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ON 3-PRIME AND QUASI 3-PRIMARY IDEALS OF TERNARY SEMIRINGS

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Abstract

The purpose of this paper is to introduce the concept of 3-prime ideal as a generalization of prime ideal. Further, we generalize the concepts of 3-prime ideal and primary ideal, namely as quasi 3-primary ideal in a commutative ternary semiring with zero. The relationship among prime ideal, 3-prime ideal, primary ideal, quasi primary and quasi 3-primary ideal are investigated. Various results and examples concerning 3-prime ideals and quasi 3-primary ideals are given. Analogous theorems to the primary avoidance theorem for quasi 3-primary ideals are also studied.

Keywords: ternary semiring, regular ternary semiring, 3-prime ideals, quasi 3-primary ideals.

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1. Introduction

The concept of the ternary algebraic system was first introduced by Lehmer [8] in 1932 which is a generalization of abelian groups. In 1971, Lister [7] introduced ternary rings. To generalize the ternary rings, Dutta and Kar [3] introduced the notion of ternary semirings in 2003. A ternary semiring is an algebraic system consisting of a set S together with a binary operation '+', called addition, and a ternary multiplication, denoted by juxtaposition, which forms a commutative

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semigroup relative to addition, a ternary semigroup relative to multiplication and the left, right, lateral distributive laws hold, i.e., for all $a, b, c, d \in S$, $(a+b)cd = acd + bcd$, $a(b+c)d = abd + acd$, $ab(c+d) = abc + abd$. If there exists an element e such that $eea = eae = aee = a$ for all $a \in S$, then e is called the identity element of S. If there exists an element $0 \in S$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for $x, y \in S$, then 0 is called zero of the ternary semiring S.

The notion of prime ideals and its generalization have an important place in commutative algebra, for their applications in many areas such as graph theory, coding theory, information science, algebraic geometry, topological spaces, etc. In 2016, Beddani and Messirdi [1] introduced the concept of 2-prime ideals as a generalization of prime ideals in a ring. A proper ideal P of ring R is said to be 2-prime if for all a, $b \in R$ $ab \in P$ implies either $a^2 \in P$ or $b^2 \in P$. Recall that in a commutative ternary semiring S, an ideal I is called primary if for all $a, b, c \in S$, $abc \in I$ implies $a \in I$ or $b \in I$ or $c^{2n+1} \in I$ for some $n \in Z_0^+$ and an ideal I of S is said to be quasi primary if $Rad(I)$ is a prime ideal. In [9], Koc, Tekir and Ulucak introduced a new class of ideals, an intermediate class between the class of primary ideals and the class of quasi-primary ideals in a ring and is called the class of strongly quasi primary ideals. A proper ideal P in a commutative ring R is said to be strongly quasi primary if $ab \in P$ for some $a, b \in R$ implies either $a^2 \in P$ or $b^n \in P$ for some positive integer n.

We shortly summarize the content of the paper. In the first Section, we recall some essential preliminaries. In Section 2, we introduce 3-prime ideals as a generalization of prime ideals on ternary semirings. Various properties and relationships among radical ideals, maximal ideals and irreducible ideals are studied. We give a characterization of 3-prime ideals in ternary semirings. Then we study ternary semirings, where every 3-prime ideal is prime. In Section 3, we define a quasi 3-primary ideal, which is a generalization of 3-prime ideal and is an intermediate class between 3-prime ideals and quasi-primary ideals in a ternary semiring. We show that in regular ternary semirings, the concept of 3-prime ideals, quasi 3-primary ideals and primary ideals are the same. Theorem 4.39 is a characterization for quasi 3-primary ideals on a ternary semiring. At the end, we focus on the study of the avoidance theorem for quasi 3-primary ideals by using the techniques of efficient covering (cf. Theorem 4.44) and give an extended version of the theorem (cf. Theorem 4.45).

2. Theoretical background for ternary semirings

In this section, we review some definitions and results which will be used in later sections.

Definition 2.1 [3]**.** A nonempty set S together with a binary operation called addition and a ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(1) $(abc)de = a(bcd)e = ab(cde)$,

(2) $(a + b)cd = acd + bcd$,

(3) $a(b + c)d = abd + acd$,

(4) $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in S$.

Example 1 [2]**.** Let S be a set of continuous functions $f : X \to \mathbb{R}^-$, where X is a topological space and $\mathbb{R}^-$ is the set of all negative real numbers. Define a binary addition and a ternary multiplication on S as follows: For $f, g, h \in S$ and $x \in X$,

(1) $(f + g)(x) = f(x) + g(x)$,

(2) $(fgh)(x) = f(x)g(x)h(x)$.

Then with respect to the binary addition and ternary multiplication, S forms a ternary semiring.

Let A, B and C be three subsets of S. By ABC, we mean the set of all finite sums of the form $\sum a_ib_ic_i$ with $a_i \in A, b_i \in B$ and $c_i \in C$.

Definition 2.2 [3]**.** A ternary semiring S is called a commutative ternary semiring if $abc = bac = bca$ for all a, b, $c \in S$.

