M \vee \{[x) \cap [a)\} = M \vee [x \vee a) = M \vee [1) = M$, which is a contradiction. Hence $x \in M$. Therefore $\omega(M) \subseteq M$. ∎

Proposition 2.4. *Let M be a prime filter of a lattice L. Then we have*

(1) *$\sigma(M) \subseteq \omega(M)$,*
(2) *if M is maximal, then $\sigma(M) = \omega(M)$.*

Proof. (1) Let $x \in \sigma(M)$. Then $(x)^+ \vee M = L$. Suppose $(x)^+ \subseteq M$. Then $M = L$, which is a contradiction. Hence $(x)^+ \nsubseteq M$. Thus $x \in \omega(M)$. Therefore $\sigma(M) \subseteq \omega(M)$.

(2) Since M is proper, we get $\sigma(M) \subseteq \omega(M)$. Conversely, let $x \in \omega(M)$. Then $(x)^+ \nsubseteq M$. Since M is maximal, we get $(x)^+ \vee M = L$. Thus $x \in \sigma(M)$. Therefore $\omega(M) = \sigma(M)$. ∎

Let us denote that μ is the set of all maximal filters of a lattice L. For any filter F of a lattice L, we also denote $\mu(F) = \{M \in \mu \mid F \subseteq M\}$. Since every maximal filter of a lattice is prime, by Proposition 2.12, we conclude that $\omega(M)$ is a filter such that $\omega(M) \subseteq M$ for every $M \in \mu$. Then we have the following result.

Theorem 14. *For any filter F of a lattice L, $\sigma(F) = \bigcap_{M\in\mu(F)} \omega(M)$.*

Proof. Let $x \in \sigma(F)$ and $F \subseteq M$ where $M \in \mu$. Then $L = (x)^+ \vee F \subseteq (x)^+ \vee M$. Suppose $(x)^+ \subseteq M$, then $M = L$, which is a contradiction. Hence $(x)^+ \nsubseteq M$. Thus $x \in \omega(M)$ for all $M \in \mu(F)$. Therefore $\sigma(F) \subseteq \bigcap_{M\in\mu(F)} \omega(M)$. Conversely, let $x \in \bigcap_{M\in\mu(F)} \omega(M)$. Then $x \in \omega(M)$ for all $M \in \mu(F)$. Suppose $(x)^+ \vee F \neq L$. Then there exists a maximal filter M_0 such that $(x)^+ \vee F \subseteq M_0$. Hence $(x)^+ \subseteq M_0$ and $F \subseteq M$. Since $F \subseteq M_0$, by hypothesis, we get $x \in \omega(M_0)$. Hence $(x)^+ \nsubseteq M_0$, which is a contradiction. Therefore $(x)^+ \vee F = L$. Hence $x \in \sigma(F)$. Therefore $\bigcap_{M\in\mu(F)} \omega(M) \subseteq \sigma(F)$. ∎

From the above theorem, it can be easily observed that $\sigma(F) \subseteq \omega(M)$ for every $M \in \mu(F)$. Now, in the following, a set of equivalent conditions is derived for the class of all filters of the form $\sigma(F)$ to become a sublattice to the lattice $\mathcal{F}(L)$ of all filters of L.

Theorem 15. *Let L be a lattice. Then the following assertions are equivalent:*

(1) *for any $M \in \mu$, $\omega(M)$ is maximal,*

(2) *for any $F, G \in \mathcal{F}(L)$, $F \vee G = L$ implies $\sigma(F) \vee \sigma(G) = L$,*

(3) *for any $F, G \in \mathcal{F}(L)$, $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$,*

(4) *for any two distinct maximal filters M and N, $\omega(M) \vee \omega(N) = L$,*

(5) *for any $M \in \mu$, M is the unique member of μ such that $\omega(M) \subseteq M$.*

