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A NOTE ON NOETHERIAN AND ARTINIAN HOOPS

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Abstract

The aim of this paper is defining the concepts of Noetherian and Artinian hoops by using the filter of hoop in the partial order set of all the filters of hoops and inclusion relation and find some equivalent definitions for this notion. We translate some important results from theory of rings to the case of hoop and their characterizations are established. The relation between short exact sequence on Noetherian and Artinian hoop studied and by using short exact sequence we prove that the Cartesian product of two hoops is Noetherian (Artinian) if and only if each one is a Noetherian (Artinian). By using the notion of filter in hoops, we define the notion of composition series and prove any $\vee$-hoop is Noetherian and Artinian if and only if it has composition series. Finally, Chinese Remainder theorem in hoop and the relation between maximal filter and Noetherian (Artinian) hoop are investigated.

Keywords: hoop, Noetherian hoop, Artinian hoop, filter, Chinese reminder, composition series.

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1. Introduction

Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information and fuzzy information. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered in a frame of residuated lattices. Hoops are naturally ordered commutative residuated integral monoids were originally introduced by Bosbach in [11, 12] under the name of complementary semigroups. Hoops have been studied by Blok and Ferreirim [5]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. In recent years, many mathematicians have studied various concepts on hoop, for example filters theory plays an important role in studying logical algebras. From logical point of view, filters correspond to sets of provable formula. The concept of filter, quotient algebra and homomorphism are all closely related to each other. In [4], Alavi and et al. introduced different kinds of filters on pseudo-hoop and investigate the relation between them and the quotient structure that is made by them. In [2], Aaly Kologani and et al. introduced the notion of co-annihilators on hoop and investigated some properties of it and in [8] studied the relation between hoops and other logical algebras. To read more about hoops, we suggest to reader the articles [1, 2, 3, 4, 7, 8, 9, 10, 16, 17, 22].

In mathematics, the adjective Noetherian is used to describe objects that satisfy an ascending or descending chain condition on certain kinds of subobjects, meaning that certain ascending or descending sequences of subobjects must have finite length. Noetherian objects are named after Emmy Noether, who was the first to study the ascending and descending chain conditions for rings. The ascending chain condition (ACC) and descending chain condition (DCC) are finiteness properties satisfied by some algebraic structures, most importantly ideals in certain commutative rings [11, 12]. These conditions played an important role in the development of the structure theory of commutative rings in the works of Hilbert, Noether, and Artin. The conditions themselves can be stated in an abstract form, so that they make sense for any partially ordered set.

The aim of this paper is defining the concepts of Noetherian and Artinian hoops by using the filter of hoop in the partial order set of all the filters of hoops and inclusion relation and find some equivalent definitions for this notion. We translate some important results from theory of rings to the case of hoop and their characterizations are established. The relation between short exact sequence on Noetherian and Artinian hoop studied and by using short exact sequence we prove that the Cartesian product of two hoops is Noetherian (Artinian) if and only if each one is a Noetherian (Artinian). By using the notion of filter in hoops, we define the notion of composition series and prove any $\vee$-hoop is Noetherian and Artinian if and only if it has composition series. Finally, Chinese Remainder theorem in hoop and the relation between maximal filter and Noetherian (Artinian) hoop are investigated.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

By a *hoop* we mean an algebraic structure $(H, \rightarrow, \odot, 1)$ of type $(2, 2, 0)$ in which $(H, \odot, 1)$ is a commutative monoid and, for any $x, y, z \in H$, the following assertions are valid.

(H1) $x \rightarrow x = 1$,

(H2) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$,

(H3) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

On hoop H we define $x \leq y$ if and only if $x \rightarrow y = 1$. Obviously $(H, \leq)$ is a poset. A bounded hoop is a hoop with the least element, it means that there exists $0 \in H$ such that $0 \leq x$, for any $x \in H$. Let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. If H is a bounded hoop, then we define a negation " $'$ " on H by, $x' = x \rightarrow 0$, for all $x \in H$. By a *sub-hoop* of a hoop H we mean a subset S of H which, for any $x, y \in S$, $x \rightarrow y \in S$ and $x \odot y \in S$ (see [8]).

Note. From now on, we let $(H, \odot, \rightarrow, 1)$ be a hoop and denote it by H, for short.

Proposition 1 [8]**.** *The following conditions hold for all $x, y, z \in H$.*

(i) $(H, \leq)$ is a $\wedge-$semilattice with $x \wedge y = x \odot (x \rightarrow y)$,

(ii) $x \odot y \leq x, y$ and $x \leq y \rightarrow x$,

(iii) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,

(iv) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$ and $x \odot z \leq y \odot z$,

(v) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$,

(vi) $x \rightarrow (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \rightarrow y_i)$.

Proposition 2 [8]**.** *Define the operation $\vee$ on H as follows,*

$$x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x).$$

Then for any $x, y \in H$ the following conditions are equivalent:

(i) $\vee$is associative,

(ii) $x \leq y$ implies $x \vee z \leq y \vee z$ for any $z \in H$,

(iii) $\vee$is the join operation on H.

Definition [8]**.** A hoop H is called a $\vee$-hoop, if it satisfies in the one of equivalent conditions of Proposition 2.

Proposition 3 [8]**.** *Let H be a $\vee$-hoop. Then the following conditions hold for any $x, y, z \in H$ and $n \in \mathbb{N}$:*

(i) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(ii) $(x \vee y)^n \to z = \bigwedge\{(x_1 \odot x_2 \odot \cdots \odot x_n) \to z | x_i \in \{x, y\}\}$.

(iii) $x \odot (\bigvee\limits_{i \in I} y_i) = \bigvee\limits_{i \in I} (x \odot y_i)$.

Definition [7]**.** A non-empty subset F of H is called a *filter* of H if for any $x, y \in F$, $x \odot y \in F$ and, for any $y \in H$ and $x \in F$, we have $x \leq y$ implies $y \in F$. The set of all filters of H is denoted by $\mathcal{F}(H)$.

Proposition 4 [7]**.** *Consider $\emptyset \neq F \subseteq H$. Then $F \in \mathcal{F}(H)$ if and only if $1 \in F$ and if $x \in F$ and $x \to y \in F$, then $y \in F$.*

Definition [2]**.** (i) $F \in \mathcal{F}(H)$ is called proper if $F \neq H$.

(ii) A proper filter P of H is called a *prime filter of* H if for all $x, y \in H$, $x \to y \in P$ or $y \to x \in P$. The set of all prime filters of H is denoted by $\mathcal{S}pec(H)$.

(iii) A proper filter M of H is called a *maximal filter of* H if it is not contained in any other proper filter. The set of all maximal filters of H is denoted by $\mathcal{M}ax(H)$.

Definition [7]**.** Let $\emptyset \neq X \subseteq H$. The intersection of all filters of H containing X is denoted by $\langle X \rangle$ and characterized by

$$\langle X \rangle = \{a \in H \mid \ x_1 \odot x_2 \odot \cdots \odot x_n \leq a \ \text{ for some } \ n \in \mathbb{N} \ \text{ and } \ x_1, ..., x_n \in X\}.$$

Let $F \in \mathcal{F}(H)$ and $x \in H \setminus F$. Then the generated filter of $F \cup \{x\}$ is denoted by $F\langle x \rangle$ and we define it as follows:

$$F\langle x \rangle = \{a \in H \mid \exists\ n \in \mathbb{N} \text{ such that } x^n \to a \in F\}.$$

Lemma 5 [2]**.** *(i) Let $(H, \to, \odot, 1)$ be a $\vee$-hoop. Then for any $x, y \in H$ we have $\langle x \vee y \rangle = \langle x \rangle \cap \langle y \rangle$.*

(ii) Let $(H, \to, \odot, 1)$ be a $\vee$-hoop and $F \in \mathcal{F}(H)$. Then

$$\langle F \cup \{x\} \rangle \cap \langle F \cup \{y\} \rangle = \langle F \cup \{x \vee y\} \rangle.$$

Proposition 6 [3]**.** *The algebraic structure $(\mathcal{F}(H), \wedge, \vee)$ is a lattice, where for any $F, G \in \mathcal{F}(H)$, $F \wedge G = F \cap G$ and $F \vee G = \langle F \cup G \rangle$.*

Proposition 7 [10]**.** *Let $F \in \mathcal{F}(H)$. Then for any $x, y \in H$ the relation $x \sim_F y$ if and only if $x \to y, y \to x \in F$ is a congruence relation on H. The set of all congruence relations on H is denoted by $\mathcal{C}on(H)$.*

Proposition 8 [10]**.** *Let $\frac{H}{F}=\{[x]|x\in H\}$, where $[x]=\{y\in H\mid x\sim_F y\}$. Define the operation $\otimes$ and $\rightsquigarrow$ on $\frac{H}{F}$ as follows:*

$$[x]\otimes[y]=[x\odot y] \text{ and } [x]\rightsquigarrow[y]=[x\rightarrow y].$$

Then $(\frac{H}{F},\otimes,\rightsquigarrow,F,\frac{H}{F})$ is a bounded hoop.