Definition 2.3 [3]**.** An additive subsemigroup T of S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.4 [3]**.** An additive subsemigroup I of S is called a left (resp. right, lateral) ideal of S if s_1s_2i (resp. is_1s_2, s_1is_2) $\in I$, for all $s_1, s_2 \in S$ and $i \in I$. If I is both a left and a right ideal of S, then I is called a two-sided ideal of S. If I is a left, a right and a lateral ideal of S, then I is called an ideal of S.

Definition 2.5 [2]**.** An ideal I of a ternary semiring S is said to be a k-ideal if for $x, y \in S$, $x + y \in I$ and $y \in I$ implies $x \in I$.

Definition 2.6 [3]**.** An element a in a ternary semiring S is called regular if there exists an element x in S such that $axa = a$. A ternary semiring is called regular if all of its elements are regular.

Definition 2.7 [4]**.** A proper ideal P of a ternary semiring S is called a prime ideal if for any three ideals A, B and C of S, $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Corollary 2.8 [4]. *A proper ideal P of a commutative ternary semiring S is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$ for all elements $a, b, c \in S$.*

Definition 2.9 [5]. A proper ideal Q of a ternary semiring S is called a semiprime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S.

Corollary 2.10 [5]. *A proper ideal Q of a commutative ternary semiring S is semiprime if and only if $x^3 \in Q$ implies that $x \in Q$ for any element x of S.*

Definition 2.11 [5]. Let S be a ternary semiring and A be an ideal of S. The radical of A, denoted by $Rad(A)$, is defined to be the intersection of all the prime ideals of S each of which contains A. In a commutative ternary semiring S, $Rad(A) = \{a \in S : a^{2n+1} \in A$ for some positive integer n$\}$.

Definition 2.12 [5]. A proper ideal I of a ternary semiring S is called a strongly irreducible if for any two ideals H and K of S, $H \cap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$.

Lemma 2.13 [12]. *Let S be a commutative ternary semiring and I be an ideal of S. Then $(I : a : b)$ is an ideal in S, where $(I : a : b) = \{c \in S : abc \in I\}$.*

Definition 2.14 [11]. A proper ideal P of a commutative ternary semiring S is called primary if for any $a, b, c \in S$, $abc \in P$ implies $a \in P$ or $b \in P$ or $c^{2n+1} \in P$ for some positive integer n. An ideal I of a commutative ternary semiring S is called quasi primary if $Rad(I)$ is prime.

3. On 3-prime ideals

Throughout the paper, unless otherwise stated S stands for a commutative ternary semiring with zero. Z_0^- and Z_0^+ denote the set of all negative integers with zero and the set of all positive integers with zero respectively.

Definition 3.15. An ideal I of a ternary semiring S is called a 3-prime ideal if for any x, y, $z \in S$; $xyz \in I$ implies $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$.

Example 2. In the ternary semiring Z_0^-, the ideal $8Z_0^-$ is a 3-prime ideal.

It's easy to see that in a ternary semiring, every prime ideal is 3-prime but the converse may not be true. In the above example, $8Z_0^-$ is a 3-prime but not a prime ideal of Z_0^-. If 3-prime ideal is semiprime, then the converse holds as is shown in the next result.

Theorem 3.16. *If an ideal I of a ternary semiring S is 3-prime as well as semiprime, then I is prime.*

Proof. Let $xyz \in I$ for some x, y, $z \in S$. Since I is a 3-prime ideal of S, $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$. As I is semiprime, we have $x \in I$ or $y \in I$ or $z \in I$. ■

Proposition 3.17. *If I is 3-prime, then $Rad(I)$ is prime.*

Proof. Let $xyz \in Rad(I)$ for some x, y, $z \in S$. Then $(xyz)^{2n+1} \in I$ for some $n \in Z_0^+$. Thus $x^{2n+1}y^{2n+1}z^{2n+1} \in I$, which implies $x^{2n+1} \in I$ or $y^{2n+1} \in I$ or $z^{2n+1} \in I$. So $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$. ■

The converse of the above proposition may not be true, as is shown in the following example.

Example 3. Consider the ternary subsemiring $Z_0^- \times 3Z_0^-$ of the ternary semiring $Z_0^- \times Z_0^-$. Then the ideal $32Z_0^- \times 81Z_0^-$ is not 3-prime in $Z_0^- \times 3Z_0^-$ but $Rad(32Z_0^- \times 81Z_0^-) = 2Z_0^- \times 3Z_0^-$ is a prime ideal of $Z_0^- \times 3Z_0^-$. This is because $(-4,-3)(-4,-3)(-2,-27) = (-32,-243) \in 32Z_0^- \times 81Z_0^-$ but $(-4,-3)^3 \notin 32Z_0^- \times 81Z_0^-$, $(-4,-3)^3 \notin 32Z_0^- \times 81Z_0^-$ and $(-2,-27)^3 \notin 16Z_0^- \times 81Z_0^-$.