Proof. (1)⇒(2) Assume condition (1). Then clearly $\omega(M) = M$ for all $M \in \mu$. Let $F, G \in \mathcal{F}(L)$ be such that $F \vee G = L$. Suppose $\sigma(F) \vee \sigma(G) \neq L$. Then there exists a maximal filter M such that $\sigma(F) \vee \sigma(G) \subseteq M$. Hence $\sigma(F) \subseteq M$ and $\sigma(G) \subseteq M$. Now

$$\begin{aligned}
\sigma(F) \subseteq M \Rightarrow & \bigcap_{M_i\in\mu(F)} \omega(M_i) \subseteq M \\
& \Rightarrow \omega(M_i) \subseteq M \quad \text{for some } M_i \in \mu(F) \text{ (since } M \text{ is prime)} \\
& \Rightarrow M_i \subseteq M \qquad \text{by condition (1)} \\
& \Rightarrow F \subseteq M \qquad \text{since } F \subseteq M_i.
\end{aligned}$$

Similarly, we can obtain that $G \subseteq M$. Hence $L = F \vee G \subseteq M$, which is a contradiction to the maximality of M. Therefore $\sigma(F) \vee \sigma(G) = L$.

(2)⇒(3) Assume condition (2). Let $F, G \in \mathcal{F}(L)$. Clearly $\sigma(F) \vee \sigma(G) \subseteq \sigma(F \vee G)$. Conversely, let $x \in \sigma(F \vee G)$. Then $\{(x)^+ \vee F\} \vee \{(x)^+ \vee G\} =$

$(x)^+ \vee F \vee G = L$. Hence by condition (2), we get $\sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G) = L$. Thus $x \in \sigma((x)^+ \vee F) \vee \sigma((x)^+ \vee G)$. Hence $x = r \wedge s$ for some $r \in \sigma((x)^+ \vee F)$ and $s \in \sigma((x)^+ \vee G)$. Now

$$\begin{aligned} r \in \sigma((x)^+ \vee F) &\Rightarrow (r)^+ \vee \{(x)^+ \vee F\} = L \\ &\Rightarrow L = \{(r)^+ \vee (x)^+\} \vee F \subseteq (r \vee x)^+ \vee F \\ &\Rightarrow (r \vee x)^+ \vee F = L \\ &\Rightarrow r \vee x \in \sigma(F). \end{aligned}$$

Similarly, we can get $s \vee x \in \sigma(G)$. Now, we have the following consequence

$$\begin{aligned} x &= x \vee x \\ &= (r \wedge s) \vee x \\ &= (r \vee x) \wedge (s \vee x) \end{aligned}$$

where $r \vee x \in \sigma(F)$ and $s \vee x \in \sigma(G)$. Hence $x \in \sigma(F) \vee \sigma(G)$. Thus $\sigma(F \vee G) \subseteq \sigma(F) \vee \sigma(G)$. Therefore $\sigma(F) \vee \sigma(G) = \sigma(F \vee G)$.

(3)⇒(4) Assume condition (3). Let M, N be two distinct maximal filters of L. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$, we get $N \vee [x) = L$. Since $y \notin M$, we get $M \vee [y) = L$. Now, we get

$$\begin{aligned} L &= \sigma(L) \\ &= \sigma(L \vee L) \\ &= \sigma\big(\{N \vee [x)\} \vee \{M \vee [y)\}\big) \\ &= \sigma\big(\{M \vee [x)\} \vee \{N \vee [y)\}\big) \\ &= \sigma(M \vee N) && \text{since } x \in M \text{ and } y \in N \\ &= \sigma(M) \vee \sigma(N) && \text{By condition (3)} \\ &\subseteq \omega(M) \vee \omega(N) && \text{By Proposition 2.4(1).} \end{aligned}$$

Therefore $\omega(M) \vee \omega(N) = L$.

(4)⇒(5) Assume condition (4). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $\omega(N) \subseteq M$. Since $\omega(M) \subseteq M$, by hypothesis, we get $L = \omega(M) \vee \omega(N) = M$, which is a contradiction. Hence M is the unique maximal filter such that $\omega(M) \subseteq M$.

(5)⇒(1) Let $M \in \mu$. Suppose $\omega(M)$ is not maximal. Let M_0 be a maximal filter of L such that $\omega(M) \subseteq M_0$. We have always $\omega(M_0) \subseteq M_0$, which is a contradiction. ■

Theorem 16. *Following assertions are equivalent in a lattice L:*

(1) *every filter is a σ-filter,*

(2) *every prime filter is a σ-filter,*

(3) *every prime filter is minimal.*

Proof. (1)⇒(2) It is clear.

