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SET-THEORETICAL SOLUTIONS FOR THE YANG-BAXTER EQUATION IN TRIANGLE ALGEBRAS

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Abstract

In this study, we give some fundamental set-theoretical solutions of Yang-Baxter equation in triangle algebras and state triangle algebras. We prove that the necessary and sufficient condition for certain mappings to be set-theoretical solutions of Yang-Baxter equation on these structures is that these structures must be also MTL-(state) triangle algebras, BL-(state) triangle algebras or RL-(state) triangle algebras. In accordance with these, we recursively introduce new operators $\widetilde{N}$ and $\mathfrak{M}$. Then, we define the notion of formula on triangle algebra as a classical logic structure. Moreover, we state the relationship of transferring of set-theoretical solutions of Yang-Baxter equation among (MTL,BL, RL)-(state) triangle algebras and state (MTL,BL, RL)-(state) triangle algebras. Then, we give a scheme to explain clearly these relations.

Keywords: triangle algebra, Yang-Baxter equation, set-theoretical solution, residuated lattice, state operator, (MTL,BL, RL)-triangle algebras.

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1. Introduction

Yang-Baxter equation was firstly introduced by the Nobel laureate C.N. Yang in theoretical physics [1] and by R.J. Baxter in statistical mechanics [2, 3]. The Yang–Baxter equation has been attracted more researchers' attention among in a wide range of disciplines such as knot theory, link invariants, quantum computing, braided categories, quantum groups, the analysis of integrable systems, quantum mechanics, etc. in recent years. Moreover, one of the uses of Yang-Baxter equations is pure mathematics; especially, finding set-theoretical solutions in algebraic structures. For example, Berceanu et al. examined algebraic structures arising from Yang–Baxter Systems [4]; Oner, Senturk et. al have constructed new set theoretical solutions for Yang–Baxter equation in MV-algebras [10], Belavin and Drinfeld have worked on solutions of the classical Yang–Baxter equation for simple Lie algebras [6], Senturk and Bozdağ handle Geometrical approach on set theoretical solutions of Yang-Baxter equation in Lie algebras [7], Massuyeau and Nichita consider the problem of constructing knot invariants from Yang-Baxter operators associated to (unitary associative) algebra structures [11], Gateva-Ivanova examined set theoretical solutions of the Yang–Baxter equation, braces and symmetric groups [5], Wang and Ma provide a new framework of obtaining singular solutions of the quantum Yang–Baxter equation by constructing weak quasi triangular structures [8] and Nichita and Parashar have studied Spectral-parameter dependent Yang-Baxter operators and Yang-Baxter systems from algebraic structures [9], and etc.

The notion of triangle algebras was firstly given by Van Gasse et al. as a variety of residuated lattices fitted with unary operators ν and μ together with a third angular point u that is different from 0 and 1. And also, It is shown that these algebras are served as an equational representation of interval-valued residuated lattices. These authors introduced triangle logic and proved that this logic is sound and complete corresponding with the variety of triangle algebras [12]. Moreover, triangle algebras are dissimilar to other algebraic structures. Therefore, triangle algebras have an important position in studying fuzzy logics and the related algebraic structures [16].

The notion of states on MV-algebras was introduced by Munduci [17]. It benefited from averaging processes for formulas in Łukasiewicz logics. It was not only a generalization of the usual probability measures on Boolean algebras but also a semantical interpreter of the probability of fuzzy events. An alternative approach to states on MV-algebras was given by Flamino and Montagna [18]. They put in a new unary operation σ to the language of MV-algebras, that preserves the usual properties of states. The notion of states has been used to other logical algebras. For example, Zahiri and Saeid introduced the concept of state triangle algebras [16].

In this work, this paper is organized as follows: In Section 2, we recall some notions, basic definitions, lemmas and their relevant results for Sheffer stroke basic algebras. In Section 3, we handle some fundamental set-theoretical solutions of Yang-Baxter equation on triangle algebras. We show that the necessary and sufficient condition for certain mappings to be set-theoretical solutions of Yang-Baxter equation on these structures is that these structures must be also MTL-triangle algebras, BL-algebras or RL-algebras. We constitute the set of $\nu\mu-$equal elements. Then, we define the notion of formulas on triangle algebras. In parallel with previous step, we build the set of $\nu\mu-$equal formulas. Then, we introduce recursively quasi-negation operator $\widetilde{N}$ on triangle algebras. We find equal formulas to each other by applying $\widetilde{N}$ operator. In the following step, we present a new operator $\mathfrak{M}$ to reproduce a formula by recursively rewriting elements and operations in reverse order. Moreover, we give some fundamental relations between these operators. Lastly, we show that if the the mapping $S(x, y)$ is a set-theoretical solution of the Yang-Baxter equation in triangle algebras and the equality $S(x, y) = S(N(y), N(x))$ is verified for all $x, y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(y, x)))$ and $\mathfrak{M}(S(y, x))$ are also set-theoretical solutions of the Yang-Baxter equation in triangle algebras. In section 4, we give some fundamental set-theoretical solutions of Yang-Baxter Equation in state triangle algebras. Then, we state the relationship of transferring of set-theoretical solutions of Yang-Baxter equation among (MTL,BL, RL)-triangle algebras and state (MTL,BL, RL)-triangle algebras. We give a scheme to explain clearly these relations. Then, we give some fundamental relations between the operators $\mathfrak{M}$ and $\widetilde{N}$ in triangle algebras. Moreover, we obtain that the mapping $S(x, y)$ is a the set-theoretical solution of the Yang-Baxter equation in state triangle algebras. If the equality $S(x, y) = S(N(\sigma(y)), N(\sigma(x)))$ for all $x, y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(\sigma(y),\ \sigma(x))))$ and $\mathfrak{M}(S(\sigma(y), \sigma(x)))$ are also set-theoretical solutions of the Yang-Baxter equation in state triangle algebras. Also, we give place to some attractive examples.

2. Preliminaries

In this section, we deal with some fundamental definitions, lemmas and propositions with reference to residuated lattices, triangle algebras, state triangle algebras and Yang-Baxter equation that will be used in the following sections.

Definition 1 [12]**.** A bounded commutative residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constants $0, 1$ such that:

- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
- operation $*$ is associative and commutative, with 1 is neutral element, and

- $a * b \leq c$ if and only if $a \leq b \to c$, for all $a, b, c \in L$.

The binary ordering relation $\leq$ and the unary negation operator $\neg$ are defined in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ as follows.

Definition 2 [12]**.** Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice and let $a, b \in L$. The binary relation $\leq$ defined on $\mathcal{L}$ as below

$$a \leq b \text{ if and only if } a \wedge b = a$$

is an order on $\mathcal{L}$.

Lemma 3 [12]**.** *Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice and let $a, b \in L$. Then the following conditions are equivalent to each other:*

(i) $a \wedge b = a$,

(ii) $a \vee b = b$,

(iii) $a \to b = 1$.

Definition 4 [12]**.** Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice and let $a, b \in L$. The unary operation $\neg$ defined on $\mathcal{L}$ as below

$$\neg a = a \to 0$$

is called negation a on $\mathcal{L}$.

Lemma 5 [19]**.** *Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice. Then the following equality verifies for each $a \in L$:*

$$a \to 0 = ((a \to 0) \to 0) \to 0.$$

Definition 6 [12]**.** Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice and let $a \in L$. The element a is called nilpotent element of L if $a^n = 0$ for some $n \in \mathbb{N}$.

Lemma 7 [13, 14]**.** *Let $\mathcal{L} = (L, \vee, \wedge, *, \to, 0, 1)$ be a residuated lattice. Then the following properties are verified for each $a, b, c \in L$:*

(i) $a \vee b \leq (a \to b) \to b$,

(ii) $a \to b \leq (a * c) \to (b * c)$,

(iii) $(a \to b) * (b \to c) \leq a \to c$,

(iv) If $a \leq b$, then $a * c \leq b * c$, $c \to a \leq c \to b$ and $b \to c \leq a \to c$,

(v) $a \to (b \to c) = b \to (a \to c) = (a * b) \to c$,

(vi) $a \to b \leq (b \to c) \to (a \to c)$.

Definition 8 [12]**.** Let $\mathcal{A} = (A, \vee, \wedge)$ be a lattice. The triangularization of $\mathcal{A}$, which is shown by $\mathbb{T}(\mathcal{A})$, is the structure $\mathbb{T}(\mathcal{A}) = (Int(\mathcal{A}), \vee, \wedge)$ defined by

- $Int(\mathcal{A}) = \{[a_1, a_2] | (a_1, a_2) \in A \times A \text{ and } a_1 \leq a_2\}$,
- $[a_1, a_2] \wedge [b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]$,
- $[a_1, a_2] \vee [b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2]$.