Proposition 3.18. *If $Rad(I)$ is prime and $(Rad(I))^3 \subseteq I$. Then I is 3-prime.*

Proof. Let $Rad(I)$ be prime and $(Rad(I))^3 \subseteq I$. For x, y, $z \in S$, suppose $xyz \in I$. Then $xyz \in Rad(I)$ which implies $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$. So $x^3 \in (Rad(I))^3 \subseteq I$ or $y^3 \in (Rad(I))^3 \subseteq I$ or $z^3 \in (Rad(I))^3 \subseteq I$. Hence I is 3-prime. ■

Theorem 3.19. *In a regular ternary semiring, an ideal is prime if and only if it is 3-prime.*

Proof. Clearly if an ideal I is prime then it is 3-prime.

Conversely, let I be a 3-prime ideal and $xyz \in I$. Then $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$. Suppose $x^3 \in I$. By regularity, there exist $a, b \in I$ such that $x = xaxbx$, that is, $x = abx^3 \in I$. So I is prime. ■

Proposition 3.20. *Let S be a ternary semiring. If an ideal I is a 3-prime ideal of S, then $(I : a^3 : b^3)$ is a 3-prime ideal of S, where $a, b \in S \setminus Rad(I)$.*

Proof. Let $xyz \in (I : a^3 : b^3)$ for some $x, y, z \in S$. Then $xyza^3b^3 \in I$. This implies $(xab)(yab)(zab) \in I$. Thus $(xab)^3 = x^3a^3b^3 \in I$ or $(yab)^3 = y^3a^3b^3 \in I$ or $(zab)^3 = z^3a^3b^3 \in I$. Hence $x^3 \in (I : a^3 : b^3)$ or $y^3 \in (I : a^3 : b^3)$ or $z^3 \in (I : a^3 : b^3)$ and so $(I : a^3 : b^3)$ is a 3-prime ideal of S. ■

The ternary product of 3-prime ideals may not be 3-prime, as is shown in the next example.

Example 4. In the ternary semiring Z_0^-, ternary product of the 3-prime ideals $2Z_0^-$, $3Z_0^-$ and $5Z_0^-$ is $30Z_0^-$, which is not a 3-prime ideal of Z_0^-.

Lemma 3.21. *Let P be a prime ideal and P', P'' be two ideals with $P \subseteq P'$ and $P \subseteq P''$. Then $PP'P''$ is 3-prime. Moreover, $PP'P''$ is prime if and only if $PP'P'' = P$.*

Proof. Let $abc \in PP'P''$. Then $abc \in PP'P'' \subseteq P$ which implies $a \in P$ or $b \in P$ or $c \in P$. So $a^3 \in P^3 \subseteq PP'P''$ or $b^3 \in P^3 \subseteq PP'P''$ or $c^3 \in P^3 \subseteq PP'P''$.

Now, let $PP'P''$ be a prime ideal of S. Clearly, $PP'P'' \subseteq P$. Consider $a \in P$. It follows that $a^3 \in PP'P''$. As $PP'P''$ is a prime ideal of S, we have $a \in PP'P''$ and so $P \subseteq PP'P''$. Hence $PP'P'' = P$. ■

Corollary 3.22. *If P is a prime ideal, then P^3 is a 3-prime ideal.*

Proposition 3.23. *In a ternary semiring S, every maximal ideal is 3-prime.*

Proof. If S is a ternary semiring with identity, then every maximal ideal is prime and hence 3-prime. Now suppose that S is a ternary semiring without identity and M is a maximal ideal of S. Consider $xyz \in M$ and $x^3 \notin M$, $y^3 \notin M$ for some $x, y, z \in S$. If possible, let $z^3 \notin M$. Then clearly $x, y, z \notin M$. Thus we conclude that $M + \langle x \rangle = S$, $M + \langle y \rangle = S$, $M + \langle z \rangle = S$. Now $x^3 = (m_1 + s_1 s_2 x + n_1 x)(m_2 + s_3 s_4 y + n_2 y)(m_3 + s_5 s_6 z + n_3 z)$ for some $m_1, m_2, m_3 \in M$, $s_1, s_2, s_3, s_4, s_5, s_6 \in S$ and $n_1, n_2, n_3 \in Z_0^+$. This implies $x^3 \in M$. Similarly $y^3 \in M$. But $x^3 \notin M$ and $y^3 \notin M$, hence $z^3 \in M$ and so M is a 3-prime ideal. ■

Lemma 3.24. *Let I be a 3-prime ideal of a ternary semiring S. If $abC \subseteq I$ and $a^3 \notin I$, $b^3 \notin I$ for some elements $a, b \in S$ and some ideal C, then $\{c^3 : c \in C\} \subseteq I$.*

Proof. Suppose $abC \subseteq I$ and $a^3 \notin I$, $b^3 \notin I$ for some $a, b \in S$ and some ideal C. Consider any arbitrary element $c \in C$, then $abc \in abC \subseteq I$. Since I is 3-prime, we conclude that $c^3 \in I$. Hence $\{c^3 : c \in C\} \subseteq I$. ■