Definition [10]**.** Let H_1 and H_2 be two hoops. Then a map $\phi: H_1\rightarrow H_2$ is called a *hoop homomorphism* if, for any $x,y\in H_1$

$$\phi(x\rightarrow y)=\phi(x)\rightarrow\phi(y) \text{ and } \phi(x\odot y)=\phi(x)\odot\phi(y).$$

3. Noetherian (Artinian) hoops

In this section, we define the notion of Noetherian and Artinian hoop and give some equivalent conditions for these notions. Then we define a short exact sequence of hoop and by using it we identify Noetherian and Artinian hoops. Finally, we define composition series in hoop and investigate the relation between them and Noetherian and Artinian hoops.

Definition. A hoop H is called Noetherian (Artinian) if for every increasing (decreasing) chain of its filters like $F_1\subseteq F_2\subseteq\cdots\subseteq F_n\subseteq\cdots(F_1\supseteq F_2\supseteq\cdots\supseteq F_n\supseteq\cdots)$, there exists $n\in\mathbb{N}$ such that $F_i=F_n$, for all $i\geq n$

Example 9. (i) Every finite hoop is Noetherian (Artinian).

(ii) Let $H=[0,1]$ such that for any $x,y\in H$, $x\odot y=\min\{x,y\}$ and $x\rightarrow y=1$ if $x\leq y$ and $x\rightarrow y=y$ if $x>y$. Then $(H,\odot,\rightarrow,0,1)$ is a bounded hoop. Let $F_n=[\frac{1}{n},1]$ with $n\geq 1$. Then F_n are filters of H and $F_1\subseteq F_2\subseteq\cdots\subseteq F_n\subseteq\cdots$ does not stop. Then H is not a Noetherian hoop.

(iii) Define the operations $\odot$, $\rightarrow$ and negation on $[0,1]$ as follows:

$$x\odot y=\min\{x,y\},\ \ x'=1-x,\ \ x\rightarrow y=\min\{1,1-x+y\},$$

then $\mathcal{H}=([0,1],\odot,\rightarrow,0,1)$ is a hoop. Now, we prove $([0,1],\odot,\rightarrow,0,1)$ has only trivial filters. If $I\subseteq[0,1]$ is a filter of $\mathcal{H}$ and $I\setminus\{1\}\neq\emptyset$, then we prove $I=[0,1]$. Let $I=[u,1]$ for some $u\leq 1$. Suppose $x\in[u,1)$. If $x+u\geq 1$, then $u\rightarrow(x+u-1)=1-u+(x+u-1)=x\in I$. Thus $u+(x-1)\in I$ and this is a contradiction. Hence, for any $x\in[u,1)$, $x+u<1$ and so $u=0$. Hence $([0,1],\odot,\rightarrow,0,1)$ is an Artinian and Noetherian hoop.

(iv) Let $H=[0,1]$. Define the operations $\odot$ and $\rightarrow$ on H as follows:

$$x\rightarrow y=\begin{cases}1 & \text{if } x\leq y\\ \frac{y}{x} & \text{o.w}\end{cases}$$

Then $([0,1],\odot,\rightarrow,0,1)$ is an Artinian and Noetherian hoop.

Theorem 10. *Let A be a non-empty set of filters of H. Then H is a Noetherian (Artinian) hoop if and only if A has a maximal (minimal) element.*

Proof. Let H be a Noetherian hoop and $S = \{F_i : F_i \in \mathcal{F}(H)\}$ be a non-empty set of filters of H which does not have a maximal element. Since S is a non-empty set, there exists $F_1 \in S$. In addition, from S does not have a maximal element, there exists $F_2 \in S$ such that $F_1 \subseteq F_2$. Continuing this method, we have $F_1 \subseteq F_2 \cdots \subset F_n \subseteq \cdots$ is an increasing chain of filters of H that there does not exist $n \in \mathbb{N}$ such that $F_i = F_n$, for all $i \geq n$, which is a contradiction. Hence, S has a maximal element.

Conversely, let $F_1 \subseteq F_2 \cdots \subset F_n \subseteq \cdots$ be an increasing chain of filters of H. Then define $S = \{F_i : F_i \in \mathcal{F}(H)\}$. Since S is a non-empty set, by assumption, S has a maximal element such as F_n. Then for all $i \geq n$, $F_i = F_n$. Therefore, H is a Noetherian hoop. The proof of other case is similar. ■

Theorem 11. *Any hoop H is Noetherian if and only if every filter of H is finitely generated.*

Proof. Let H be a Noetherian hoop and $F \in \mathcal{F}(H)$ which is not finitely generated. Suppose

$$S = \{G \in \mathcal{F}(H) | G \text{ is a finitely generated filter of H and } G \subseteq F\}.$$

Since $\langle 1 \rangle = \{1\} \in S$, we get $S \neq \emptyset$. Then by Theorem 10, S has a maximal element such as F_1. Thus $F_1 \subseteq F$ and $F_1 = \langle x_1, ..., x_n \rangle$, for some $x_1, ..., x_n \in H$. Since F is not finitely generated, we have $F_1 \subsetneqq F$, and there exists $x \in F \setminus F_1$ such that $F_1 \subsetneqq \langle x_1, ..., x_n, x \rangle \subset F$. Since $\langle x_1, ..., x_n, x \rangle$ is finitely generated and $F_1 \subsetneqq \langle x_1, ..., x_n, x \rangle$, we get $\langle x_1, ..., x_n, x \rangle \in S$, which is a contradiction. Therefore, F is a finitely generated filter of H.

Conversely, suppose every filter of H is finitely generated and $F_1 \subseteq F_2 \cdots \subset F_n \subseteq \cdots$ is an increasing chain of filters of H. Let $F = F_1 \cup F_2 \cup F_3 \cup ....$ Obviously, $F \in \mathcal{F}(H)$ and by assumption, F is a finitely generated filter of H. Suppose $F = \langle x_1, ..., x_n \rangle$, for some $x_1, ..., x_n \in H$. Since $F = \bigcup_{i \in I} F_i$ and $x_1, ..., x_n \in F$, we get that there exist $i_1, ..., i_n \in \mathbb{N}$ such that $x_j \in F_{i_j}$. Now, by property of chain, there exists $m \in \mathbb{N}, 1 \leq m \leq n$ such that $x_1, ..., x_n \in F_{i_m}$. Thus $F = \langle x_1, ..., x_n \rangle \subseteq F_{i_m} \subseteq F$. Hence, $F_{i_m} = F$ for all $t \geq i_m$. Therefore, H is a Noetherian hoop. ■

Theorem 12. *Suppose every increasing chain of finitely generated filters of H stops. Then H is a Noetherian hoop.*

Proof. Assume H is not a Noetherian hoop. Then by Theorem 11, there exists $F \in \mathcal{F}(H)$ which is not finitely generated. Thus $F \neq \langle 1 \rangle = \{1\}$ and there exists

$x_1 \in F \setminus \{1\}$ such that $\langle x_1 \rangle \subsetneqq F$ and since F is not finitely generated $F \neq \langle x_1 \rangle$. Thus there exists $x_2 \in F \setminus \langle x_1 \rangle$ where $\langle x_1, x_2 \rangle \subsetneq F$. By continuing this method, we have $\langle x_1 \rangle \subsetneqq \langle x_1, x_2 \rangle \subsetneqq \cdots$ which is a proper increasing chain of finitely generated filters of H that does not stop, which is a contradiction. Therefore, H is a Noetherian hoop. ■

Lemma 13. *Let $F, G \in \mathcal{F}(H)$ such that $F \subseteq G$. Then $\frac{x}{F} \in \frac{G}{F}$ if and only if $x \in G$. In addition, $\frac{G}{F} \in \mathcal{F}(\frac{H}{F})$.*