(2)⇒(3) Assume that every prime filter is a σ-filter. Let P be a prime filter of L. Since P is proper, there exists $c \in L$ such that $c \notin P$. In view of condition (2), P is a σ-filter of L. Hence $\sigma(P) = P$. Let $x \in P = \sigma(P)$. Then $(x)^+ \vee P = L$ and thus $c \in (x)^+ \vee P$. Then $c = a \wedge b$ for some $a \in (x)^+$ and $b \in P$. Since $a \in (x)^+$, we get $x \vee a = 1$. Suppose $a \in P$. Since P is prime and $b \in P$, we get $c = a \wedge b \in P$ which is a contradiction. Thus $a \notin P$. This means that $x \vee a = 1$ for some $a \notin P$. Therefore P is minimal.

(3)⇒(1) Assume that every prime filter is minimal. Let F be a filter of L. Clearly $\sigma(F) \subseteq F$. Conversely, let $x \in F$. Suppose $(x)^+ \vee F \neq L$. Then there exists a prime filter P such that $(x)^+ \vee F \subseteq P$. Hence $(x)^+ \subseteq P$ and $F \subseteq P$. By our assumption, P is minimal. Since $x \in F \subseteq P$, by Theorem 2, there exists $y \notin P$ such that $x \vee y = 1$. Hence $y \in (x)^+ \subseteq P$, which is a contradiction. Thus $(x)^+ \vee F = L$. Therefore F is a σ-filter of L. ■

Proposition 2.5. *Let I be an ideal of a lattice L. For any $x, y \in L$, define a binary relation ψ_I on L by $(x, y) \in \psi_I$ if and only if $x \vee a = y \vee a$ for some $a \in I$. Then ψ_I is a congruence on L with I as a congruence class modulo ψ_I.*

For any distributive lattice L, it can be shown that the quotient algebra $L_{/\psi_I}$ is also a distributive lattice with respect to the following operations

$$[x]_{\psi_I} \wedge [y]_{\psi_I} = [x \wedge y]_{\psi_I} \text{ and } [x]_{\psi_I} \vee [y]_{\psi_I} = [x \vee y]_{\psi_I}$$

where $[x]_{\psi_I}$ is the congruence class of x modulo ψ_I. It can be routinely verified that the mapping $\Psi : L \longrightarrow L_{/\psi_I}$ defined by $\Psi(x) = [x]_{\psi_I}$ is a homomorphism. For any $x, y \in L$, it is clear that $x \leq y$ implies $[x]_{\psi_I} \subseteq [y]_{\psi_I}$. Hence $\left(L_{/\psi_I}, \cap, \vee\right)$ is a lattice in which $[0]_{\psi_I}$ is the smallest element and $[1]_{\psi_I}$ is the greatest element.

Definition 17. Let I be an ideal of a lattice L. For any filter F of L, define $\bar{F} = \{[x]_{\psi_I} \mid x \in F\}$.

By the nature of congruences of lattices, it can be easily observed that $\bar{F}$ is a filter in $L_{/\psi_I}$ whenever F is a filter in L.

Definition 18. Let I be an ideal of a lattice L. For any $a \in L$, define $(a)^{\Delta} = \left\{[x]_{\psi_I} \in L_{/\psi_I} \mid [a]_{\psi_I} \vee [x]_{\psi_I} = [1]_{\psi_I}\right\}$.

Clearly $(0)^{\Delta} = \{1\}$ and $(1)^{\Delta} = L_{/\psi_I}$.

Lemma 19. *Let I be an ideal of a lattice L. For any $a \in L$,*

(1) *for each* $x \in L$, $x \in (a)^+$ *implies* $[x]_{\psi_I} \in (a)^\Delta$,

(2) $(a)^\Delta$ *is a filter of* $L_{/\psi_I}$.

Proof. Routine verification. ■

Definition 20. Let I be an ideal of a lattice L. For any filter F of $L_{/\psi_I}$, define $\sigma(F) = \{[x]_{\psi_I} \mid (x)^\Delta \vee F = L_{/\psi_I}\}$.

Lemma 21. *Let I be an ideal of a lattice L. For any filter F of $L_{/\psi_I}$,*

(1) $\sigma(F) \subseteq F$,

(2) $\sigma(F)$ *is a filter of* $L_{/\psi_I}$.