The set $D_{\mathcal{A}} = \{[a, a] | a \in A\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

Definition 9 [12]**.** A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \to, 0, 1)$ is a residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the following conditions:

$$
\begin{array}{ll}
(T_1)\nu a \leq a, & (T_1')a \leq \mu a, \\
(T_2)\nu a \leq \nu\nu a, & (T_2')\mu\mu a \leq \mu a, \\
(T_3)\nu(a \wedge b) = \nu a \wedge \nu b, & (T_3')\mu(a \wedge b) = \mu a \wedge \mu b, \\
(T_4)\nu(a \vee b) = \nu a \vee \nu b, & (T_4')\mu(a \vee b) = \mu a \vee \mu b, \\
(T_5)\nu u = 0, & (T_5')\mu u = 1, \\
(T_6)\nu\mu a = \mu a, & (T_6')\mu\nu a = \nu a, \\
(T_7)\nu(a \to b) \leq \nu a \to \nu b, & \\
(T_8)(\nu a \leftrightarrow \nu b) * (\mu a \leftrightarrow \mu b) \leq (a \leftrightarrow b), & \\
(T_9)\nu a \to \nu b \leq \nu(\nu a \to \nu b). &
\end{array}
$$

Proposition 10 [12]**.** *Suppose* $(A, \vee, \wedge, \to, 0, 1)$ *is a residuated lattice such that* $\neg$ *is involutive. If there exists an element* u *in* A *such that* $\neg u = u$*, if* ν *is a unary operator on* A *that satisfies* $(T_1) - (T_6)$*,* (T_8)*,* (T_9) *and if* $(\nu a \leftrightarrow \nu b) * (\nu\neg a \leftrightarrow \nu\neg b) \leq a \leftrightarrow b$*, then* $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ *is a triangle algebra if we define* $\mu a = \neg\nu\neg a$.

Definition 11 [15]**.** A triangle algebra A is called an $MTL-$triangle algebra if $(a \to b) \vee (b \to a) = 1$. An $MTL-$triangle algebra A is called $BL-$triangle algebra if $a \wedge b = a * (a \to b)$ for all $a, b \in A$.

Definition 12 [16]**.** A triangle algebra A is called an $RL-$triangle algebra if $a \wedge b = a * (a \to b)$ for all $a, b \in A$.

Proposition 13 [20]**.** *In a triangle algebra* $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$*, the implication* $\to$ *and the product* $*$ *are completely determined by their action on* $E(A)$ *and the value of* $u * u$*, where* $E(A) = \{a \in A | \nu a = a\}$*. More specifically:*

- $\nu(a \to b) = (\nu a \to \nu b) \wedge (\mu a \to \mu b)$.
- $\mu(a \to b) = (\mu a \to (\mu(u * u) \to \mu b)) \wedge (\nu a \to \mu b)$.
- $\nu(a * b) = \nu a * \nu b$.
- $\mu(a * b) = (\nu a * \mu b) \vee (\mu a * \nu b) \vee (\mu a * \mu b * \mu(u * u))$.

Definition 14 [16]**.** Let A be triangle algebra. A mapping $\sigma : A \to A$ such that, for all $x, y \in A$, we have

(ST_1) $\sigma(0) = 0$,

(ST_2) $\sigma(x \to y) = \sigma(x) \to \sigma(x \vee y)$,

(ST_3) $\sigma(x * y) = \sigma(x) * \sigma(x \to (x * y))$,

(ST_4) $\sigma(\sigma(x) * \sigma(y)) = \sigma(x) * \sigma(y)$,

(ST_5) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$,

(ST_6) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$,

(ST_7) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$,

(ST_8) $\sigma(\nu x) = \nu(\sigma(x))$,

(ST_9) $\sigma(\mu x) = \mu(\sigma(x))$

is said to be a state operator on A and the pair (A, σ) is said to be a state triangle algebra.

Lemma 15 [16]**.** *Let (A, σ) be state triangle algebra. Then, we have following statements for all $x, y \in A$:*

(i) $\sigma(1) = 1$,

(ii) $\sigma(\neg x) = \neg\sigma(x)$,

(iii) *If $x \leq y$, then $\sigma(x) \leq \sigma(y)$,*

(iv) $\sigma(\sigma(x)) = \sigma(x)$,

(v) $\sigma(A) = \{x \in A | x = \sigma(x)\}$,

(vi) $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ *if and only if* $\sigma(y \to x) = \sigma(y) \to \sigma(x)$,

(vii) *if A is linear and faithful, then $\sigma(x) = x$ for all $x \in A$.*

Let F be a field where tensor products are defined and W be a F-space. The mapping over $W \otimes W$ is denoted by δ. The twist map on this structure is given by $\delta(w_1 \otimes w_2) = w_2 \otimes w_1$ and the identity map on F is defined by $I : W \to W$; for $S : W \otimes W \to W \otimes W$ a $F-$linear map, let $S^{12} = S \otimes I$, $S^{13} = (I \otimes \delta)(S \otimes I)(\delta \otimes I)$ and $S^{23} = I \otimes S$.

Definition 16 [21]**.** A Yang-Baxter operator is an invertible $F-$linear map $S : W \otimes W \to W \otimes W$ that verifies the braid condition (called the Yang–Baxter equation):

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}. \tag{1}$$

If S verifies Equation (1), then both $S \circ \mu$ and $\mu \circ S$ verify the quantum Yang–Baxter equation (QYBE):

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}. \tag{2}$$

Lemma 17 [21]**.** *Equations (1) and (2) are equivalent to each other.*

To construct set-theoretical solutions of Yang–Baxter equation in Lie algebras, we need the following definition.

Definition 18 [21]**.** Let A be a set and $S : A \times A \to A \times A$, $S(a_1, a_2) = (a_1', a_2')$ be a map. The map S is set-theoretical solution of Yang–Baxter equation if it verifies the following equality:

$$S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}, \tag{3}$$

which is also equivalent to

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12}, \tag{4}$$

where

$$\begin{aligned} S^{12} : A^3 \to A^3, &\quad S^{12}(a_1, a_2, a_3) = (a_1', a_2', a_3), \\ S^{23} : A^3 \to A^3, &\quad S^{23}(a_1, a_2, a_3) = (a_1, a_2', a_3'), \\ S^{13} : A^3 \to A^3, &\quad S^{13}(a_1, a_2, a_3) = (a_1', a_2, a_3'). \end{aligned}$$

3. Set-Theoretical Solutions for Yang-Baxter Equation in Triangle Algebras

In this part of this paper, we handle some fundamental set-theoretical solutions of Yang-Baxter equation on triangle algebras. We show that the necessary and sufficient condition for certain mappings to be set-theoretical solutions of Yang-Baxter

equation on these structures is that these structure must be also MTL-triangle algebras, BL-triangle algebras or RL-triangle algebras. We constitute the set of $\nu\mu-$equal elements. Then, we define the notion of formulas on triangle algebras. In parallel with previous step, we build the set of $\nu\mu-$equal formulas. Then, we introduce recursively quasi-negation operator $\widetilde{N}$ on triangle algebras. We find equal formulas to each other by applying $\widetilde{N}$ operator. In the following step, we present a new operator $\mathfrak{M}$ to reproduce a formula by recursively rewriting elements and operations in reverse order. Moreover, we give some fundamental relations between these operators. Lastly, we show that if the the mapping $S(x, y)$ is a set-theoretical solution of the Yang-Baxter equation in triangle algebras and the equality $S(x, y) = S(N(y), N(x))$ is verified for all $x, y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(y, x)))$ and $\mathfrak{M}(S(y, x))$ are also set-theoretical solutions of the Yang-Baxter equation in triangle algebras.

Lemma 19. *Let $\mathcal{A}$ be a residuated lattice. Then the following mappings are set-theoretical solutions of Yang–Baxter equation where u is constant (uncertainty, $u \neq 0$, $u \neq 1$):*

(i) if S is an identity map,

(ii) if $S(a, b) = (0, 0)$,

(iii) if $S(a, b) = (1, 1)$,

(iv) if $S(a, b) = (u, u)$,

(v) if $S(a, b) = (a, u)$,

(vi) if $S(a, b) = (u, b)$,

(vii) if $S(a, b) = (a, 0)$,

(viii) if $S(a, b) = (a, 1)$,

(ix) if $S(a, b) = (0, b)$,

(x) if $S(a, b) = (1, b)$.

Proof. The proof of parts $(i) - (iv)$ are straightforward.