Proof. Let $\frac{x}{F} \in \frac{G}{F}$. Then there exists $a \in G$ such that $\frac{x}{F} = \frac{a}{F}$ and so $x \to a, a \to x \in F \subseteq G$. Since $a \in G$ and $G \in \mathcal{F}(H)$, we get $x \in G$. By the similar way, the proof of other side is clear. Since $F \subseteq G$, we have $\frac{1}{F} \in \frac{G}{F}$. Let $x, y \in H$ such that $\frac{x}{F}, \frac{x}{F} \to \frac{y}{F} \in \frac{G}{F}$. Then $x, x \to y \in G$. Since $G \in \mathcal{F}(H)$, we get $y \in G$. Hence, $\frac{y}{F} \in G$. ■

Theorem 14. *Let $F \in \mathcal{F}(H)$. Then $\frac{H}{F}$ is a Noetherian (Artinian) hoop if and only if H is a Noetherian (Artinian) hoop.*

Proof. Let H be a Noetherian (Artinian) hoop and $\frac{F_1}{F} \subseteq \frac{F_2}{F} \subseteq \cdots \frac{F_n}{F} \subseteq \cdots$ be an increasing chain of filters of $\frac{H}{F}$. Then $F \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ is an increasing chain of filters of H. Since H is a Noetherian hoop, there exists $n \in \mathbb{N}$ such that for all $i \geq n$, $F_i = F_n$. Then for all $i \geq n$, $\frac{F_i}{F} = \frac{F_n}{F}$. Therefore, $\frac{H}{F}$ is a Noetherian hoop.

Conversely, let $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ be an increasing chain of filters of H. If $F_1 = \{1\}$, since $\frac{F_i}{\{1\}} \cong F_i$ then the proof is clear. Let $F_1 \neq \{1\}$. Since $F_1 \subseteq F_i$ for any $2 \leq i \leq n$, by Lemma 13, $\frac{F_1}{F_1} \subseteq \frac{F_2}{F_1} \subseteq \cdots \subseteq \frac{F_n}{F_1} \cdots$ is an increasing chain of filters of $\frac{H}{F_1}$. Since $\frac{H}{F_1}$ is a Noetherian hoop, there exists $n \in \mathbb{N}$ such that for $i \geq n$, $\frac{F_i}{F_1} = \frac{F_n}{F_1}$. Hence for any $x \in F_i$, $\frac{x}{F_1} \in \frac{F_i}{F_1} = \frac{F_n}{F_i}$ we have $x \in \frac{F_n}{F_i}$ by Lemma 13, $x \in F_n$ so $F_i \subseteq F_n$ by the similar way $F_n \subseteq F_i$ thus for all $i \geq n$, $F_i = F_n$. Therefore, H is a Noetherian hoop.

The proof of other case is similar. ■

Proposition 15. *Let S be a sub-hoop of H. Then the set of all filters of S is $\mathcal{F}(S) = \{F \cap S | F \in \mathcal{F}(H)\}$.*

Proof. Let S be a sub-hoop of H and K be a filter of S. Clearly $K \subseteq \langle K \rangle \cap S$. Let $x \in \langle K \rangle \cap S$. Since $x \in \langle K \rangle$, by Definition 2, there exist $x_1, x_2, \cdots, x_n \in K$ and $n \in \mathbb{N}$ such that $x_1 \odot x_2 \odot \cdots \odot x_n \leq x$. Since K is a filter of S, we get $x_1 \odot x_2 \odot \cdots \odot x_n \in K$ and so $x \in K$. Thus $x \in K \cap S = K$. Hence $K = \langle K \rangle \cap S$. Therefore, $\mathcal{F}(S) = \{F \cap S | F \in \mathcal{F}(H)\}$. ■

Corollary 16. *Any sub-hoop of Noetherian (Artinian) hoop H is Noetherian (Artinian).*

Definition. Let H_1, H_2 and H_3 be hoops. A sequence $1 \longrightarrow H_1 \xrightarrow{\phi} H_2 \xrightarrow{\psi} H_3 \longrightarrow 1$ is called a *short exact sequence of hoops* if ϕ is one-to-one, ψ is onto and $ker(\psi) = Im(\phi)$.

Example 17. Let $H_1 = \{0, a, b, c, d, 1\}$ and $H_2 = \{0, 1\}$ be two sets such that $0 \leq a \leq c \leq 1$, $0 \leq b \leq d \leq 1$ and $0 \leq b \leq c \leq 1$. Then the Cayley tables are as follows:

$\rightarrow_{H_1}$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot_{H_1}$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow_{H_2}$	0	1
0	1	1
1	0	1

$\odot_{H_2}$	0	1
0	0	0
1	0	1

Then $(H_1, \rightarrow_{H_1}, \odot_{H_1}, 1_{H_1})$ and $(H_2, \rightarrow_{H_2}, \odot_{H_2}, 1_{H_2})$ are hoops. By routine calculations, we get $F = \{a, c, 1\}$ is a filter of H_1. Define a map $\psi : H_1 \rightarrow H_2$ by $\psi(0) = \psi(b) = \psi(d) = 0$ and $\psi(1) = \psi(c) = \psi(a) = 1$. Easily we can check ψ is a hoop homomorphism. Thus a sequence $1 \longrightarrow F \xrightarrow{\phi} H_1 \xrightarrow{\psi} H_2 \longrightarrow 1$ is a short exact sequence of hoops, where ϕ is an identity map.

Proposition 18. *Let $\phi : H_1 \rightarrow H_2$ be a hoop homomorphism such that $F \in \mathcal{F}(H_1)$ and $G \in \mathcal{F}(H_2)$. Then the following statements hold:*

(i) If ϕ is a surjective hoop homomorphism such that $ker(\phi) \subseteq F$, then $\phi(F) \in \mathcal{F}(H_2)$.

(ii) $\phi^{-1}(G) \in \mathcal{F}(H_1)$.

(iii) $ker(\phi) = \{x \in H_1 | \phi(x) = 1\} \in \mathcal{F}(H_1)$.

Proof. (i) Obviously, $1 = \phi(1) \in \phi(F)$. Let $x, y \in \phi(F)$. Then there exist $a, b \in F$ such that $\phi(a) = x$ and $\phi(b) = y$. Since $F \in \mathcal{F}(H_1)$, clearly $a \odot b \in F$, and so $x \odot y = \phi(a) \odot \phi(b) = \phi(a \odot b) \in \phi(F)$. Let $x, y \in H_2$ such that $x \leq y$ and $x \in \phi(F)$. Thus there is $a \in F$ such that $\phi(a) = x$ and since ϕ is surjective, there exists $b \in H_1$ such that $\phi(b) = y$. Since $x \leq y$, we have $\phi(a) \leq \phi(b)$ and so $\phi(a \rightarrow b) = \phi(a) \rightarrow \phi(b) = 1$. Thus $a \rightarrow b \in ker\phi \subseteq F$. From $F \in \mathcal{F}(H_1)$ and $a \in F$, we get $b \in F$ and so $y = \phi(b) \in \phi(F)$. Therefore, $\phi(F) \in \mathcal{F}(H_2)$.

(ii) Obviously, $1 \in \phi^{-1}(G)$. Let $x, x \rightarrow y \in \phi^{-1}(G)$. Then $\phi(x), \phi(x) \rightarrow \phi(y) \in G$. Since $G \in \mathcal{F}(H_2)$ and $\phi(x) \in G$, we have $\phi(y) \in G$, and so $y \in \phi^{-1}(G)$. Therefore, $\phi^{-1}(G) \in \mathcal{F}(H_1)$.