Theorem 3.25. *Let I be a proper ideal of a ternary semiring S with identity. Then I is a 3-prime ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, we have $\{a^3 : a \in I_1\} \subseteq I$ or $\{b^3 : b \in I_2\} \subseteq I$ or $\{c^3 : c \in I_3\} \subseteq I$.*

Proof. Suppose that the condition holds and $abc \in I$ for some a, b, c in S. Then $(SSa)(SSb)(SSc) \subseteq I$ and so by the given condition $\{x^3 : x \in SSa\} \subseteq I$ or $\{y^3 : y \in SSb\} \subseteq I$ or $\{z^3 : z \in SSc\} \subseteq I$. Thus $a^3 \in I$ or $b^3 \in I$ or $c^3 \in I$.

Conversely, suppose I is a 3-prime ideal of S and $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3. Also, suppose that $\{a^3 : a \in I_1\} \nsubseteq I$ and $\{b^3 : b \in I_2\} \nsubseteq I$. Then there exist $i_1 \in I_1$, and $i_2 \in I_2$ such that $i_1^3, i_2^3 \notin I$. By Lemma 3.24, $\{c^3 : c \in I_3\} \subseteq I$. ■

Theorem 3.26. *Let $f : S \longrightarrow T$ be a ternary homomorphism of ternary semirings. Then the following statements hold.*

(1) *If J is a 3-prime ideal of T, then $f^{-1}(J)$ is a 3-prime ideal of S.*

(2) *Let f be a ternary epimorphism and I be a k-ideal of S with $\{x \in S :$ for some $a, b \in S$, $x = a + b$ and $f(a) = f(b)\} \subseteq I$, then $f(I)$ is a 3-prime ideal of T if I is a 3-prime ideal of S.*

Proof. (1) Let $xyz \in f^{-1}(J)$ for some $x, y, z \in S$. Then $f(xyz) = f(x)f(y)f(z) \in J$, which implies $(f(x))^3 = f(x^3) \in J$ or $(f(y))^3 = f(y^3) \in J$ or $(f(z))^3 = f(z^3) \in J$. Thus $x^3 \in f^{-1}(J)$ or $y^3 \in f^{-1}(J)$ or $z^3 \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a 3-prime ideal of S.

(2) Let $xyz \in f(I)$ for some $x, y, z \in S$. Then there exist a, b, c $\in S$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. So $xyz = f(a)f(b)f(c) = f(abc) \in f(I)$. Then $f(abc) = f(i)$ for some $i \in I$. Thus $abc + i \in I$. Hence $abc \in I$, since I is a k-ideal of S and $i \in I$. So $a^3 \in I$ or $b^3 \in I$ or $c^3 \in I$. Therefore $f(a^3) = (f(a))^3 = x^3 \in f(I)$ or $f(b^3) = (f(b))^3 = x^3 \in f(I)$ or $f(c^3) = (f(c))^3 = x^3 \in f(I)$. Consequently, $f(I)$ is a 3-prime ideal of T. ■

Proposition 3.27. *If an ideal I is strongly irreducible in a regular ternary semiring S, then I is 3-prime.*

Proof. Assume that S is a regular ternary semiring and I is a strongly irreducible ideal of S. Suppose that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ for some $a, b, c \in S$. We have to show that $c^3 \in I$. On the contrary, assume that $c^3 \notin I$. Then I is properly contained in $(I + \langle a^3 \rangle) \cap (I + \langle b^3 \rangle) \cap (I + \langle c^3 \rangle)$. So there exists an element $x \in (I + \langle a^3 \rangle) \cap (I + \langle b^3 \rangle) \cap (I + \langle c^3 \rangle)$ such that $x \notin I$. Since S is regular, we have $x \in (I + \langle a^3 \rangle)(I + \langle b^3 \rangle)(I + \langle c^3 \rangle) = (I + \langle a^3 \rangle) \cap (I + \langle b^3 \rangle) \cap (I + \langle c^3 \rangle)$. Thus for some $i_1, i_2, i_3 \in I$ and $r_1, r_2, s_1, s_2, t_1, t_2 \in S$ $x = (i_1 + r_1 r_2 a^3)(i_2 + s_1 s_2 b^3)(i_3 + t_1 t_2 c^3) \in I$, which is a contradiction. Therefore I is a 3-prime ideal of S. ■

Definition 3.28. A ternary semiring S is called a 3-P-ternary semiring if every 3-prime ideal of S is prime.

Example 5. Every regular ternary semiring is a 3-P-ternary semiring.

Definition 3.29. Let A be an ideal of a ternary semiring S. A 3-prime ideal I containing A, is called a minimal 3-prime ideal over A if for any 3-prime ideal Q, $A \subseteq Q \subseteq I$ implies $Q = I$.