(v) Let S^{12} and S^{23} be defined as follows:

$$S^{12}(a_1, a_2, a_3) = (a_1, u, a_3),$$
$$S^{23}(a_1, a_2, a_3) = (a_1, a_2, u).$$

We satisfy the equation $S^{12} \circ S^{23} \circ S^{12} = S^{23} \circ S^{12} \circ S^{23}$, for all $(a_1, a_2, a_3) \in A^3$.

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(a_1, a_2, a_3) &= S^{12}(S^{23}(S^{12}(a_1, a_2, a_3))) \\
&= S^{12}(S^{23}(a_1, u, a_3)) \\
&= S^{12}(a_1, u, u) \\
&= (a_1, u, u) \\
&= S^{23}(a_1, u, u) \\
&= S^{23}(S^{12}(a_1, a_2, u)) \\
&= S^{23}(S^{12}(S^{23}(a_1, a_2, a_3))) \\
&= (S^{23} \circ S^{12} \circ S^{23})(a_1, a_2, a_3).
\end{aligned}$$

Therefore, $S(a, b) = (a, u)$ is a set-theoretical solution of Yang-Baxter equation in triangle algebras. The proof of parts (vi)–(x) is similar to proof of (v). ■

Let $\mathcal{A}$ be a triangle algebra. The mapping N and N_u are defined on A as

$$N(x) := x \to 0 \quad \text{and} \quad N_u(x) := x \to u$$

for each $x \in A$.

Lemma 20. *Let $\mathcal{A}$ be a residuated lattice. The mapping N and N_u are antitone on A.*

Proof. Let $x, y \in A$ and $x \leq y$. By the help of Lemma 7 (iv), we obtain $y \to 0 \leq x \to 0$. So, we have $N(y) \leq N(x)$. Then, N is an antitone mapping.

Similarly, we get that N_u is also an antitone mapping. ■

Lemma 21. *Let $\mathcal{A}$ be a residuated lattice. The mappings $S_1(x, y) = (N(y), N(x))$ and $S_2(x, y) = (N_u(y), N_u(x))$ verify the braid condition on this structure. Then, they are set-theoretical solutions of the Yang-Baxter equation on $\mathcal{A}$.*

Lemma 22. *Let $\mathcal{A}$ be a residuated lattice and let $a = N(x)$ and $b = N(N(y))$. Then the following identity holds:*

$$N(a) \to N(b) = b \to a.$$

Proof. Assume that $a, b \in A$ such that $a = N(x)$ and $b = N(N(y))$. By using Lemma 7, we obtain

$$\begin{aligned}
N(a) \to N(b) &= N(N(x)) \to N(N(N(y))) \\
&= ((x \to 0) \to 0) \to (((y \to 0) \to 0) \to 0) \\
&= ((y \to 0) \to 0) \to (((x \to 0) \to 0) \to 0) \\
&= b \to a.
\end{aligned}$$

■

Lemma 23. *Let $\mathcal{A}$ be a residuated lattice. The mappings*

(i) $S(x,y) = (1, x \wedge y)$

(ii) $S(x,y) = (x \wedge y, 1)$

(iii) $S(x,y) = (x \vee y, 0)$

(iv) $S(x,y) = (0, x \vee y)$,

(v) $S(x,y) = (x \vee y, x \wedge y)$

(v) $S(x,y) = (x \wedge y, x \vee y)$

verify the braid condition on this structure. Therefore, they are set-theoretical solutions of Yang-Baxter equation on residuated lattices.

Proposition 24. *If a mapping verifies the braid condition on residuated lattices, then it also verifies the braid condition on triangle algebras.*

Proposition 25. *Let $\mathcal{A}$ be a triangle algebra. The mappings*

(i) $S(x,y) = (\nu(x \vee y), \nu(x \wedge y))$,

(ii) $S(x,y) = (\mu(x \vee y), \mu(x \wedge y))$,

(iii) $S(x,y) = (\nu(x \wedge y), \nu(x \vee y))$,

(iv) $S(x,y) = (\mu(x \wedge y), \mu(x \vee y))$

verify the braid condition on this structure. Therefore, they are set-theoretical solutions of Yang-Baxter equation on $\mathcal{A}$.

Lemma 26. *Let $\mathcal{A}$ be a triangle algebra. The mappings*

(i) $S(x,y) = ((x \to y) \to (y \to x), 1)$

(ii) $S(x,y) = (1, (x \to y) \to (y \to x))$

verify the braid condition on this structure if and only if $\mathcal{A}$ is also an $MTL-$triangle algebra.

Proof. (i) ($\Rightarrow$:) Assume that the mapping $S(x,y) = ((x \to y) \to (y \to x), 1)$ verifies the braid condition on this structure. Then, we have

$$\begin{aligned}
& (S^{12} \circ S^{23} \circ S^{12})(x,y,z) \\
= \; & S^{12}(S^{23}(S^{12}(x,y,z))) \\
= \; & S^{12}(S^{23}((x \to y) \to (y \to x), 1, z)) \\
= \; & S^{12}((x \to y) \to (y \to x), (1 \to z) \to (z \to 1), 1) \\
= \; & S^{12}((x \to y) \to (y \to x), 1, 1) \\
= \; & (((x \to y) \to (y \to x) \to 1) \to (1 \to (x \to y) \to (y \to x)), 1, 1) \\
= \; & ((x \to y) \to (y \to x), 1, 1) \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
& (S^{23} \circ S^{12} \circ S^{23})(x,y,z) \\
= \; & S^{23}(S^{12}(S^{23}(x,y,z))) \\
= \; & S^{23}(S^{12}(x, (y \to z) \to (z \to y), 1)) \\
= \; & S^{23}((x \to ((y \to z) \to (z \to y))) \to (((y \to z) \to (z \to y)) \to x), 1, 1) \\
= \; & ((x \to ((y \to z) \to (z \to y))) \to (((y \to z) \to (z \to y)) \to x), 1, 1). \qquad (6)
\end{aligned}$$

Since this mapping satisfy the braid condition on this structure, the Equation (5) and the Equation (6) must be equal to each other. This condition is only verified when the equation $(a \to b) \vee (b \to a) = 1$ is correct for each $a, b \in A$. By Definition 11, we obtain that the structure $\mathcal{A}$ is also an $MTL-$triangle algebra.

($\Leftarrow$:) Assume that $\mathcal{A}$ is also an $MTL-$triangle algebra. Then, we have $(a \to b) \vee (b \to a) = 1$ for each $a, b \in A$. So, we get $S(x,y) = ((x \to y) \to (y \to x), 1) = (1,1)$. Then, it easily show that the mapping $S(x,y)$ is verified braid condition on $\mathcal{A}$.

(ii) We also obtain same conclusion for the mapping $S(x,y) = (1, (x \to y) \to (y \to x))$ by using similar procedure in (i). ■

Proposition 27. *Let $\mathcal{A}$ be a triangle algebra. The mappings*

(i) $S(x,y) = (\nu y, \nu x)$,

(ii) $S(x,y) = (\mu y, \mu x)$

verify the braid condition on this structure. Therefore, they are set-theoretical solutions of Yang-Baxter equation on $\mathcal{A}$.

Proof. (i) We show that this mapping satisfies the braid condition on this structure. Then, we get the following equalities:

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\
&= S^{12}(S^{23}(\nu y, \nu x, z)) \\
&= S^{12}(\nu y, \nu z, \nu\nu x) \\
&= (\nu\nu z, \nu\nu y, \nu\nu x) \\
&= S^{23}(\nu\nu z, \nu x, \nu y) \\
&= S^{12}(S^{23}(x, \nu z, \nu y)) \\
&= S^{23}(S^{12}(S^{23}(x,y,z))) = (S^{23} \circ S^{12} \circ S^{23})(x,y,z).
\end{aligned}$$

(ii) By the help of similar method in (i), we can show that the mapping $S(x,y) = (\mu y, \mu x)$ satisfies the braid condition. ■

Using Lemma 23, Definition 9 and Proposition 27, we obtain Proposition 25.

Lemma 28. *Let $\mathcal{A}$ be a triangle algebra. The mappings*

(i) $S(x,y) = (x * (x \to y), 1)$,

(ii) $S(x,y) = (1, x * (x \to y))$,

(iii) $S(x,y) = (y * (y \to x), 1)$,

(iv) $S(x,y) = (1, y * (y \to x))$,

(v) $S(x,y) = (\nu x * (\nu x \to \nu y), 1)$,

(vi) $S(x,y) = (1, \nu x * (\nu x \to \nu y))$,

(vii) $S(x,y) = (\nu y * (\nu y \to \nu x), 1)$,

$(viii)$ $S(x,y) = (1, \nu y * (\nu y \to \nu x))$,

(ix) $S(x,y) = (\mu x * (\mu x \to \mu y), 1)$,

(vi) $S(x,y) = (1, \mu x * (\mu x \to \mu y))$,

(vii) $S(x,y) = (\mu y * (\mu x \to \mu x), 1)$,

$(viii)$ $S(x,y) = (1, \mu y * (\mu x \to \mu x))$

verify the braid condition on this structure if and only if the structure $\mathcal{A}$ is also an $RL-$triangle algebra or a $BL-$triangle algebra.