(iii) Clearly $\phi(1) = 1$, thus $1 \in ker(\phi)$. Let $x, x \to y \in ker(\phi)$. Then $\phi(x) = 1$ and $\phi(x \to y) = \phi(x) \to \phi(y) = 1$. Thus $\phi(x) \leq \phi(y)$ and $\phi(x) = 1$. Hence $\phi(y) = 1$ and $y \in ker(\phi)$. Therefore, $ker(\phi) \in \mathcal{F}(H_1)$. ∎

Theorem 19. *Let $1 \longrightarrow H_1 \xrightarrow{\phi} H_2 \xrightarrow{\psi} H_3 \longrightarrow 1$ be a short exact sequence of hoops. Then H_1 and H_3 are Noetherian hoops if and only if H_2 is a Noetherian hoop.*

Proof. ($\Rightarrow$) Let $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ be an increasing chain of filters of H_2. Since ψ is a surjective hoop homomorphism and $\ker \phi \subseteq Im\psi$, we have $\psi(F_1) \subseteq \psi(F_2) \subseteq \cdots \subseteq \psi(F_n) \subseteq \cdots$ is an increasing chain of filters of H_3 and $\phi^{-1}(F_1) \subseteq \phi^{-1}(F_2) \subseteq \cdots \subseteq \phi^{-1}(F_n) \subseteq \cdots$ is an increasing chain of filters of H_1. Since H_1 and H_3 are Noetherian hoops, there exist $m, k \in \mathbb{N}$ such that $\psi(F_i) = \psi(F_m)$ and $\phi^{-1}(F_j) = \phi^{-1}(F_k)$ for all $i \geq m$ and $j \geq k$. Let $l = \max\{m, k\}$. Clearly, for all $i \geq l$, we have $F_l \subseteq F_i$. It is enough to prove $F_i \subseteq F_l$ for all $i \geq l$. Let $x \in F_i$ for $i \geq l$. Then $\psi(x) \in \psi(F_i) = \psi(F_l)$, thus there exists $a \in F_l$ such that $\psi(x) = \psi(a)$. It follows that $\psi(a \to x) = \psi(a) \to \psi(x) = 1$, that is $a \to x \in ker(\psi) = Im(\phi)$. Hence there exists $b \in H_1$ such that $a \to x = \phi(b)$. Moreover, since F_i is a filter of H_2, $x \in F_i$ and $x \leq a \to x$, we get $a \to x \in F_i$. Then $\phi(b) \in F_i$ implies $b \in \phi^{-1}(F_i) = \phi^{-1}(F_l)$ and so $\phi(b) \in F_l$. Hence, $a \to x \in F_l$. Now, since $a \in F_l$ and F_l is a filter of H_2, we get $x \in F_l$. Then $F_i \subseteq F_l$, and so $F_i = F_l$ for all $i \geq l$. Therefore, H_2 is Noetherian.

($\Leftarrow$) Let H_2 be a Noetherian hoop. Then by first isomorphism theorem, we have $\frac{H_2}{ker(\psi)} \cong H_3$. Thus by Theorem 14, H_3 is a Noetherian hoop. Since ϕ is a hoop homomorphism, $H_1 \cong \phi(H_1)$ and $\phi(H_1)$ is a subalgebra of H_2, by Corollary 16, we get H_1 is a Noetherian hoop. ∎

Corollary 20. *Let $F \in \mathcal{F}(H)$ and S be a sub-hoop of H such that $F \subseteq S$. Then F and $\frac{S}{F}$ are Noetherian (Artinian) if and only if S is Noetherian (Artinian) hoop.*

Proof. Since $1 \longrightarrow F \xrightarrow{i} S \xrightarrow{\psi} \frac{S}{F} \longrightarrow 1$ is a short exact sequence of sub-hoops where i is identity and ψ is a natural homomorphism, by Theorem 19 the proof is clear. ∎

Proposition 21. *Let H be a Noetherian hoop and $\pi : H \to H$ be an onto homomorphism. Then π is one-to-one homomorphism.*

Proof. Let $x \in ker(\pi)$. Since $ker(\pi) \in \mathcal{F}(H)$, and the composition of homomorphism is a homomorphism we can see that $ker(\pi^n)$ is filter. Let $x \in ker(\pi^i)$ for any $1 \leq i \leq n$. Then $\pi^i(x) = 1$ and so $\pi(\pi^i(x)) = 1$. Thus $x \in ker(\pi^{i+1})$. Hence, $ker(\pi^i) \subseteq ker(\pi^{i+1})$. Suppose $ker(\pi) \subseteq ker(\pi^2) \subseteq \cdots \subseteq ker(\pi^n) \cdots$ be an increasing chain of filters of H. Since H is Noetherian and $ker(\pi^i) \in \mathcal{F}(H)$, there exists $n \in \mathbb{N}$ such that $ker(\pi^i) = ker(\pi^n)$, for all $i \geq n$. Let $x \in ker(\pi)$. Since π^n is onto, there exists $y \in H$ such that $x = \pi^n(y)$. Then $\pi(x) = \pi^{n+1}(y) = 1$ and so $y \in ker(\pi^{n+1}) = ker(\pi^n)$. Hence $x = \pi^n(y) = 1$. Therefore, $ker(\pi) = \{1\}$ and π is a one-to-one hoop homomorphism. ■

Proposition 22. *Let $\phi : H_1 \to H_2$ be a surjective homomorphism. If H_1 is Noetherian (Artinian), then H_2 is, too.*

Proof. Let $G \in \mathcal{F}(H_2)$. Then by Theorem 11, it is enough to show that G is a finitely generated filter of H_2. By Proposition 18, $F = \phi^{-1}(G) \in \mathcal{F}(H_1)$. Since H_1 is a Noetherian hoop, we get F is finitely generated. Suppose that there exist $x_1, x_2, \cdots x_n \in H_1$ such that $F = \langle x_1, x_2, \cdots, x_n \rangle$. Now, we prove $G = \langle \phi(x_1), \phi(x_2), \cdots, \phi(x_n) \rangle$. For this, let

$B = \{y \in H_2 |$ There exist $x_1, \cdots, x_n \in F$ such that $\phi(x_1) \odot \phi(x_2), \cdots \odot \phi(x_n) \leq y\}$,

and $y \in B$. Then $\phi(x_1) \odot \phi(x_2), \cdots \odot \phi(x_n) \leq y$. Since $x_1, x_2, \cdots, x_n \in F$ and $F \in \mathcal{F}(H_1)$, we get $x_1 \odot x_2, \cdots \odot x_n \in F$. Then $\phi(x_1 \odot x_2, \cdots \odot x_n) \in G$. Since ϕ is a hoop homomorphism, we have

$$\phi(x_1 \odot x_2 \odot \cdots \odot x_n) = \phi(x_1) \odot \phi(x_2) \odot \cdots \odot \phi(x_n) \leq y.$$

Moreover, from $G \in \mathcal{F}(H_2)$, we get $y \in G$ and so $B \subseteq G$.

Conversely, let $a \in G$. Since preimage of any filter of H_2 is a filter of H_1, we have $\phi^{-1}(a) \in F$. Moreover, since $F \in \mathcal{F}(H_1)$ and F is finitely generated, there exist $x_1, x_2, \cdots, x_n \in F$ such that $x_1 \odot x_2, \cdots \odot x_n \leq \phi^{-1}(a)$. Thus

$$\phi(x_1 \odot x_2 \odot \cdots \odot x_n) \leq a \quad , \quad \phi(x_1) \odot \phi(x_2) \odot \cdots \odot \phi(x_n) \leq a$$

Hence $a \in B$, and so

$$G = \{y \in H_2 | \phi(x_1) \odot \phi(x_2) \odot \cdots \odot \phi(x_n) \leq y\}.$$

Therefore, G is finitely generated. ■

Theorem 23. *Let $F, G \in \mathcal{F}(H_1)$ and $\phi : H_1 \to H_2$ be a hoop homomorphism such that $ker(\phi) \subseteq G$. If $\langle \phi(F) \rangle = \langle \phi(G) \rangle$, then $F = G$.*

Proof. Suppose $F, G \in \mathcal{F}(H_1)$ and $\langle \phi(F) \rangle = \langle \phi(G) \rangle$. If $x \in F$, then $\phi(x) \in \langle \phi(F) \rangle = \langle \phi(G) \rangle$. By Definition 2, there exist $n \in \mathbb{N}$ and $x_1, \cdots x_n \in G$ such that $\phi(x_1) \odot \phi(x_2) \odot \cdots \odot \phi(x_n) \leq \phi(x)$. Then $(\phi(x_1) \odot \phi(x_2) \odot \cdots \odot \phi(x_n)) \to \phi(x) = 1$. Since ϕ is a hoop homomorphism, we have $\phi((x_1 \odot x_2 \odot \cdots \odot x_n) \to x) = 1$, and so

$$(x_1 \odot x_2 \odot \cdots \odot x_n) \to x \in ker(\phi)$$

Since $ker(\phi) \subseteq G$, we get $(x_1 \odot x_2 \odot \cdots \odot x_n) \to x \in G$. In addition, since for $n \in \mathbb{N}$, we have $x_1, \cdots x_n \in G$ and $G \in \mathcal{F}(H_1)$, then $x \in G$ and so $F \subseteq G$. By the similar way, we can prove $G \subseteq F$. Therefore, $F = G$ ■

Definition. If $(H_1, \odot_{H_1}, \to_{H_1}, 1)$ and $(H_2, \odot_{H_2}, \to_{H_2}, 1)$ are hoops, then $(H_1 \times H_2, \otimes, \rightsquigarrow, 1_{H_1\times H_2})$ is called a *Cartesian product of hoops*, where;

$$(x,z)\otimes(y,w) = (x \odot_{H_1} y, z \odot_{H_2} w) \text{ and } (x,z) \rightsquigarrow (y,w) = (x \to_{H_1} y, z \to_{H_2} w).$$

for any $(x,z),(y,w) \in H_1 \times H_2$.