Proposition 3.30. *A ternary semiring S is a 3-P-ternary semiring if and only if every 3-prime ideal is semiprime.*

Proof. Follows from Theorem 3.16. ■

Theorem 3.31. *A ternary semiring S is a 3-P-ternary semiring if and only if every prime ideal is idempotent and every 3-prime ideal is of the form P^3, for some prime ideal P of S.*

Proof. Let S be a 3-P-ternary semiring and P' be a prime ideal of S. By Corollary 3.22, P'^3 is a 3-prime ideal. Thus P'^3 is prime and so $P' \subseteq P'^3$. Also $P'^3 \subseteq P'$ and hence $P'^3 = P'$. Now, consider any 3-prime ideal P'' of S, then P'' is prime. So we have P'' is idempotent as it is needed.

Conversely, let I be a 3-prime ideal of S. Then I is of the form $I = P'^3$ for some idempotent prime ideal P', it follows that $I = P'$, as required. ■

Theorem 3.32. *Let S be a ternary semiring with unique maximal ideal M. Then for any prime ideal P of S, P^2M is a 3-prime ideal of S. Moreover, P^2M is prime if and only if $P^2M = P$.*

Proof. Since $P \subseteq M$, the proof follows from the Lemma 3.21. ■

Theorem 3.33. *Let S be a ternary semiring with unique maximal ideal M, then S is a 3-P-ternary semiring if and only if for for every 3-prime ideal I, $I^2M = Rad(I)$.*

Proof. Suppose for every 3-prime ideal I, $I^2M = Rad(I)$. Thus $I \subseteq Rad(I) = I^2M \subseteq I$. So $I = Rad(I)$. Hence I is prime. The converse part follows from the Theorem 3.32. ■

Theorem 3.34. *Let S be a ternary semiring with unique maximal ideal M and P be a prime ideal of S. If $(Rad(I))^3 \subseteq I$ for any 3-prime ideal I of S, then the following are equivalent.*

(i) *For every minimal 3-prime ideal I over P^3, if P is minimal prime over I, then $I^2M = P$.*

(ii) *For every minimal 3-prime ideal I over P^3 such that $I \subseteq P$, then $I = P$.*

Proof. (i) $\Longrightarrow$ (ii) Let I be a minimal 3-prime ideal over P^3 and $I \subseteq P$. We claim that P is a minimal prime ideal over I. If $I \subseteq J \subseteq P$, for some prime ideal J of S. Then for any $x \in P$, $x^3 \in P^3 \subseteq I \subseteq J$. Thus $x \in J$. So $J = P$ and hence P is minimal. By (i), $I^2M = P$. Thus $P = I^2M \subseteq I \subseteq P$ and so $I = P$.

(ii) $\Longrightarrow$ (i) Assume that I is a minimal 3-prime ideal over P^3 and P is a minimal prime ideal over I. Since $Rad(I)$ is a prime ideal and $P^3 \subseteq I \subseteq Rad(I)$, it follows that $P = Rad(I)$. By hypothesis, $P^3 \subseteq I \subseteq P$ and so $I = P$. Also $P^3 \subseteq P^2M \subseteq P = I$ and by Theorem 3.32, P^2M is 3-prime. Therefore $P^2M = P^2I = P$, as required. ■

ON QUASI 3-PRIMARY IDEALS

Definition 4.35. An ideal I of a ternary semiring S is called a quasi 3-primary ideal if for any $a, b, c \in S$, $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ implies there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$.

It can be easily obtained by the definition that every 3-prime ideal is a quasi 3-primary ideal. The following example shows that the converse may not be true.

Example 6. Consider the ternary subsemiring $Z_0^- \times 3Z_0^-$ of the ternary semiring $Z_0^- \times Z_0^-$. Then the ideal $\{0\} \times 81Z_0^-$ is strongly quasi primary, but not 3-prime in $Z_0^- \times 3Z_0^-$, since $(0, -81) = (-6, -9)(-5, -3)(0, -3) \in \{0\} \times 81Z_0^-$ and $(-6, -9)^3 \notin \{0\} \times 81Z_0^-$, $(-5, -3)^3 \notin \{0\} \times 81Z_0^-$ and $(0, -3)^3 \notin \{0\} \times 81Z_0^-$ but $(0, -3)^5 \in \{0\} \times 81Z_0^-$.

Theorem 4.36. *Let S be a regular ternary semiring, then an ideal I is quasi 3-primary if and only if I is 3-prime.*

Proof. Let I be a quasi 3-primary ideal of S. Assume that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$. Then there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$. Since S is a regular ternary semiring, there exists $x \in S$ such that $c = xc^{2n+1} \in I$. So $c^3 \in I$ and hence I is a 3-prime ideal of S. ■

Theorem 4.37. *If I is a quasi 3-primary ideal of ternary semiring S, then I is a quasi primary ideal.*

Proof. Let $abc \in Rad(I)$ for some $a, b, c \in S$ and $a \notin Rad(I)$, $b \notin Rad(I)$. Then there exists an integer $n \in Z_0^+$ such that $(abc)^{2n+1} = a^{2n+1}b^{2n+1}c^{2n+1} \in I$. Since I is a quasi 3-primary ideal and $a \notin Rad(I)$, $b \notin Rad(I)$, so we have $c^{(2m+1)(2n+1)} \in I$ for some integer $m \in Z_0^+$. This implies $c \in Rad(I)$ and so I is a quasi primary ideal of S. ■

The converse may not be true as is shown in the following example.