Proof. (i) ($\Rightarrow$:) Assume that the mapping $S(x,y) = (x * (x \to y), 1)$ verifies the braid condition on this structure. Then, we have

$$\begin{aligned} & (S^{12} \circ S^{23} \circ S^{12})(x,y,z) \\ = \; & S^{12}(S^{23}(S^{12}(x,y,z))) \\ = \; & S^{12}(S^{23}(x * (x \to y), 1, z)) \\ = \; & S^{12}(x * (x \to y), 1 * (1 \to z), 1) \\ = \; & ((x * (x \to y)) * ((x * (x \to y)) \to (1 * (1 \to z))), 1, 1) \end{aligned} \tag{7}$$

and

$$\begin{aligned} & (S^{23} \circ S^{12} \circ S^{23})(x,y,z) \\ = \; & S^{23}(S^{12}(S^{23}(x,y,z))) \\ = \; & S^{23}(S^{12}(x, y * (y \to z), 1)) \\ = \; & S^{23}(x * (x \to (y * (y \to z))), 1, 1) \\ = \; & (x * (x \to (y * (y \to z))), 1, 1). \end{aligned} \tag{8}$$

Since this mapping satisfy the braid condition on this structure, the Equation (7) and the Equation (8) must be equal to each other. This condition is only verified when the equation $a \wedge b = a * (a \to b)$ is correct for each $a, b \in A$. By Definition Definition 11 and Definition 12, we obtain that the structure $\mathcal{A}$ is also an $RL-$triangle algebra or $BL-$triangle algebra.

($\Leftarrow$:) Let the structure $\mathcal{A}$ be also an $RL-$triangle algebra or $BL-$triangle algebra. Then, we have $a \wedge b = a * (a \to b)$ for each $a, b \in A$. So, we get $S(x,y) = (x * (x \to y), 1) = (x \wedge y, 1)$. By the help of Lemma 23, it easily show that the mapping $S(x,y)$ is verified braid condition on $\mathcal{A}$.

(ii)–(viii): Since the proof of these parts follow a similar procedure to the proof of (i), we omit them. ■

In a triangle algebra, the model operators ν and μ correspond to the necessity and possibility, respectively. By taking into consideration these, we can define a set of formulas which are equal under the necessity and possibility operators in triangle algebras.

Definition 29. Let $\mathcal{A}$ be a triangle algebra. For an element $x \in A$, if $\nu x = \mu x$, then it is called $\nu\mu - equal$ element. The set of $\nu\mu - equal$ elements is illustrated with the following set:

$$E_{\nu\mu} =: \{x \in A | \nu x = \mu x\}.$$

During this paper, a formula is said to be atomic on triangle algebra, if it contains no connectives.

Lemma 30. *If the formula ψ is not atomic on triangle algebra, then there are formulas $\phi \to \theta$, $\phi * \theta$, $\phi \vee \theta$, $\phi \wedge \theta$, $\nu(\phi)$ or $\mu(\phi)$ which are equal to ψ such that $\phi, \theta \in \mathcal{A}$.*

Example 31. Let $\mathcal{A}$ be a triangle algebra. Then,

- $\psi(x,y) := x$, $\psi(x,y) := y$, $\psi(x,y) := u$ or $\psi(x,y) := 1$ and etc. are atomic formulas because there is no operator in them.
- $\psi(x,y) := x \to y$, $\psi(x,y) := x \to (y * x)$, $\psi(x,y) := 0 \to 1$ or $\psi(x,y) := \mu y * (\mu x \to \mu x)$ and etc. are not atomic formulas because they contain operators.

For a formula $\psi \in A$, if $\nu\psi = \mu\psi$, then it is called $\nu\mu - equal$ formula. The set of $\nu\mu - equal$ formulas is illustrated with the following set:

$$E_{\nu\mu}^{FOR} =: \{\psi \in A | \nu\psi = \mu\psi\}.$$

Lemma 32. *Let $\mathcal{A}$ be a triangle algebra. If $x \in E_{\nu\mu}$, then $x = \nu x = \mu x$. And also, if $\psi \in E_{\nu\mu}^{FOR}$, then $\psi = \nu\psi = \mu\psi$.*

Proposition 33. *Let $\mathcal{A}$ be a triangle algebra. The mapping $S(x,y) = (\mu\psi(y), \nu\phi(x))$ verifies the braid condition on this structure where $\nu\phi(\mu\psi(y)) = \mu\psi(\nu\phi(y))$ for each $y \in A$. Therefore, this mapping is a set-theoretical solution of Yang-Baxter equation on $\mathcal{A}$.*

Proof. We show that this mapping satisfies the braid condition on this structure. Then, we obtain the following eqaulities:

$$\begin{aligned}
(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\
&= S^{12}(S^{23}(\mu\psi(y), \nu\phi(x), z)) \\
&= S^{12}(\mu\psi(y), \mu\psi(z), \nu\phi(\nu\phi(x))) \\
&= (\mu\psi(\mu\psi(z)), \nu\phi(\mu\psi(y)), \nu\phi(\nu\phi(x))) \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}
(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\
&= S^{23}(S^{12}(x, \mu\psi(z), \nu\phi(y))) \\
&= S^{23}(\mu\psi(\mu\psi(z)), \nu\phi(x), \nu\phi(y)) \\
&= (\mu\psi(\mu\psi(z)), \mu\psi(\nu\phi(y)), \nu\phi(\nu\phi(x))). \qquad (10)
\end{aligned}$$

By the hypothesis, we have $\nu\phi(\mu\psi(y)) = \mu\psi(\nu\phi(y))$ for each $y \in A$. Therefore, we verify the braid condition from the Equality (9) and (10). As a result, this mapping is a set-theoretical solution of Yang-Baxter equation on $\mathcal{A}$. ∎

Proposition 34. *Let $\mathcal{A}$ be a triangle algebra. The mappings $(i)-(ii)$ in Proposition 27 and the following mappings*

(iii) $S(x,y)=(\nu y,\mu x)$,

(iv) $S(x,y)=(\mu y,\nu x)$

verify the braid condition on this structure where $x,y\in E_{\nu\mu}$. Therefore, the Yang-Baxter equation has a set-theoretical solution in triangle algebra.

Proof. It is clear from Proposition 27 and Lemma 32. ■

Proposition 35. *Let $\mathcal{A}$ be a triangle algebra. The mappings $(i)-(iv)$ in Proposition 25 and the following mappings*

(v) $S(x,y)=(\nu(x\vee y),\mu(x\wedge y))$,

(vi) $S(x,y)=(\mu(x\vee y),\nu(x\wedge y))$,

(vii) $S(x,y)=(\nu(x\wedge y),\mu(x\vee y))$,

$(viii)$ $S(x,y)=(\mu(x\wedge y),\nu(x\vee y))$

verify the braid condition on this structure where $x,y\in E_{\nu\mu}$. Therefore, they are set-theoretical solutions of Yang-Baxter equation on triangle algebras.

Let $\mathcal{A}$ be a triangle algebra. The mapping N^{FOR} are defined as

$$N^{FOR}(\psi):=\psi\to 0$$

for each formula ψ on $\mathcal{A}$.

Theorem 36. *Let $\mathcal{A}$ be a triangle algebra. For all formula ψ on $\mathcal{A}$, we have $\nu(N^{FOR}(\psi))\leq N^{FOR}(\nu(\psi))$.*

Proof. Assume that ψ be a formula on $\mathcal{A}$. Then, we have following inequality by the help of Definition 9:

$$\begin{aligned}\nu(N^{FOR}(\psi)) &= \nu(\psi\to 0)\\ &\leq \nu(\psi)\to\nu 0\\ &= \nu(\psi)\to 0\\ &= N^{FOR}(\nu(\psi)).\end{aligned}$$

■

Definition 37. Let $\mathcal{A}$ be a triangle algebra. The unary quasi-negation operator $\widetilde{N}$ is recursively defined as follows:

$$\widetilde{N}(\psi) = \begin{cases} N(\psi), & \text{if } \psi \text{ is atomic,} \\ \widetilde{N}(\phi) \to \widetilde{N}(\theta), & \text{if } \psi = \theta \to \phi, \\ \widetilde{N}(\phi) * \widetilde{N}(\theta), & \text{if } \psi = \theta * \phi, \\ \widetilde{N}(\phi) \wedge \widetilde{N}(\theta), & \text{if } \psi = \theta \wedge \phi, \\ \widetilde{N}(\phi) \vee \widetilde{N}(\theta), & \text{if } \psi = \theta \vee \phi, \\ \nu(\widetilde{N}(\phi)), & \text{if } \psi = \nu(\phi), \\ \mu(\widetilde{N}(\phi)), & \text{if } \psi = \mu(\phi) \end{cases}$$

for any formula ψ, ϕ and θ on $\mathcal{A}$.