Proposition 24. *Let H_2 and H_1 be two hoops. Then $K \in \mathcal{F}(H_1 \times H_2)$ if and only if there exist $F \in \mathcal{F}(H_1)$ and $G \in \mathcal{F}(H_2)$ such that $K = F \times G$.*

Proof. Let $K \in \mathcal{F}(H_1 \times H_2)$ such that $K = F \times G$, where $F = \{x \in H_1 | (x,z) \in K,\ \text{for some } z \in H_2\}$ and $G = \{w \in H_2 | (y,w) \in K,\ \text{for some } y \in H_1\}$. Suppose $x, y \in F$. Then there exist $z, w \in H_2$ such that $(x,z),(y,w) \in K$. Since $K \in \mathcal{F}(H_1 \times H_2)$, we have $(x \odot y, z \odot w) = (x,z) \odot (z,w) \in K$, and so $x \odot y \in F$. Now suppose $x \leq y$ and $x \in F$. Then there exists $z \in H_2$ such that $(x,z) \in K$. Since $(x,z) \leq (y,z)$ and $K \in \mathcal{F}(H_1 \times H_2)$, we get $(y,z) \in K$, and so $y \in F$. Hence, $F \in \mathcal{F}(H_1)$. By a similar way, we can prove that $G \in \mathcal{F}(H_2)$. ■

Theorem 25. *The hoops H_1 and H_2 are Noetherian (Artinian) if and only if $H_1 \times H_2$ is a Noetherian (Artinian) hoop.*

Proof. Let $1 \longrightarrow H_1 \xrightarrow{\phi} H_1 \times H_2 \xrightarrow{\psi} H_2 \longrightarrow 1$ be a short sequence of hoops. It is clear that ϕ is one-to-one and ψ is surjective. Then this sequence is a short exact sequence of hoops and by Theorem 19, the proof is clear. ■

Lemma 26. *If H is a $\vee$-hoop such that for any $x, y \in H$, $(x \to y) \vee (y \to x) = 1$, then $P \in Spec(H)$ if and only if $x \in P$ or $y \in P$.*

Proof. Consider P is a prime filter of H and $x \vee y \in P$ such that $x \notin P$ and $y \notin P$. Since P is prime, we have $x \to y \in P$ or $y \to x \in P$. Suppose $x \to y \in P$. By Proposition 2, $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ and so $((x \to y) \to y) \wedge ((y \to x) \to x) \leq (x \to y) \to y$. From $P \in \mathcal{F}(H)$ and $x \vee y \leq (x \to y) \to y$, we get $(x \to y) \to y \in P$. As $P \in \mathcal{F}(H)$ and $x \to y \in P$, we obtain $y \in P$, which is a contradiction.
Conversely, since $(x \to y) \vee (y \to x) = 1 \in P$ for any $x, y \in H$, by (i) the proof is clear. ■

Note. Let H be a $\vee$-hoop. Then a subset $S \subseteq H$ is a $\vee$-closed subset if $x \vee y \in S$ for any $x, y \in S$.

Proposition 27. *Let H be a $\vee$-hoop. If F is a proper filter of H and S is a $\vee$-closed subset of H such that $S \cap F = \emptyset$, then F is contained in a prime filter P of H such that $S \cap P = \emptyset$, and $F \subseteq P$.*

Proof. Let $\Gamma = \{G \in \mathcal{F}(H) | F \subseteq G, G \cap S = \emptyset\}$. Since $F \in \Gamma$, we get $\Gamma \neq \emptyset$. Consider $\{G_i\}_{i\in I}$ is a family of filters of H such that $G_i \in \Gamma$ for any $i \in I$. By Zorn's Lemma $(\Gamma, \subseteq)$ has a maximal element such as $P = \bigcup_{i\in I} G_i$. Now, we prove P is a prime filter of H. Clearly P is a proper filter of H. Suppose $x \vee y \in P$ such that $x \notin P$ and $y \notin P$. Since $F \subseteq \langle P \cup \{x\}\rangle$, $F \subseteq \langle P \cup \{y\}\rangle$, and P is a maximal element of Γ, we get $\langle P \cup \{x\}\rangle \notin \Gamma$ and $\langle P \cup \{y\}\rangle \notin \Gamma$. Thus $\langle P \cup \{x\}\rangle \cap S \neq \emptyset$ and $\langle P \cup \{y\}\rangle \cap S \neq \emptyset$. So there exist $a \in \langle P \cup \{x\}\rangle \cap S$ and $b \in \langle P \cup \{y\}\rangle \cap S$. Since S is $\vee$-close, we have $a \vee b \in S$. Also, by Lemma 5(ii) we have $a \vee b \in \langle P \cup \{x\}\rangle \cap \langle P \cup \{y\}\rangle = \langle P \cup \{x \vee y\}\rangle = P$. Hence, $P \cap S \neq \emptyset$, which is a contradiction. Thus $x \in P$ or $y \in P$. If $x \in P$, then since for any $y \in H$, we have $x \leq y \to x$, we obtain $y \to x \in P$. Hence by Lemma 26, P is a prime filter of H. ■

Corollary 28. *Let H be a $\vee$-hoop. Then*

(i) If F is a filter of a $\vee$-hoop H and $x \in H \setminus F$, then there exists a prime filter P of H such that $F \subseteq P$ and $x \notin P$.

(ii) Every proper filter of hoop H can be extend to a maximal filter of hoop H.

Proof. (i) Clearly $S = \{x\}$ is a $\vee$-closed subset of H. Thus by Proposition 27, the proof is completed.

(ii) Let F be a proper filter of H. Then there exists $x \in H \setminus F$ and by (i), F contained in a prime filter P such that $x \notin P$. Suppose

$$X = \{G | P \subseteq G, \text{G is a proper filter of H}\}.$$

By Zorn's Lemma $(\Gamma, \subseteq)$ has a maximal element such as $M = \bigcup\{G | G \in X\}$. Obviously, by (i), M is a maximal filter of H. ■

Proposition 29. *Let H be a $\vee$-hoop. Every proper filter F of H is intersection of all prime filters including F.*

Proof. Let F be a proper filter of H and $\{P_i\}_{i\in I}$ be the set of all prime filters of H such that for any $i \in I$, $F \subseteq P_i$. So $F \subseteq \bigcap_{i\in I} P_i$. Suppose $x \in \bigcap_{i\in I} P_i$ and $x \notin F$. Then by Corollary 28, there exists a prime filter of H such as P_j such that $F \subseteq P_j$ and $x \notin P_j$. Moreover, since $x \in \bigcap_{i\in I} P_i \subseteq P_j$, we get $x \in P_j$ which is a contradiction. Hence, every proper filter F of H is intersection of all prime filters including F. ■

Proposition 30. *Let H be a $\vee$-hoop. Then $\mathcal{M}ax(H) \subseteq \mathcal{S}pec(H)$.*

Proof. Let $M \in \mathcal{M}ax(H)$. Then M is a proper filter of H. By Proposition 29, there exists a prime filter P of H such that $M \subseteq P$. Since M is a maximal filter and $P \in \mathcal{S}pec(H)$, we get $M = P$. Hence $M \in \mathcal{S}pec(H)$. Therefore, $\mathcal{M}ax(H) \subseteq \mathcal{S}pec(H)$. ■

Lemma 31. *Let H be a $\vee$-hoop and $I, J \in \mathcal{F}(H)$ such that $I \cap J \subseteq P$, where $P \in \mathcal{S}pec(H)$. Then $I \subseteq P$ or $J \subseteq J$.*

Proof. Let $P \in \mathrm{Spec}(H)$ such that for $I, J \in \mathcal{F}(H)$, we have $I \cap J \subseteq P$. If $I \nsubseteq P$ and $J \nsubseteq P$, then there exist $x \in I \setminus P$ and $y \in J \setminus P$. Since $I, J \in \mathcal{F}(H)$, we have $x \vee y \in I \cap J \subseteq P$. In addition, $P \in \mathrm{Spec}(H)$, and so $x \in P$ or $y \in P$, which is a contradiction. Hence, $I \subseteq P$ or $J \subseteq J$. ■