Example 7. Consider the ternary subsemiring $2Z_0^- \times Z_0^-$ of the ternary semiring $Z_0^- \times Z_0^-$. Then the ideal $16Z_0^- \times 81Z_0^-$ is quasi primary, since $Rad(16Z_0^- \times 81Z_0^-) = 2Z_0^- \times 3Z_0^-$ is prime on $2Z_0^- \times Z_0^-$. But this ideal is not quasi 3-primary, as $(-2, -27)(-4, -3)(-2, -4) = (-16, -324) \in 16Z_0^- \times 81Z_0^-$, where $(-2, -27)^3 \notin 16Z_0^- \times 81Z_0^-$, $(-4, -3)^3 \notin 16Z_0^- \times 81Z_0^-$ and $(-2, -4)^{2n+1} \notin 16Z_0^- \times 81Z_0^-$ for any $n \in Z_0^+$.

Proposition 4.38. *In a ternary semiring S, I is a quasi 3-primary ideal if and only if $Rad(I)$ is a 3-prime ideal.*

Proof. Suppose I is a quasi 3-primary ideal of S. Then $Rad(I)$ is a prime ideal of S, thus $Rad(I)$ is a 3-prime ideal.

Conversely, assume that $Rad(I)$ is a 3-prime ideal of S. Let $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ for some $a, b, c \in S$. Since $abc \in I \subseteq Rad(I)$, we have $c^3 \in Rad(I)$. Thus there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$ and hence I is a quasi 3-primary ideal of S. ■

In a commutative regular ternary semiring, every non-zero proper ideal is semiprime. Hence it can be easily shown that in a regular ternary semiring the concept of prime ideal, 3-prime ideal, primary ideal, quasi 3-primary and quasi primary ideal are the same.

The following example shows that the intersection of quasi 3-primary ideals may not be a quasi 3-primary ideal.

Example 8. In the ternary semiring Z_0^-, the intersection of quasi 3-primary ideals $3Z_0^-$, $5Z_0^-$ and $2Z_0^-$ is $30Z_0^-$, which is not a quasi 3-primary ideal.

Theorem 4.39. *Let S be a ternary semiring with identity and I be a proper ideal of S, then the following are equivalent.*

(i) *I is a quasi 3-primary ideal of S.*

(ii) *For any $a, b \in S$, if $\langle a\rangle \nsubseteq (I : a : a)$ and $\langle b\rangle \nsubseteq (I : b : b)$, then $(I : a : b) \subseteq Rad(I)$.*

(iii) *For any three ideals J, K, L of S, $JKL \subseteq I$, $\{a^3 : a \in J\} \nsubseteq I$ and $\{b^3 : b \in K\} \nsubseteq I$ implies $K \subseteq Rad(I)$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose I is a quasi 3-primary ideal of S and $\langle a\rangle \nsubseteq (I : a : a)$, $\langle b\rangle \nsubseteq (I : b : b)$. Then $a^3 \notin I$ and $b^3 \notin I$. We have to show $(I : a : b) \subseteq Rad(I)$. Take $c \in (I : a : b)$. Then $abc \in I$. Also $a^3 \notin I$ and $b^3 \notin I$. Thus there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$ and hence $(I : a : b) \subseteq Rad(I)$.

(ii) $\Longrightarrow$ (iii) Consider $JKL \subseteq I$, $\{a^3 : a \in J\} \nsubseteq I$ and $\{b^3 : b \in K\} \nsubseteq I$ for some ideals J, K, L of S. Then $a \in J$ and $b \in K$ such that a^3, $b^3 \notin I$ and so $\langle a\rangle \nsubseteq (I : a : a)$ and $\langle b\rangle \nsubseteq (I : b : b)$. Then by (ii), $(I : a : b) \subseteq Rad(I)$. For any arbitrary element $c \in K$, $abc \in JKL \subseteq I$. So $c \in (I : a : b) \subseteq Rad(I)$. This yields that $K \subseteq Rad(I)$.

(iii) $\Longrightarrow$ (i) Assume that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$. Then $\{x^3 : x \in \langle a\rangle\} \nsubseteq I$ and $\{y^3 : y \in \langle b\rangle\} \nsubseteq I$. Since $abc \in \langle a\rangle \langle b\rangle \langle c\rangle \subseteq I$, by (iii) there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I$. So I is a quasi 3-primary ideal of S. ∎

Proposition 4.40. *Let I be a quasi 3-primary ideal of ternary semiring S with identity and $\langle a\rangle = \langle a^3\rangle$ for $a \in S$. If $a \notin (I : a : a)$, then $(I : a : a)$ is a quasi 3-primary ideal of S.*