Example 38. Let $\mathcal{A}$ be a triangle algebra. The formula $\psi(x, y) = (\mu a \to (\mu(u * u) \to \mu b)) \wedge (\nu a \to \mu b)$ is given. Then, we have the following equality by the Definition 37:

$$\begin{aligned} & \widetilde{N}(\psi(x, y)) \\ = \; & \widetilde{N}((\mu a \to (\mu(u * u) \to \mu b)) \wedge (\nu a \to \mu b)) \\ = \; & \widetilde{N}(\nu a \to \mu b) \wedge \widetilde{N}(\mu a \to (\mu(u * u) \to \mu b)) \\ = \; & (\widetilde{N}(\mu b) \to \widetilde{N}(\nu a)) \wedge (\widetilde{N}(\mu(u * u) \to \mu b) \to \widetilde{N}(\mu a)) \\ = \; & (\mu(\widetilde{N}(b)) \to \nu(\widetilde{N}(a))) \wedge ((\widetilde{N}(\mu b) \to \widetilde{N}(\mu(u * u))) \to \mu(\widetilde{N}(a))) \\ = \; & (\mu(\widetilde{N}(b)) \to \nu(\widetilde{N}(a))) \wedge ((\mu(\widetilde{N}(b)) \to \mu(\widetilde{N}(u * u))) \to \mu(\widetilde{N}(a))) \\ = \; & (\mu(\widetilde{N}(b)) \to \nu(\widetilde{N}(a))) \wedge ((\mu(\widetilde{N}(b)) \to \mu(\widetilde{N}(u) * \widetilde{N}(u))) \to \mu(\widetilde{N}(a))). \end{aligned}$$

Theorem 39. *Let $\mathcal{A}$ be a triangle algebra. The mapping N^{FOR} is an antitone on $\mathcal{A}$ but the mapping $\widetilde{N}$ is not an antitone on $\mathcal{A}$.*

Proof. Let ψ and θ be two formulas on $\mathcal{A}$. Assume that $\psi \leq \theta$. Then, we have $\psi \to \theta = 1$. By using Lemma 3 and Lemma 7, we get following result:

$$\begin{aligned} 1 \geq N^{FOR}(\theta) \to N^{FOR}(\psi) &= (\theta \to 0) \to (\psi \to 0) \\ &\geq \psi \to \theta = 1. \end{aligned}$$

Then, we get $N^{FOR}(\theta) \to N^{FOR}(\psi) = 1$. This means that $N^{FOR}(\theta) \leq N^{FOR}(\psi)$. So, the mapping N^{FOR} is an antitone.

Let ψ and θ be two formulas on $\mathcal{A}$ such that $\psi \leq \theta$. Then, we have $\psi \to \theta = 1$. Assume that $\widetilde{N}(\theta) \leq \widetilde{N}(\psi)$. We attain $\widetilde{N}(\theta) \to \widetilde{N}(\psi) = 1$. By using Definition 37, we get $\widetilde{N}(\psi \to \theta) = \widetilde{N}(1) = 1$. This is contradiction since $\widetilde{N}(1) = N(1) = 0 \neq 1$. So, the mapping $\widetilde{N}$ is not an antitone. ■

Definition 40. Let $\mathcal{A}$ be a triangle algebra. The binary quasi-negation operator $\widetilde{N}$ is defined as follows:

$$\widetilde{N}(\phi, \theta) = (\widetilde{N}(\phi), \widetilde{N}(\theta))$$

holds for any formulas $\phi, \theta \in \mathcal{A}$.

Lemma 41. *Let $\mathcal{A}$ be a triangle algebra. The mapping $S(x, y) = \widetilde{N}(1 \to y, 1 \to x)$ verifies the braid condition on this structure. Therefore, this mapping is a set-theoretical solution of Yang-Baxter equation on $\mathcal{A}$.*

Proof. Using Definition 40 and Definition 37, we get

$$S(x, y) = \widetilde{N}(1 \to x, 1 \to y) = ((x \to 0) \to 0, (y \to 0) \to 0).$$

We show that this mapping verifies the braid condition. Then, we have

$$\begin{aligned}
& (S^{12} \circ S^{23} \circ S^{12})(x, y, z) \\
= \; & S^{12}(S^{23}(S^{12}(x, y, z))) \\
= \; & S^{12}(S^{23}((x \to 0) \to 0, (y \to 0) \to 0, z)) \\
= \; & S^{12}((x \to 0) \to 0, (((y \to 0) \to 0) \to 0) \to 0, (z \to 0) \to 0) \\
= \; & ((((x \to 0) \to 0) \to 0) \to 0, (((((y \to 0) \to 0) \to 0) \to 0) \to 0) \to 0, \\
& (z \to 0) \to 0) \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
& (S^{23} \circ S^{12} \circ S^{23})(x, y, z) \\
= \; & S^{23}(S^{12}(S^{23}(x, y, z))) \\
= \; & S^{23}(S^{12}(x, (y \to 0) \to 0, (z \to 0) \to 0)) \\
= \; & S^{23}((x \to 0) \to 0, (((y \to 0) \to 0) \to 0) \to 0, (z \to 0) \to 0) \\
= \; & ((x \to 0) \to 0, (((((x \to 0) \to 0) \to 0) \to 0) \to 0) \to 0, \\
& (((z \to 0) \to 0) \to 0) \to 0). \qquad (12)
\end{aligned}$$

By the help of Lemma 5, we get the equality of (11) and (12). So, it verifies the braid condition on this structure. Therefore, this mapping is a set-theoretical solution of Yang-Baxter equation on $\mathcal{A}$. ■

Example 42. Let $\mathcal{A}$ be a triangle algebra. The mappings

(*i*) $S_1(x, y) = (1 \to x, 1 \to y)$

(*ii*) $S_2(x, y) = ((x \to 0) \to y, 0)$

are two set-theoretical solutions of the Yang-Baxter equation in triangle algebras. The mapping

$$\begin{aligned} T_1(x,y) &= \widetilde{N}(S_1(x,y)) \\ &= \widetilde{N}(1 \to x, 1 \to y) \\ &= (\widetilde{N}(1 \to y), \widetilde{N}(1 \to x)) \\ &= ((y \to 0) \to 0, (x \to 0) \to 0) \end{aligned}$$

is also a set-theoretical solution of the Yang-Baxter equation in triangle algebras. We show that the mapping $T_1(x,y)$ verifies the braid condition on this structure:

$$\begin{aligned} & (T_1^{12} \circ T_1^{23} \circ T_1^{12})(x,y,z) \\ = \; & T_1^{12}(T_1^{23}(T_1^{12}(x,y,z))) \\ = \; & T_1^{12}(T_1^{23}(((y \to 0) \to 0, (x \to 0) \to 0, z)) \\ = \; & T_1^{12}((y \to 0) \to 0, (z \to 0) \to 0, (((x \to 0) \to 0) \to 0) \to 0) \\ = \; & ((((z \to 0) \to 0) \to 0) \to 0, (((y \to 0) \to 0) \to 0) \to 0, \\ & (((x \to 0) \to 0) \to 0) \to 0) \end{aligned} \tag{13}$$

and

$$\begin{aligned} & (T_1^{23} \circ T_1^{12} \circ T_1^{23})(x,y,z) \\ = \; & T_1^{23}(T_1^{12}(T_1^{23}(x,y,z))) \\ = \; & T_1^{23}(T_1^{12}((x, (z \to 0) \to 0, (y \to 0) \to 0)) \\ = \; & T_1^{23}((((z \to 0) \to 0) \to 0) \to 0, (x \to 0) \to 0, (y \to 0) \to 0) \\ = \; & ((((z \to 0) \to 0) \to 0) \to 0, (((y \to 0) \to 0) \to 0) \to 0, \\ & (((x \to 0) \to 0) \to 0) \to 0) \end{aligned} \tag{14}$$