Proof. Routine verification. ■

Proposition 2.6. *Let P be a prime filter and I an ideal of a lattice L such that $P \cap I = \emptyset$. Then the following conditions hold:*

(1) $x \in P$ *if and only if* $[x]_{\psi_I} \in \bar{P}$,

(2) $\bar{P} \cap \bar{I} = \emptyset$,

(3) *If P is a prime filter of L, then $\bar{P}$ is a prime filter of $L_{/\psi_I}$,*

(4) *If P is a σ-filter of L, then $\bar{P}$ is a σ-filter of $L_{/\psi_I}$.*

Proof. (1) Clearly $x \in P$ implies $[x]_{\psi_I} \in \bar{P}$. Conversely, let $[x]_{\psi_I} \in \bar{P}$. Then $[x]_{\psi_I} = [t]_{\psi_I}$ for some $t \in P$. Hence $(x, t) \in \psi_I$. Thus $x \vee a = t \vee a \in P$ for some $a \in I$. Since $P \cap I = \emptyset$, we get $a \notin P$. Since $x \vee a \in P$ and $a \notin P$, we must have $x \in P$.

(2) Suppose $\bar{P} \cap \bar{I} \neq \emptyset$. Choose $[x]_{\psi_I} \in \bar{P} \cap \bar{I}$. By using (1), we get $x \in P$ and $[x]_{\psi_I} \in \bar{I}$. Hence

$$\begin{aligned} [x]_{\psi_I} \in \bar{I} &\Rightarrow [x]_{\psi_I} = [y]_{\psi_I} && \text{for some } y \in I \\ &\Rightarrow (x, y) \in \psi_I \\ &\Rightarrow x \vee a = y \vee a && \text{for some } a \in I \\ &\Rightarrow x \vee a \in I && \text{since } y \vee a \in I \\ &\Rightarrow x \vee a \in P \cap I && \text{since } x \in P \end{aligned}$$

which is a contradiction to $P \cap I = \emptyset$. Therefore $\bar{P} \cap \bar{I} = \emptyset$.

(3) Since P is a filter of L, it is clear that $\bar{P}$ is a filter of $L_{/\psi_I}$. Since P is a proper filter of L, by (1), we get that $\bar{P}$ is a proper filter in $L_{/\psi_I}$. Let $[x]_{\psi_I}, [y]_{\psi_I} \in L_{/\psi_I}$. Then

$$\begin{aligned} [x]_{\psi_I} \vee [y]_{\psi_I} \in \bar{P} &\Rightarrow [x \vee y]_{\psi_I} \in \bar{P} \\ &\Rightarrow x \vee y \in P && \text{from (1)} \\ &\Rightarrow x \in P \text{ or } y \in P \\ &\Rightarrow [x]_{\psi_I} \in \bar{P} \text{ or } [y]_{\psi_I} \in \bar{P}. \end{aligned}$$

Therefore $\bar{P}$ is a prime filter in $L_{/\psi_I}$.

(4) Suppose that P is a σ-filter of L. Clearly $\bar{P}$ is a filter of $L_{/\psi_I}$. Clearly $\sigma(\bar{P}) \subseteq \bar{P}$. Let $[x]_{/\psi_I} \in \bar{P}$. Then $x \in P = \sigma(P)$. Hence $(x)^+ \vee P = L$. Let $[a]_{\psi_I} \in L_{/\psi_I}$ be arbitrary. For this $a \in L$, we get $a = b \wedge c$ for some $b \in (x)^+$ and $c \in P$. Since $c \in P$, we get $[c]_{\psi_I} \in \bar{P}$. Since $b \in (x)^+$, we get $[b]_{\psi_I} \in (x)^\Delta$. Hence $[a]_{\psi_I} = [b \wedge c]_{\psi_I} = [b]_{\psi_I} \cap [c]_{\psi_I} \in (x)^\Delta \vee \bar{P}$. Hence $L_{/\psi_I} \subseteq (x)^\Delta \vee \bar{P}$. Therefore $\bar{P}$ is a σ-filter of $L_{/\psi_I}$. ■