Theorem 32. *Let H be an Artinian $\vee$-hoop. Then $\mathcal{M}ax(H)$ is a finite set.*

Proof. Let

$$S = \{F \in \mathcal{F}(H) \mid F \text{ is an intersection of finitely many maximal filters of } H\}.$$

If $\mathcal{M}ax(H)$ is an empty set, then $\mathcal{M}ax(H)$ is finite and the proof is clear. If $\mathcal{M}ax(H)$ is a non-empty set, then there exists a maximal filter of H such as M such that $M \in S$, and so S is a non-empty set. Thus, by Theorem 10, we get S has a minimal element. Suppose G is a minimal element of S. Then there exist $M_1, M_2 \cdots M_n \in \mathcal{M}ax(H)$ such that $G = M_1 \cap M_2 \cap \cdots \cap M_n$. Now, let $M \in \mathcal{M}ax(H)$. Then $M \cap G \subseteq G$ and so $M \cap G = M \cap M_1 \cap M_1 \cap \cdots \cap M_n \in S$. Since G is a minimal element of S and $M \cap G \subseteq G$, we get $M \cap G = G$. Thus $G = M_1 \cap M_2 \cap \cdots \cap M_n \subseteq M$. Since $M \in \mathcal{M}ax(H)$, by Proposition 30, we get $M \in \mathcal{S}pec(H)$ and by Lemma 31, there exists $i \in \mathbb{N}$, such that $M_i \subseteq M$. Since $M, M_i \in \mathcal{M}ax(H)$, we obtain $M = M_i$. Hence $\mathcal{M}ax(H) = \{M_1, M_2 \cdots M_n\}$ and it is a finite set. ■

In the following example, we show that every filter of Noetherian hoop H is not an intersection of finitely number of prime filters of H.

Example 33. Let $H = \{0, a, b, c, 1\}$ be a set. Define the operations $\rightarrow$ and $\odot$ on H as follow:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	0	0	1
b	c	0	1	0	1
c	c	0	0	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

Then $(H, \odot, \rightarrow, 1)$ is a hoop. By a routine calculate the set of all filters and primes filters of H are:

$$\mathcal{F}(H) = \{\{1\}, \{a, 1\}, \{b, 1\}, \{0, a, b, c, 1\}\} \text{ and } \mathcal{S}pec(H) = \emptyset.$$

Theorem 34. *Let H be a Noetherian $\vee$-hoop such that for any $x, y \in H$, $(x \to y) \vee (y \to x) = 1$. Then every filter of H is an intersection of finitely number of prime filters of H.*

Proof. Let

$$S = \{G \in \mathcal{F}(H) \mid G \text{ is not an intersection of finitely number of prime filters of } H\}.$$

If S is a non-empty set, since H is a Noetherian $\vee$-hoop, then by Theorem 10, S has a maximal element G. According to definition of set S, clearly G is not a prime filter of H. Thus there exist $x, y \in H$ such that $x \to y \notin G$ and $y \to x \notin G$. So $G \subsetneqq \langle G \cup \{x \to y\}\rangle$ and $G \subsetneqq \langle G \cup \{y \to x\}\rangle$. Since G is a maximal element of S, $\langle G \cup \{x \to y\}\rangle \notin S$ and $\langle G \cup \{y \to x\}\rangle \notin S$. Now, there exist $P_1, P_2, \cdots, P_n, P'_1, P'_2, \cdots, P'_m \in \mathcal{S}pec(H)$ such that

$$\langle G \cup \{x \to y\}\rangle = P_1 \cap P_2 \cap \cdots \cap P_n \quad , \quad \langle G \cup \{y \to x\}\rangle = P'_1 \cap P'_2 \cap \cdots \cap P'_n$$

By Remark 5,

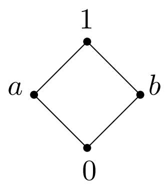
$$G = \langle G \cup \{x \to y\}\rangle \cap \langle G \cup \{y \to x\}\rangle = P_1 \cap P_2 \cap \cdots \cap P_n \cap P'_1 \cap P'_2 \cap \cdots \cap P'_m$$

which is a contradiction. Hence S is an empty set. Therefore, every filter of H is an intersection of finitely number of prime filters of H. ∎

Definition. Let $(A, \leq)$ be an order set and $B, C \in \mathcal{P}(A)$ where $\mathcal{P}(A)$ is the power set of A. Then B *is covered by* C if $B \subseteq C$ and there is no $D \subseteq A$ such that $B \subseteq D \subseteq C$.

Similarly we can define covered elements if sets are singletone.

Example 35. Let $H = \{0, a, b, 1\}$ be a set such that $0 \leq a, b \leq 1$ with the following Hasse diagram.



$\to$	0	a	b	c
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
c	0	a	b	1

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	1

According to Definition 3 clearly, 0 covered by a and b.

Definition. Let $F \in \mathcal{F}(H)$. Then an increasing sequence of filters $\{F_i | i = 1, 2, \cdots, n\}$ of H such that $\{1\} = F_1 \subseteq F_2 \subseteq \cdots F_{n-1} \subseteq F_n = F$ is called an *F-chain* of H.

Example 36. Let H be the hoop as in Example 35. Consider $F_1 = \{1\}$ and $F_2 = \{a, 1\}$. Then it is clear that the sequence $\{F_i | i = 1, 2\}$ is an F-*chain* of H.

Theorem 37. *Let $F, G \in \mathcal{F}(H)$ such that $F \subseteq G$. Then the followings statements are equivalent:*

(i) F is covered by G,

(ii) $\langle F \cup \{x\} \rangle = G$ for all $x \in G \setminus F$,

(iii) $\langle \frac{x}{F} \rangle = \frac{G}{F}$ for all $x \in G \setminus F$.

Proof. $(i) \Rightarrow (ii)$ Let $x \in G \setminus F$ and F covered by G. Since $F \subseteq \langle F \cup \{x\} \rangle \subseteq G$ by Definition 3, we get $\langle F \cup \{x\} \rangle = G$.

$(ii) \Rightarrow (iii)$ Let $\frac{a}{F} \in \frac{G}{F}$. Then by Lemma 13, we have $a \in G$. Since by (ii), $\langle F \cup \{x\} \rangle = G$, by Definition 2, there exist $u \in F$ and $n \in \mathbb{N}$ such that $(u \odot x^n) \to a \in F$. Since $u \in F$, we get $x^n \to a \in F$, and so $\frac{G}{F} \subseteq \langle \frac{x}{F} \rangle$. By the similar way, $\langle \frac{x}{F} \rangle \subseteq \frac{G}{F}$. Hence, $\langle \frac{x}{F} \rangle = \frac{G}{F}$.

$(iii) \Rightarrow (i)$ Let $F \subseteq K \subseteq G$, for $K \in \mathcal{F}(H)$. If $F \neq K$, then there exists $x \in K \setminus F$. Since $K \subseteq G$ and $x \in K \setminus F$, we get $x \in G \setminus F$. Then by assumption $\langle \frac{x}{F} \rangle = \frac{G}{F}$. Let $a \in G$. By Definition 2, $\frac{x^n}{F} \to \frac{a}{F} = \frac{1}{F}$, for some $n \in \mathbb{N}$. It follows that $x^n \to a \in F \subseteq K$. Thus from $x \in K$, we conclude $a \in K$. Therefore, $K = G$ and so F is covered by G. ∎

Definition. An F-chain $\{F_i | i = 1, 2, \cdots, n\}$ is called a *composition series for F* if for any $0 \leq i \leq n - 1$, F_i is covered by F_{i+1} in ordered set $(\mathcal{F}(H), \subseteq)$. The smallest length of a composition series for F is denoted by $le(F)$. We denoted $le(F) = \infty$ if F has no composition series.

Example 38. Let H be the hoop as in Example 35. Suppose an F-chain $F = \{F_i | 1 \leq i \leq 3\}$ such that $F_1 = \{1\}, F_2 = \{a, 1\}$ and $F_3 = \{0, a, b, 1\}$. Clearly F is a composition series for F_3.