Proof. Suppose I is a quasi 3-primary ideal of S. Here $\langle a\rangle \nsubseteq (I : a : a)$, since $a \notin (I : a : a)$. So by Theorem 4.39, $(I : a : a) \subseteq Rad(I)$. Thus $(I : a : a) = Rad(I)$. Consider $xyz \in (I : a : a)$ and $z^{2n+1} \notin (I : a : a)$ for some $x, y, z \in S$ and any $n \in Z_0^+$. Whence $(xa^2)yz = xyza^2 \in I$ and $z^{2n+1} \notin I$ implies $(xa^2)^3 \in I$ or $y^3 \in I$. That is $x^3 \in (I : a^3 : a^3) = (I : a : a)$ or $y^3 \in I \subseteq (I : a : a)$. Hence $(I : a : a)$ is a quasi 3-primary ideal of S. ∎

Proposition 4.41. *Suppose that I_1 and I_2 are two ideals of ternary semiring S_1 and S_2 respectively. Consider the ternary semiring $S = S_1 \times S_2$, then the followings hold.*

(i) *$I_1 \times S_2$ is a quasi 3-primary ideal of S if and only if I_1 is a quasi 3-primary ideal of S_1.*

(ii) *$S_1 \times I_2$ is a quasi 3-primary ideal of S if and only if I_2 is a quasi 3-primary ideal of S_2.*

Proof. (i) Suppose that $I_1 \times S_2$ is a quasi 3-primary ideal of S, $abc \in I_1$ for some $a, b, c \in S_1$ and $a^3 \notin I_1$, $b^3 \notin I_1$. Then we have $(abc, 0) = (a, 0)(b, 0)(c, 0) \in I_1 \times S_2$ and $(a, 0)^3 = (a^3, 0) \notin I_1 \times S_2$, $(b, 0)^3 = (b^3, 0) \notin I_1 \times S_2$. So we conclude that there exists an integer $n \in Z_0^+$ such that $(c, 0)^{2n+1} = (c^{2n+1}, 0) \in I_1 \times S_2$. Thus there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I_1$.

Conversely, assume that I_1 is a quasi 3-primary ideal of S_1. Let $(a, x)(b, y)(c, z) \in I_1 \times S_2$ and $(a, x)^3 \notin I_1 \times S_2$, $(b, y)^3 \notin I_1 \times S_2$. This implies $abc \in I_1$ and $a^3 \notin I_1$, $b^3 \notin I_1$. So there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \in I_1$. Hence $(c, z)^{2n+1} = (c^{2n+1}, z^{2n+1}) \in I_1 \times S_2$. Therefore $I_1 \times S_2$ is a quasi 3-primary ideal of S.

(ii) The proof is similar to (i). ■

Definition 4.42. Let $I, I_1, I_2, \dots, I_n$ be ideals of a ternary semiring S. The collection $\{I_1, I_2, \dots, I_n\}$ is said to be a cover of I if $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$. We call such a cover of I efficient, if I is not contained in the union of any $n-1$ ideals of $I_1, I_2, \dots, I_n$.

Lemma 4.43. *Let $\{I_1, I_2, \dots, I_n\}$ be an efficient covering of the ideal I, where $I_1, I_2, \dots, I_n$ are k-ideals of ternary semiring S and $n > 1$. If $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, then no I_j is quasi 3-primary ideal of S.*

Proof. We first show that for efficient covering $I \subseteq I_1 \cup I_2 \cup \cdots \cup I_n$ of I, $(\bigcap_{i \neq k} I_i) \cap I = (\bigcap_{i=1}^n I_i) \cap I$ for all k. Let $x \in (\bigcap_{i \neq k} I_i) \cap I$. Since the cover is efficient, there exists $x_k \in I_k \cap I$ such that $x_k \notin (\bigcup_{i \neq k} I_i) \cap I$. Now consider the element $x + x_k$ in I. If $x + x_k \in I_i$ for $i \neq k$, then $x_k \in I_i$ for all $i \neq k$, which is a contradiction. Then $x + x_k \in I_k$ and thus $x \in I_k$. So $(\bigcap_{i \neq k} I_i) \cap I = (\bigcap_{i=1}^n I_i) \cap I$. If possible, let I_j be a quasi 3-primary ideal of S for some $j = 1, 2, \dots, n$. Since $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$ we have $I = \bigcup_{i=1}^n (Rad(I_i) \cap I)$. Since $\{Rad(I_i) \cap I : 1 \leq i \leq n\}$ is also an efficient covering of I, there exists an element $x_i \in I \setminus Rad(I_i)$. This yields that $x_i^3 \notin I_i$ for each $i = 1, 2, \dots, n$. Also $Rad(I_i) \nsubseteq Rad(I_j)$ for each $i \neq j$. Hence there exist $y_i \in Rad(I_i) \setminus Rad(I_j)$ for every $i \neq j$. Thus $y_i^{2n_i+1} \in I_i$ but $y_i^{2n_i+1} \notin I_j$ for some $n_i \in N$ and $i \neq j$. Consider $y = (y_1)^{n+1} y_2 \cdots y_{j-1} y_{j+1} \cdots y_n$. Since $Rad(I_i)$ is prime, we have $y \notin Rad(I_j)$. Assume that $k = \max\{2n_1 + 1, 2n_2 + 1, \dots, 2n_{j-1} + 1, 2n_{j+1} + 1, \dots, 2n_n + 1\}$, then $y^k \in I_i$ for every $i \neq j$ but $y^k \notin I_j$. Now $y^k x_j x_j \in I \cap I_i$ for every $i \neq j$ but $y^k x_j x_j \notin I \cap I_j$. Since $y^k x_j x_j \in I_j$ and $x_j^3 \notin I_j$, there exists an integer $n \in Z_0^+$ such that $(y^k)^{(2n+1)} \in I_j$, that is, $y \in Rad(I_j)$, a contradiction. Thus $y^k x_j x_j \in I \cap (\bigcap_{i \neq j}^n I_i)$ but $y^k x_j x_j \notin I \cap I_j$, which also contradicts $(\bigcap_{i \neq k} I_i) \cap I = (\bigcap_{i=1}^n I_i) \cap I$. Therefore I_j is not a quasi 3-primary ideal of S. ■