Since the Equation (13) and the Equation (14) are equal to each other, the mapping $T_1(x,y)$ verifies the braid condition on this structure. But the mapping

$$\begin{aligned} T_2(x,y) &= \widetilde{N}(S_2(x,y)) \\ &= \widetilde{N}((x \to 0) \to y, 0) \\ &= (\widetilde{N}(0), \widetilde{N}((x \to 0) \to y)) \\ &= (1, (y \to 0) \to (x \to 0)) \end{aligned}$$

is not a set-theoretical solution of the Yang-Baxter equation in triangle algebras. Because, we get

$$\begin{aligned}
& (T_2^{12} \circ T_2^{23} \circ T_2^{12})(x,y,z) \\
= \; & T_2^{12}(T_2^{23}(T_2^{12}(x,y,z))) \\
= \; & T_2^{12}(T_2^{23}((1,(y \to 0) \to (x \to 0), z)) \\
= \; & T_2^{12}(1,1,(z \to 0) \to (((y \to 0) \to (x \to 0)) \to 0)) \\
= \; & (1,1,(z \to 0) \to (((y \to 0) \to (x \to 0)) \to 0)) \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
& (T_2^{23} \circ T_2^{12} \circ T_2^{23})(x,y,z) \\
= \; & T_2^{23}(T_2^{12}(T_2^{23}(x,y,z))) \\
= \; & T_2^{23}(T_2^{12}((x,1,(z \to 0) \to (y \to 0))) \\
= \; & T_2^{23}(1,(1 \to 0) \to (x \to 0),(z \to 0) \to (y \to 0)) \\
= \; & T_2^{23}(1,1,(z \to 0) \to (y \to 0)) \\
= \; & (1,1,(((z \to 0) \to (y \to 0)) \to 0) \to (1 \to 0)). \qquad (16)
\end{aligned}$$

Since the Equation (15) and the Equation (16) are not equal to each other, the mapping $T_2(x,y)$ does not verify the braid condition.

Lemma 43. *Let $\mathcal{A}$ be a triangle algebra and let ψ be any formula from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then the following identity*

$$\widetilde{N}(\psi(\widetilde{N}(x,y))) = \begin{cases} \psi(y,x), & \text{if } \psi \text{ is atomic,} \\ \psi(N(y),N(x)), & \text{if } \psi \in \{\theta \to \phi, \theta * \phi, \theta \wedge \phi, \theta \vee \phi, \nu(\phi), \mu(\phi)\} \\ & \text{such that } \theta \text{ and } \phi \text{ are any atomic formulas,} \end{cases}$$

is satisfied for each $(x,y) \in A \times A$.

Proof. Let x, y and k be any three elements in A. Assume that ψ is an atomic formula from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then $\psi(x,y) = x$ (first projection mapping), $\psi(x,y) = y$ (second projection mapping) or $\psi(x,y) = k$ (constant mapping). We assume that $\psi(x,y) = x$. Then we have

$$\begin{aligned}
\widetilde{N}(\psi(\widetilde{N}(x,y))) &= \widetilde{N}(\psi(\widetilde{N}(y),\widetilde{N}(x))) \\
&= \widetilde{N}(\psi(N(y),N(x))) \\
&= \widetilde{N}(N(y)) \\
&= N(N(y)) \\
&= y \\
&= \psi(y,x).
\end{aligned}$$

For other parts of atomic formulas could be similarly obtained.

Let (x, y) be any element in $A \times A$ and ψ be a formula from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. We assume that ψ is not atomic formula, then we have $\psi \in \{\theta \to \phi, \theta * \phi, \theta \wedge \phi, \theta \vee \phi, \nu(\phi), \mu(\phi)\}$. Assume that $\psi = \phi \to \theta$. Then, we get

$$\begin{aligned}
\widetilde{N}(\psi(\widetilde{N}(x,y))) &= \widetilde{N}(\psi(\widetilde{N}(y), \widetilde{N}(x))) \\
&= \widetilde{N}(\psi(N(y), N(x))) \\
&= \widetilde{N}(\phi(N(y), N(x)) \to \theta(N(y), N(x))) \\
&= N(\theta(N(y), N(x))) \to N(\phi(N(y), N(x))) \\
&= \phi(N(y), N(x)) \to \theta(N(y), N(x)) \\
&= \psi(N(y), N(x)).
\end{aligned}$$

Similarly, we obtain $\widetilde{N}(\psi(\widetilde{N}(x,y))) = \psi(N(y), N(x))$ where $\psi \in \{\theta \to \phi, \theta * \phi, \theta \wedge \phi, \theta \vee \phi, \nu(\phi), \mu(\phi)\}$. ■

Definition 44. Let $\mathcal{A}$ be a triangle algebra. Then the mapping $\mathfrak{M}$ is defined on A as follows

$$\mathfrak{M}(\psi(x,y)) = \begin{cases}
\psi(y,x), & \text{if } \psi \text{ is atomic,} \\
\mathfrak{M}(\theta(y,x)) \to \mathfrak{M}(\phi(y,x)), & \text{if } \psi(x,y) = \phi(x,y) \to \theta(x,y), \\
\mathfrak{M}(\theta(y,x)) * \mathfrak{M}(\phi(y,x)), & \text{if } \psi(x,y) = \phi(x,y) * \theta(x,y), \\
\mathfrak{M}(\theta(y,x)) \wedge \mathfrak{M}(\phi(y,x)), & \text{if } \psi(x,y) = \phi(x,y) \wedge \theta(x,y), \\
\mathfrak{M}(\theta(y,x)) \vee \mathfrak{M}(\phi(y,x)), & \text{if } \psi(x,y) = \phi(x,y) \vee \theta(x,y), \\
\mathfrak{M}(\nu(\phi(y,x))), & \text{if } \psi(x,y) = \nu(\phi(y,x)) \\
\mathfrak{M}(\mu(\phi(y,x))), & \text{if } \psi(x,y) = \mu(\phi(y,x))
\end{cases}$$

for any formula ψ, ϕ and $\theta \in \mathcal{A}$.

Example 45. Let $\mathcal{A}$ be a triangle algebra and let the formula $\psi(a,b) = ((a \to b) \to a) \to (b \to a)$ be given. Then, the image of ψ under the mapping $\mathfrak{M}$ is found as below:

For this aim, we firstly determine $\phi(x,y) = (x \to y) \to x$ and $\theta(x,y) = y \to x$. So, we obtain $\phi(y,x) = (y \to x) \to y$ and $\theta(y,x) = x \to y$ by substituting $[x := y]$ and $[y := x]$ simultaneously.

$$\begin{aligned}
\mathfrak{M}(\psi(x,y)) &= \mathfrak{M}(\theta(y,x)) \to \mathfrak{M}(\phi(y,x)) \\
&= \mathfrak{M}(x \to y) \to \mathfrak{M}((y \to x) \to y) \\
&= (\mathfrak{M}(y) \to \mathfrak{M}(x)) \to (\mathfrak{M}(y) \to \mathfrak{M}(y \to x)) \\
&= (y \to x) \to (y \to (\mathfrak{M}(x) \to \mathfrak{M}(y))) \\
&= (y \to x) \to (y \to (x \to y)).
\end{aligned}$$

Consequently, the mapping $\mathfrak{M}$ gives a new formula which is obtained by rewriting all binary connectives in inverse order and by substituting $[x := y]$ and $[y := x]$ in formula simultaneously.

Theorem 46. *Let $\mathcal{A}$ be a triangle algebra.Then, the following identity*

$$\mathfrak{M}(\psi(x,y)) = \widetilde{N}(\psi(\widetilde{N}(y,x)))$$

for all $\psi(x,y) \in \mathcal{A}$.

Proof. We prove this theorem by using induction on formulas.

- Assume that $\psi(x,y)$ is an atomic formula on $\mathcal{A}$. Then, the equality is clearly obtained from the Definition 44 and the Lemma 43.
- Assume that $\psi(x,y) = \phi(x,y) \to \theta(x,y)$ such that $\mathfrak{M}(\phi(x,y)) = \widetilde{N}(\phi(\widetilde{N}(y,x)))$ and $\mathfrak{M}(\theta(x,y)) = \widetilde{N}(\theta(\widetilde{N}(y,x)))$. By using hypothesis, we get

$$\begin{aligned}
\mathfrak{M}(\psi(x,y)) &= \mathfrak{M}(\phi(x,y) \to \theta(x,y)) \\
&= \mathfrak{M}(\theta(x,y)) \to \mathfrak{M}(\phi(x,y)) \\
&= \widetilde{N}(\theta(\widetilde{N}(y,x))) \to \widetilde{N}(\phi(\widetilde{N}(y,x))) \\
&= \widetilde{N}(\phi(\widetilde{N}(y,x)) \to \theta(\widetilde{N}(y,x))) \\
&= \widetilde{N}(\psi(\widetilde{N}(y,x))).
\end{aligned}$$

Similarly, we obtain $\mathfrak{M}(\psi(x,y)) = \widetilde{N}(\psi(\widetilde{N}(y,x)))$ where $\psi \in \{\theta \to \phi, \theta * \phi, \theta \wedge \phi, \theta \vee \phi, \nu(\phi), \mu(\phi)\}$.