Theorem 39. *Let $F, G \in \mathcal{F}(H)$ such that $F \subset G$ and G has a composition series. Then $le(F) < le(G)$.*

Proof. Let $le(G) = n$. Then there is a composition series $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ for G. Thus $\{1\} = G_0 \cap F \subseteq G_1 \cap F \subseteq \cdots \subseteq G_n \cap F = F$. Consider $x \in (G_{i+1} \cap F) \setminus (G_i \cap F)$ for $0 \leq i \leq n$. If $x \in G_i$, then since $x \in G_{i+1} \cap F$, we have $x \in G_i \cap F$, which is a contradiction. Hence, $x \notin G_i$. Then by Theorem 37, $\langle G_i \cup \{x\} \rangle = G_{i+1}$. Let $z \in G_i \cap F$. Then $z \in \langle G_i \cup \{x\} \rangle$ and by Definition 2, there exist $n \in \mathbb{N}$ such that $x^n \to z \in G_i$. Since $z \in F$, by Proposition 1(vi), $x^n \to z \in F \cap G_i$. Hence, $z \in \langle (G_i \cap F) \cup \{x\} \rangle$ and $\langle (G_i \cap F) \cup \{x\} \rangle = G_{i+1} \cap F$. Now, by Theorem 37, $G_i \cap F$ is covered by $G_{i+1} \cap F$. By repeating this method, the sequence $\{1\} = G_0 \cap F \subseteq G_1 \cap F \subseteq \cdots \subseteq G_n \cap F = F$, is a composition series for F. Hence $le(F) \leq le(G)$. Now, suppose $le(F) = le(G)$. A chain

$\{1\} = G_0 \cap F \subseteq G_1 \cap F \subseteq \cdots \subseteq G_n \cap F = F$ is a composition series by length n for F. By assumption, $F \subset G$, and so

$$\{1\} = G_0 \cap F \subseteq G_1 \cap F \subseteq \cdots \subseteq G_n \cap F = F \subset G$$

is a composition series for G, where $le(G) = n + 1$, which is a contradiction. ■

Theorem 40. *Let $F \in \mathcal{F}(H)$ such that $le(F) = n$, for some $n \in \mathbb{N}$. Then the length of any composition series for F is n.*

Proof. Let $\{1\} = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = F$ be a composition series for F. Since $le(F) = n$, by Definition 3, we get $n \leq m$. Thus by Theorem 39, $0 = le(F_0) < le(F_1) < \cdots < le(F_{m-1}) < le(F) = n$. By adding only one unit to each $le(F_i)$, $1 \leq i \leq n$, we get $le(F)$ at least is m. Hence $m \leq n$ and the length of every composition series for F is n. ■

Theorem 41. *Let H be a $\vee$-hoop. Then H is a Noetherian and Artinian $\vee$-hoop if and only if $le(H)$ is finite.*

Proof. Let H be a $\vee$-hoop. If H is a finite hoop, then the proof is clear. Suppose H is an infinite Noetherian and Artinian $\vee$-hoop. If $\{1\}$ is a maximal filter of H, then $\{1\} \subseteq H$ is a composition series for H and $le(H)$ is finite. Suppose $\{1\}$ is not a maximal filter of H. By Theorem 25, $\mathcal{M}ax(H)$ is a finite set. Let $\mathcal{M}ax(H) = \{M_1, M_2, \cdots, M_n\}$. Assume $M_i \in \mathcal{M}ax(H)$ has a composition series. Let $\{1\} = F_0 \subset F_1 \subset \cdots \subset F_j = M_i$ be a composition series for M_i. Since M_i is a maximal filter of H, we get $\{1\} = F_0 \subset F_1 \subset \cdots \subset F_j = M_i \subset H$ is a composition series for H. Thus $le(H)$ is finite. In the other case, suppose for any $1 \leq i \leq n$, $le(M_i) = \infty$. Consider the set $\mathcal{V} = \{F \in \mathcal{F}(H) | le(F) = \infty\}$. Clearly, since $M_i \in \mathcal{V}$, we get $\mathcal{V}$ is a non-empty set. Since H is an Artinean hoop, by Theorem 10, every non-empty set of filter of H has minimal element, thus $\mathcal{V}$ has a minimal element K. Let $\mathcal{U} = \{F \in \mathcal{F}(H) | F \subset K\}$. Since $\{1\} \in \mathcal{U}$, we get $\mathcal{U}$ is a non-empty set and since H is a Noetherian hoop, by Theorem 10, $\mathcal{U}$ has a maximal element such as K'. Since $K' \subset K$ and K is a minimal element in $\mathcal{V}$, we have $K' \notin \mathcal{V}$. Suppose $le(K') = m$ for some $m \in \mathbb{N}$ and $\{1\} = K'_0 \subset K'_1 \subset \cdots \subset K'_m = K'$ is a composition series for K'. Hence, $\{1\} = K'_0 \subset K'_1 \subset \cdots \subset K'_m \subset K$ is a composition series for K, which is a contradiction. Therefore, $le(H)$ is finite.

Conversely, by Theorem 39, the length of every chain of filters of H is finite and H is a Noetherian and Artinian hoop. ■

Theorem 42. *Let $F \in \mathcal{F}(H)$. If $le(H)$ is finite, then $le(\frac{H}{F})$ is finite. Moreover $le(H) = le(F) + le(\frac{H}{F})$.*

Proof. Suppose $le(H)$ is finite. Then by Theorems 14 and 41, we have $le(\frac{H}{F})$ is finite. Moreover, by Theorem 39, we get $le(F)$ is finite. Let $m, n \in \mathbb{N}$ such that $le(F) = n$ and $le(\frac{H}{F}) = m$. Consider $\{1\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ as a composition series for F. By Lemma 13, for any $1 \leq i \leq m$, there exists $K_i \in \mathcal{F}(H)$ such that $F \subseteq K_i$ and $\frac{K_i}{F} \in \mathcal{F}(\frac{H}{F})$. Suppose

$$\{\frac{1}{F}\} = \frac{K_0}{F} \subset \frac{K_1}{F} \subset \frac{K_2}{F} \subset \cdots \subset \frac{K_m}{F} = \frac{H}{F}$$

is a composition series for $\frac{H}{F}$. Now, we get

$$\{1\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset K_1 \subset K_2 \subset \cdots \subset K_m = H$$

is a composition series for H. Hence, by Theorem 40, $le(H) = le(F) + le(\frac{H}{F})$. ■

Definition. The intersection of all maximal filters of hoop H is called a *radical of H* and is denoted by $Rad(H)$. It means that

$$Rad(H) = \bigcap_{M \in \mathcal{M}ax(H)} M.$$

Example 43. Let H be a hoop as in Example 35. Clearly $\mathcal{M}ax = \{\{a, 1\}, \{b, 1\}\}$ and so $Rad(H) = \{1\}$.

Lemma 44. *Let H be bounded and $F, G \in \mathcal{F}(H)$ such that $\langle F \cup G \rangle = H$. Then there exists $x \in H$ such that $x \sim_F 1$ and $x \sim_G 0$, where $\sim$ is a congruence relation on H by F and G, respectively.*

Proof. Since $0 \in H = \langle F \cup G \rangle$ there exist $x \in F$ and $y \in G$ such that $x \odot y = 0$. Since $x \in F$, clearly, $x \sim_F 1$. By Proposition 1(viii), since $x \odot y \leq 0$, we get $y \leq x'$. Moreover, $y \in G$, $G \in \mathcal{F}(H)$ and $y \leq x'$, then $x' \in G$. Hence, $(0 \to x) \odot (x \to 0) = x' \in G$, and so $x \sim_G 0$. ■

Example 45. Let H be a hoop as in Example 35. Obviously, $H = \langle \{a, 1\} \cup \{b, 1\} \rangle$. So there exist $F, G \in \mathcal{F}(H)$ such that $\langle F \cup G \rangle = H$.