Corollary 22. *Let P and Q be two prime filters of a lattice of L such that $P \cap I = \emptyset$ and $Q \cap I = \emptyset$. Then $P \subseteq Q$ if and only if $\bar{P} \subseteq \bar{Q}$.*

Proof. From Proposition 2.6(1), it is clear. ■

Proposition 2.7. *Let I be an ideal of a lattice L. For any prime filter R of $L_{/\psi_I}$, there exists a prime filter P of L such that $P \cap I = \emptyset$ and $\bar{P} = R$.*

Proof. Let R be a prime filter of $L_{/\psi_I}$. Consider $P = \{x \in L \mid [x]_{\psi_I} \in R\}$. Since R is a filter of $L_{/\psi_I}$, we get that P is a filter of L. Let $x, y \in L$ be such that $x \vee y \in P$. Then $[x]_{\psi_I} \vee [y]_{\psi_I} = [x \vee y]_{\psi_I} \in R$. Since R is prime, we get either $[x]_{\psi_I} \in R$ or $[y]_{\psi_I} \in R$. Hence either $x \in P$ or $y \in P$. Therefore P is a prime filter of L. Clearly $\overline{P} = R$. Suppose $P \cap I \neq \emptyset$. Choose $a \in P \cap I$. Then $[a]_{\psi_I} \in R$ and $a \in I$. Let $[y]_{\psi_I} \in L_{/\psi_I}$ be an arbitrary element. Now for any $a \in I$ and $y \in L$, we have

$$\begin{aligned} a \vee y = a \vee y \vee a &\Rightarrow (y, y \vee a) \in \psi_I \\ &\Rightarrow [y]_{\psi_I} = [y \vee a]_{\psi_I} \\ &\Rightarrow [y]_{\psi_I} = [y]_{\psi_I} \vee [a]_{\psi_I} \in R \qquad \text{since } R \text{ is a filter} \\ &\Rightarrow [y]_{\psi_I} \in R. \end{aligned}$$

Hence $L_{/\psi_I} \subseteq R$, which is a contradiction. Therefore $P \cap I = \emptyset$. ■

Theorem 23. *Let I an ideal of a lattice L. Every prime filter of L is a σ-filter if and only if every prime filter of $L_{/\psi_I}$ is a σ-filter.*

Proof. Assume that every prime filter of L is a σ-filter. By Theorem 16, every prime filter of L is minimal. Let R be a prime filter of $L_{/\psi_I}$. By Proposition 2.7, there exists a prime filter P of L such that $P \cap I = \emptyset$ and $\bar{P} = R$. By the assumption, P is minimal. Let $[x]_{\psi_I} \in \bar{P} = R$. Then $x \in P$. Since P is minimal, there exists $y \notin P$ such that $x \vee y = 1$. Hence $[y]_{\psi_I} \notin \bar{P} = R$ and $[x]_{\psi_I} \vee [y]_{\psi_I} = [x \vee y]_{\psi_I} = [1]_{\psi_I}$. Thus R is minimal in $L_{/\psi_I}$. Therefore, by Theorem 16, R is a σ-filter in $L_{/\psi_I}$.

Conversely, assume that every prime filter of $L_{/\psi_I}$ is a σ-filter. By Theorem 16, every prime filter of $L_{/\psi_I}$ is minimal. Let P be a prime filter of L. Take

$I = L - P$. Then clearly P is a prime filter of L such that $P \cap I = \emptyset$. Then by Theorem 2.6, $\bar{P}$ is a prime filter of $L_{/\psi_I}$. By our assumption, $\bar{P}$ is minimal in $L_{/\psi_I}$. Suppose P is not minimal in L. Then there exists a prime filter Q of L such that $Q \subseteq P$. Since $P \cap I = \emptyset$, we get $Q \cap I = \emptyset$. Hence $\bar{Q}$ is a prime filter of $L_{/\psi_I}$. Since $Q \subseteq P$. By Corollary 22, we get $\bar{Q} \subseteq \bar{P}$. This contradicts the minimality of $\bar{P}$ in $L_{/\psi_I}$. Hence P is minimal prime filter of L. By Theorem 16, P is a σ-filter in L. ∎

Acknowledgements

The author would like to thank the referee for his valuable suggestions and comments.

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Received 30 April 2022
Revised 13 February 2023
Accepted 13 February 2023

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