So, the identity $\mathfrak{M}(\psi(x,y)) = \widetilde{N}(\psi(\widetilde{N}(y,x)))$ holds for all $\psi(x,y) \in \mathcal{A}$. ■

Theorem 47. *Let $\mathcal{A}$ be a triangle algebra. Assume that the mapping $S(x,y)$ is a set-theoretical solution of the Yang-Baxter equation in triangle algebras. If the equality $S(x,y) = S(N(y),N(x))$ for all $x,y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(y,x)))$ and $\mathfrak{M}(S(y,x))$ are also set-theoretical solutions of the Yang-Baxter equation in triangle algebras.*

4. Set-Theoretical Solutions of Yang-Baxter Equation in State Triangle Algebras

In this section, we handle some fundamental set-theoretical solutions of Yang-Baxter Equation in state triangle algebras. Then, we state the relationship of transferring of set-theoretical solutions of Yang-Baxter equation among (MTL,BL, RL)-triangle algebras and state (MTL,BL, RL)-triangle algebras. Then, we give a scheme to explain clearly these relations. Then, we give some fundamental

relations between the operators $\mathfrak{M}$ and $\widetilde{N}$ in triangle algebras. Moreover, we obtain that the mapping $S(x,y)$ is a the set-theoretical solution of the Yang-Baxter equation in state triangle algebras. If the equality $S(x,y) = S(N(\sigma(y)), N(\sigma(x)))$ for all $x, y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(\sigma(y),\ \sigma(x))))$ and $\mathfrak{M}(S(\sigma(y), \sigma(x)))$ are also set-theoretical solutions of the Yang-Baxter equation in state triangle algebras.

Lemma 48. *Let (A, σ) be a state triangle algebra. The mappings*

(i) $S(x,y) = (\sigma x, \sigma y)$,

(ii) $S(x,y) = (\sigma y, \sigma x)$

verify the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in state triangle algebra.

Proof. (i) We show that this mapping satisfies the braid condition on this structure. Then, we obtain the following eqaulities:

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(\sigma(x), \sigma(y), z)) \\ &= S^{12}(\sigma(x), \sigma(\sigma(y)), \sigma(z)) \\ &= (\sigma(\sigma(x)), \sigma(\sigma(\sigma(y))), \sigma(z)) \qquad (17)\end{aligned}$$

and

$$\begin{aligned}(S^{23} \circ S^{12} \circ S^{23})(x,y,z) &= S^{23}(S^{12}(S^{23}(x,y,z))) \\ &= S^{23}(S^{12}(x, \sigma(y), \sigma(z))) \\ &= S^{23}(\sigma(x), \sigma(\sigma(y)), \sigma(z)) \\ &= (\sigma(x), \sigma(\sigma(\sigma(y))), \sigma(\sigma(z))). \qquad (18)\end{aligned}$$

By Lemma 15, we have $\sigma(x) = \sigma(\sigma(x))$ and $\sigma(z) = \sigma(\sigma(z))$.Therefore, we verify the braid condition from the equality (17) and (18). As a result, this mapping is a set-theoretical solution in state triangle algebra.

(ii) We show that this mapping satisfies the braid condition on this structure. Then, we get the following equalities:

$$\begin{aligned}(S^{12} \circ S^{23} \circ S^{12})(x,y,z) &= S^{12}(S^{23}(S^{12}(x,y,z))) \\ &= S^{12}(S^{23}(\sigma(y), \sigma(x), z)) \\ &= S^{12}(\sigma(y), \sigma(z), \sigma(\sigma(x))) \\ &= (\sigma(\sigma(z)), \sigma(\sigma(y)), \sigma(\sigma(x))) \\ &= S^{23}(\sigma(\sigma(z)), \sigma(x), \sigma(y)) \\ &= S^{12}(S^{23}(x, \sigma(z), \sigma(y))) \\ &= S^{23}(S^{12}(S^{23}(x,y,z))) = (S^{23} \circ S^{12} \circ S^{23})(x,y,z).\end{aligned}$$

■

Proposition 49. *Let (A, σ) be a state triangle algebra. The mappings*

(i) $S(x, y) = (\sigma(\nu y), \sigma(\nu x))$,

(ii) $S(x, y) = (\sigma(\mu y), \sigma(\mu x))$

verify the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in state triangle algebra.

Proof. It is straightforward from Lemma 22 and Definition 14. ■

Proposition 50. *Let (A, σ) be a state triangle algebra. The mappings*

(i) $S(x, y) = (\sigma(\nu(\sigma(x \vee y))), \sigma(\nu(\sigma(x \wedge y))))$,

(ii) $S(x, y) = (\sigma(\mu(\sigma(x \vee y))), \sigma(\mu(\sigma(x \wedge y))))$,

(iii) $S(x, y) = (\sigma(\nu(\sigma(x \wedge y))), \sigma(\nu(\sigma(x \vee y))))$,

(iv) $S(x, y) = (\sigma(\mu(\sigma(x \wedge y))), \sigma(\mu(\sigma(x \vee y))))$

(v) $S(x, y) = (\sigma(\sigma(\nu(x \vee y))), \sigma(\sigma(\nu(x \wedge y))))$,

(vi) $S(x, y) = (\sigma(\sigma(\mu(x \vee y))), \sigma(\sigma(\mu(x \wedge y))))$,

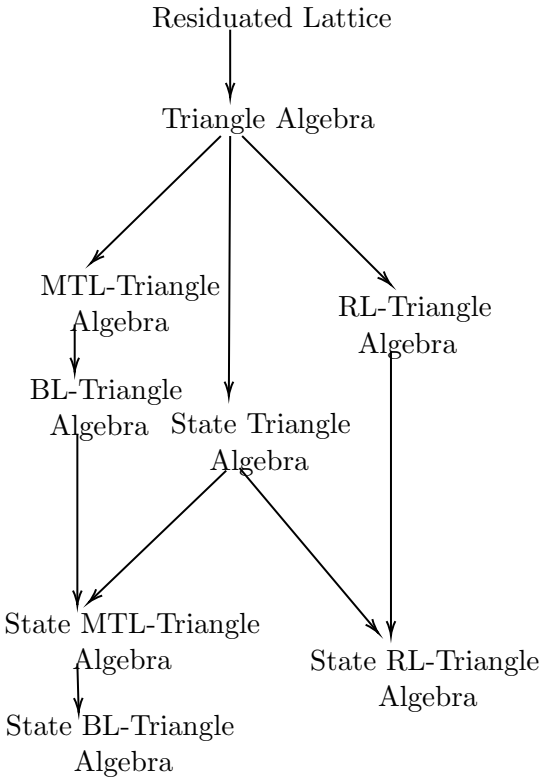
(vii) $S(x, y) = (\sigma(\sigma(\nu(x \wedge y))), \sigma(\sigma(\nu(x \vee y))))$,

$(viii)$ $S(x, y) = (\sigma(\sigma(\mu(x \wedge y))), \sigma(\sigma(\mu(x \vee y))))$

verify the braid condition on this structure. Therefore, the Yang-Baxter equation has a set-theoretical solution in state triangle algebra.

Proof. It is shown by Proposition 25 and Definition 14. ■

Now, we can give the following scheme to explain clearly the relationship of transferring of set-theoretical solutions of Yang-Baxter equation among (MTL,BL, RL)-triangle algebras and state (MTL,BL, RL)-triangle algebras.



Example 51. Let $L = \{0, x, y, z, t, 1\}$. The relations of elements in A are given as Figure 1

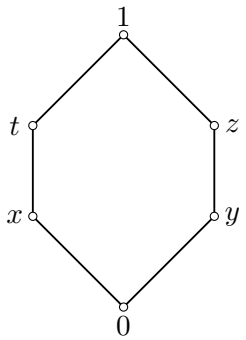


Figure 1.: Hasse Diagram of L

and the binary operations $\wedge, \vee, *$ and $\rightarrow$ on this structure are defined as the Cayley Table 1, 2, 3 and 4.
So, the algebraic structure $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice. By the help of this residuated lattice, we construct a triangle algebra.