Theorem 46. *Let H be bounded and $\mathcal{M}ax(H) = \{M_1, M_2, \cdots, M_n\}$. Then a mapping $CR : H \to \prod_{i=1}^{n} \frac{H}{M_i}$ define by $CR(x) = (\frac{x}{M_1}, \frac{x}{M_2}, \cdots, \frac{x}{M_n})$ is a surjective hoop homomorphism.*

Proof. Since CR is a product of the natural homomorphisms $CR_i : H \to \frac{H}{M_i}$ such that $CR_i(x) = \frac{x}{M_i}$ where $1 \leq i \leq n$, clearly we have CR is a hoop homomorphism. Now, we prove CR is a surjective homomorphism. Let

$$y = (\frac{x_1}{M_1}, \frac{x_2}{M_2}, \cdots, \frac{x_n}{M_n}) \in \prod_{i=1}^{n} \frac{H}{M_i}$$

such that $\frac{x_i}{M_i} \in \frac{H}{M_i}$ for all $1 \leq i \leq n$. Clearly, $x_i \in H \setminus M_i$. If $x_i \in M_i$, then $\frac{x_i}{M_i} = \frac{1}{M_i}$ in other word $x_i \sim_{M_i} 1$, $1 \leq i \leq n$. Now, we try to find an element $z \in H$ such that $CR(z) = y$. Since for every $1 \leq i \leq n$, M_i are maximal filters of H, we get $\langle M_i \cup M_j \rangle = H$ for any $1 \leq i \neq j \leq n$. By Lemma 44, for any $1 \leq i \neq j \leq n$, there is an element $a_{i,j} \in H$ such that $a_{i,j} \sim_{M_i} 1$ and $a_{i,j} \sim_{M_J} 0$. Thus $a_{i,j} \in M_i$ and $a'_{i,j} \in M_j$. Consider

$$\begin{aligned} r_1 &= a_{1,2} \odot a_{1,3} \odot a_{1,n}, \\ r_2 &= a_{2,1} \odot a_{2,3} \odot a_{2,n}, \\ &\vdots \\ r_n &= a_{n,1} \odot a_{n,2} \odot a_{n,n-1}. \end{aligned}$$

Then for any $1 \leq i \neq j \leq n$, since M_i is a maximal filter of H and $a_{i,j} \in M_i$, we get $r_i \in M_i$. By Proposition 1(iii), $r_i \leq a_{i,j}$ and so $a'_{i,j} \leq r'_i$. Moreover, from M_j is a maximal filter of H and $a'_{i,j} \in M_j$, we have $r'_j \in M_j$. Since $M_j \in \mathcal{F}(H)$ and $r'_i \in M_j$ we obtain $r_i \sim_{M_i} 1$ and $r_i \sim_{M_j} 0$. Let $z = ((x_1 \odot r_1)' \odot (x_2 \odot r_2)' \odot \cdots \odot (x_n \odot r_n)')'$. According to Lemma 44, it is enough to prove $(x_i \to z) \odot (z \to x_i) \in M_i$ for any $1 \leq i \leq n$. By using (H3), we have

$$\begin{aligned} & (x_i \odot r_i) \odot (x_i \odot [(x_1 \odot r_1)' \odot \cdots \odot (x_n \odot r_n)']) = 0 \\ \Leftrightarrow \quad & x_i \odot r_i \leq (x_i \odot [(x_1 \odot r_1)' \odot \cdots \odot (x_n \odot r_n)'])' \\ \Leftrightarrow \quad & x_i \odot r_i \leq x_i \to ([(x_1 \odot r_1)' \odot \cdots \odot (x_i \odot r_i)'] \to 0) \\ \Leftrightarrow \quad & x_i \odot r_i \leq x_i \to z. \end{aligned}$$

Since $x_i, r_i \in M_i$ and $M_i \in \mathcal{F}(H)$, we have $x_i \to z \in M_i$. Moreover, by Proposition 1(vi), $x_i \leq z \to x_i$. Since $M_i \in \mathcal{F}(H)$ and $x_i \in M_i$, we obtain $z \to x_i \in M_i$. Hence, by Definition 4, $(x_i \to z) \odot (z \to x_i) \in M_i$, and so $\frac{z}{M_i} = \frac{x_i}{M_i}$. Therefore, $CR(z) = (\frac{z}{M_1}, \frac{z}{M_2}, \cdots, \frac{z}{M_n}) = (\frac{x_1}{M_1}, \frac{x_2}{M_2}, \cdots, \frac{x_n}{M_n})$, and so CR is a surjective hoop homomorphism. ■

Corollary 47. *If H is bounded, then* $\frac{H}{Rad(H)} \cong \prod_{i=1}^{n} \frac{H}{M_i}$, *where* $\mathcal{M}ax(H) = \{M_1, M_2, \cdots, M_n\}$.

Proof. Let $x \in H$ and for every $1 \leq i \leq n$, $M_i \in \mathcal{M}ax(H)$ such that $CR(x) = (\frac{1}{M_1}, \frac{1}{M_2}, \cdots, \frac{1}{M_n}) = 1_{\prod_{i=1}^{n} \frac{H}{M_i}}$. By definition of CR we have $(\frac{x}{M_1}, \frac{x}{M_2}, \cdots, \frac{x}{M_n}) = (\frac{1}{M_1}, \frac{1}{M_2}, \cdots, \frac{1}{M_n})$, and so $x \sim_{M_i} 1$ for any $1 \leq i \leq n$. Thus for any $1 \leq i \leq n$, we get $x \in M_i$ and so $x \in \bigcap_{i=1}^{n} M_i = Rad(H)$. By Theorem 46 and Proposition 21, since CR is surjective, we get CR is one-to-one and $ker(CR) = Rad(H)$. Hence, by using the first isomorphism theorem, we obtain $\frac{H}{Rad(H)} \cong \prod_{i=1}^{n} \frac{H}{M_i}$. ∎

Definition. A hoop H is called a *simple hoop* if $\mathcal{F}(H) = \{H, \{1\}\}$.

Example 48. Let H_2 be a hoop as in Example 17. Clearly H is a simple hoop.

Note. Let $F, G \in \mathcal{F}(H)$. An interval of $[F, G]$ is denoted by $K \in \mathcal{F}(H)$ where $F \subseteq K \subseteq G$.

Theorem 49. *Let $M \in \mathcal{F}(H)$. Then $\frac{H}{M}$ is a simple hoops if and only if $M \in \mathcal{M}ax(H)$.*

Proof. Suppose $\frac{H}{M}$ is not simple. Then there exists $\frac{K}{M} \in \mathcal{F}(\frac{H}{M})$ such that $\frac{1}{M} \neq \frac{K}{M} \neq \frac{H}{M}$, and so $1 \subset K \subset H$. Hence, $M \notin \mathcal{M}ax(H)$, which is a contradiction.

The proof of converse is similar. ∎

Theorem 50. *Let H be bounded such that $Rad(H) = \{1\}$. Then the following statements hold:*

(i) H is up to isomorphism a finite product of some simple hoop if and only if H is an Artinian hoop.

(ii) If $\mathcal{M}ax(H)$ is finite, then H is an Artinian hoop.

(iii) If H is an Artinian hoop, then H is Noetherian.

Proof. (i) Let H be an Artinian hoop and $Rad(H) = \{1\}$. By Theorem 32, we get $\mathcal{M}ax(H)$ is finite. Moreover, by Corollary 47, we have $H \cong \prod_{i=1}^{n} \frac{H}{M_i}$. Since M_i is a maximal filter of H, for every $1 \leq i \leq n$, we have $\frac{H}{M_i}$ is a simple hoop. Hence, H is a finite direct product of simple hoops.

Conversely, suppose $H \cong \prod_{i=1}^{n} H_i$ such that for every $1 \leq i \leq n$, H_i is a simple hoop. Then for every $1 \leq i \leq n$, $\mathcal{F}(H_i) = \{\{1\}, H_i\}$ and by Proposition 24, we get $\mathcal{F}(\prod_{i=1}^{n} H_i)$ is finite. Hence, H is an Artinian hoop.

(ii) By (i) the proof is clear.

(iii) Let H be an Artinian hoop. By (i), H is a finite direct product of simple hoops and by Proposition 24, we get $\mathcal{F}(H)$ is finite. Therefore, H is a Noetherian hoop. ∎

Theorem 51. *Let H be a $\vee$-hoop. Then $\mathcal{M}ax(H)$ is finite if and only if every properly increasing chain of filters of $\frac{H}{Rad(H)}$ is finite.*

Proof. Suppose $\mathcal{M}ax(H)$ if finite. Assume $\mathcal{M}ax(H) = \{M_1, M_2, \cdots, M_n\}$, then by Theorem 14, every properly increasing chain of filters of $\frac{H}{Rad(H)}$ is finite. For the converse by Theorem 32 and Proposition 24, the proof is clear. ■

4. Conclusions and future works

In this paper, the notion of Noetherian and Artinian hoops are defined and characterized by using the filters of hoops. Then the relation between Noetherian and Artinian hoops are investigated. Also, the notion of a short exact sequence is introduced and the relation between a short exact sequence and Noetherian and Artinian hoops are investigated. The concept of composition series is defined and proved every $\vee$-hoop is Noetherian and Artinian hoop if it has a finite composition series. Finally, we investigate the condition that proved H is up to isomorphism a finite product of some simple hoop if and only if H is an Artinian hoop.

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