By using Lemma 4.43, we obtain the following Theorem.

Theorem 4.44. *Let I be an arbitrary ideal in a commutative ternary semiring S and $I_1, I_2, \ldots, I_n$ be k-ideals of S such that at least $n-2$ of which are quasi 3-primary ideals. If $\{I_1, I_2, \ldots, I_n\}$ be a cover of I and $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, then $I \subseteq I_i$ for some i.*

Proof. We may assume that the cover is efficient since the hypothesis remains valid if one reduces the covering to an efficient covering. Then $n \neq 2$. Since $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, by Lemma 4.43, we have $n < 2$. Therefore $n = 1$ and hence $I \subseteq I_i$ for some i. ■

Theorem 4.45. *Let S be a commutative ternary semiring and $I_1, I_2, \ldots, I_n$ be quasi 3-primary k-ideals of S such that $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for all $i \neq j$. Let I be an ideal of S such that $aSS + I \nsubseteq \bigcup_{i=1}^n I_i$ for some $a \in S$. Then there exists an element $c \in I$ such that $a + c \notin \bigcup_{i=1}^n I_i$.*

Proof. Assume that a lies in all of $I_1, I_2, \ldots, I_k$ but none of $I_{k+1}, \ldots, I_n$. If $k = 0$, then $a + 0 \notin \bigcup_{i=1}^n I_i$. So consider $k \geq 1$. Now $I \nsubseteq \bigcup_{i=1}^k Rad(I_i)$. If $I \subseteq \bigcup_{i=1}^k Rad(I_i)$, by Theorem 4.44, $I \subseteq Rad(I_i)$ for some $1 \leq i \leq k$, which contradicts the hypothesis that $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for all $i \neq j$. So there exists an element $p \in I$ such that $p \notin \bigcup_{i=1}^k Rad(I_i)$. Also, $I_{k+1} \cap \cdots \cap I_n \nsubseteq Rad(I_1) \cup Rad(I_2) \cup \cdots \cup Rad(I_k)$. If $I_{k+1} \cap \cdots \cap I_n \subseteq Rad(I_1) \cup Rad(I_2) \cup \cdots \cup Rad(I_k)$, then by Theorem 4.44, we get $I_{k+1} \cap \cdots \cap I_n \subseteq Rad(I_j)$ for some $1 \leqslant j \leqslant k$. Thus $(Rad(I_{k+1}))^{n-k} \cap \cdots \cap Rad(I_n) = Rad((I_{k+1})^{n-k} \cap \cdots \cap I_n) \subseteq Rad(I_{k+1} \cap \cdots \cap I_n) \subseteq Rad(I_j)$ and since $Rad(I_j)$ is a prime ideal of S, we conclude that $Rad(I_l) \subseteq Rad(I_j)$ for $k+1 \leqslant l \leqslant n$, so $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for $i \neq j$, which contradicts the hypothesis. Thus there exists $q \in I_{k+1} \cap \cdots \cap I_n$ such that $q \notin Rad(I_1) \cup Rad(I_2) \cup \cdots \cup Rad(I_k)$. Consider the element $c = ppq \in I$. Then $c \in I_{k+1} \cap \cdots \cap I_n$ but $c \notin I_1 \cup I_2 \cup \cdots \cup I_k$. If $c \in I_1 \cup I_2 \cup \cdots \cup I_k$, then $c = ppq \in I_i$ for some $1 \leqslant i \leqslant k$. Also $p^3 \notin I_i$. Since I_i is a quasi 3-primary ideal, there exists an integer $n \in Z_0^+$ such that $q^{2n+1} \in I_i$, a contradiction. Hence $c \in \bigcup_{j=k+1}^n I_j \setminus \bigcup_{i=1}^k I_i$. Again, as $a \in \bigcup_{i=1}^k I_i \setminus \bigcup_{j=k+1}^n I_j$, it follows that $a + c \notin \bigcup_{i=1}^n I_i$. ■

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