$\wedge$	0	x	y	z	t	1
0	0	0	0	0	0	0
x	0	x	0	0	x	x
y	0	0	y	y	0	y
z	0	0	y	z	0	z
t	0	x	0	0	t	t
1	0	x	y	z	t	1

Table 1.: $\wedge-$operation on L

and

$\vee$	0	x	y	z	t	1
0	0	x	y	z	t	1
x	x	x	1	1	t	1
y	y	1	y	z	1	1
z	z	1	z	z	1	1
t	t	t	1	1	t	1
1	1	1	1	1	1	1

Table 2.: $\vee-$operation on L

$*$	0	x	y	z	t	1
0	0	0	0	0	0	0
x	0	x	0	0	x	x
y	0	0	y	y	0	y
z	0	0	t	z	0	z
t	0	x	0	z	t	t
1	0	x	y	z	t	1

Table 3.: $*-$operation on L

and

$\rightarrow$	0	x	y	z	t	1
0	1	1	1	1	1	1
x	z	1	z	z	1	1
y	t	t	1	1	t	1
z	x	x	1	1	t	1
t	y	1	y	z	1	1
1	0	x	y	z	t	1

Table 4.: $\rightarrow-$operation on L

Let $A = ([0,0],[0,x],[0,y],[0,z],[0,t],[0,1],[x,t],[x,1],[y,y],[y,z],[y,1],$ $[t,t],[t,1],[z,z],[z,1],[1,1])$. If the operators ν, μ, $\vee$, $\wedge$, $*$ and $\rightarrow$ are defined as follows:

$$\nu[a,b] = [a,a],$$
$$\mu[a,b] = [b,b],$$
$$[a,b] \vee [c,d] = [a \vee c, b \vee d]$$
$$[a,b] \wedge [c,d] = [a \wedge c, b \wedge d],$$
$$[a,b] * [c,d] = [a * c, b * d]$$
$$[a,b] \rightarrow [c,d] = [(a \rightarrow c) \wedge (b \rightarrow d), b \rightarrow d].$$

The algebraic structure $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with the smallest element $[0,0]$, the greatest element $[1,1]$ and new constant $[0,1]$. The Hasse diagram of A is given in Figure 2.

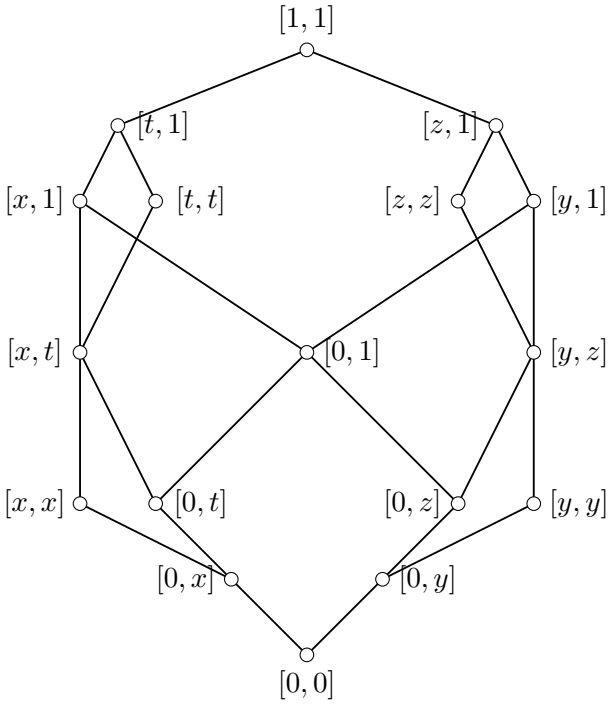


Figure 2.: Hasse Diagram of A.

Also, the state operator σ is defined as

$$\sigma(a) = \begin{cases} [0,0], & a \in \{[0,0],[0,x],[0,y],[0,z],[0,t],[0,1]\}, \\ [x,x], & a \in \{[x,0],[x,t],[x,1]\}, \\ [y,y], & a \in \{[y,y],[y,z],[y,1]\}, \\ [t,t], & a \in \{[t,t],[t,1]\}, \\ [z,z], & a \in \{[z,z],[z,1]\}, \\ [1,1], & a \in \{[1,1]\}. \end{cases}$$

Then the structure (A, σ) is a state triangle algebra. Moreover, we get for each $a, b \in A$:

$$\sigma(a * b) = \sigma(a) * \sigma(b) \quad \text{and} \quad \sigma(a \to b) = \sigma(a) \to \sigma(b).$$

Then, the state mapping σ is also an endomorphism. In addition to this, we attain

$$Ker(\sigma) = \{[1,1]\}.$$

So, we obtain the state operator σ is faithful. In accordance with these, all of the set-theoretical solutions given in Section 3 and Section 4 could be verified on this example.

Lemma 52. *Let $\mathcal{A}$ be a state triangle algebra. The mappings*

(i) $S(x,y) = (\sigma(x) * (\sigma(x) \to \sigma(y)), 1)$,

(ii) $S(x,y) = (1, \sigma(x) * (\sigma(x) \to \sigma(y)))$,

(iii) $S(x,y) = (\sigma(y) * (\sigma(y) \to \sigma(x)), 1)$,

(iv) $S(x,y) = (1, \sigma(y) * (\sigma(y) \to \sigma(x)))$,

(v) $S(x,y) = (\sigma(\nu x) * (\sigma(\nu x) \to \sigma(\nu y)), 1)$,

(vi) $S(x,y) = (1, \sigma(\nu x) * (\sigma(\nu x) \to \sigma(\nu y)))$,

(vii) $S(x,y) = (\sigma(\nu y) * (\sigma(\nu y) \to \sigma(\nu x)), 1)$,

$(viii)$ $S(x,y) = (1, \sigma(\nu y) * (\sigma(\nu y) \to \sigma(\nu x)))$,

(ix) $S(x,y) = (\sigma(\mu x) * (\sigma(\mu x) \to \sigma(\mu y)), 1)$,

(vi) $S(x,y) = (1, \sigma(\mu x) * (\sigma(\mu x) \to \sigma(\mu y)))$,

(vii) $S(x,y) = (\sigma(\mu y) * (\sigma(\mu y) \to \sigma(\mu x)), 1)$,

$(viii)$ $S(x,y) = (1, \sigma(\mu y) * (\sigma(\mu y) \to \sigma(\mu x)))$

verify the braid condition if and only if $\mathcal{A}$ is a state $RL-$triangle algebra or a state $BL-$triangle algebra.

Proof. It is obtained from Definition 14 and Lemma 28. ■

Lemma 53. *Let $\mathcal{A}$ be a state triangle algebra and let ψ be any formula from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then the following identity*

$$\widetilde{N}(\psi(\widetilde{N}(\sigma(x), \sigma(y)))) = \begin{cases} \psi(\sigma(y), \sigma(x)), & \text{if } \psi \text{ is atomic,} \\ \\ \psi(N(\sigma(y)), N(\sigma(x))), & \begin{array}{l} \text{if } \psi \in \{\theta \to \phi, \theta * \phi, \theta \wedge \phi, \\ \theta \vee \phi, \nu(\phi), \mu(\phi)\} \text{ such that} \\ \theta \text{ and } \phi \text{ are any atomic formulas,} \end{array} \end{cases}$$

is satisfied for each $(x,y) \in A \times A$.

Proof. The proof is verified by using Definition 14, Lemma 15 and Lemma 43. ■

Theorem 54. *Let $\mathcal{A}$ be a state triangle algebra.Then, the following identity*

$$\mathfrak{M}(\psi(\sigma(x),\sigma(y))) = \widetilde{N}(\psi(\widetilde{N}(\sigma(y),\sigma(x))))$$

for all $\psi(x,y) \in \mathcal{A}$.

Proof. It is obtained from Definition 14, Lemma 15 and Theorem 46. ■

Theorem 55. *Let $\mathcal{A}$ be a state triangle algebra. Assume that the mapping $S(x,y)$ is a the set-theoretical solution of the Yang-Baxter equation in state triangle algebras. If the equality $S(x,y) = S(N(\sigma(y)), N(\sigma(x)))$ for all $x,y \in A$, then the mappings $\widetilde{N}(S(\widetilde{N}(\sigma(y),\sigma(x))))$ and $\mathfrak{M}(S(\sigma(y),\sigma(x)))$ are also set-theoretical solutions of the Yang-Baxter equation in state triangle algebras.*

Proof. It is straightforward from Definition 14, Lemma 15 and Theorem 47. ■

5. Conclusion

In this paper, by taking into consideration the previous research of triangle algebras, we present some fundamental set-theoretical solutions of Yang-Baxter equation in triangle algebras and state triangle algebras. We showed that the necessary and sufficient condition for certain mappings to be set-theoretical solutions of Yang-Baxter equation on these structures is that these structures must be also MTL-(state) triangle algebras, BL-(state) triangle algebras or RL-(state) triangle algebras. In accordance with these, we recursively introduce new operators $\widetilde{N}$ and $\mathfrak{M}$. Then, we define the notion of formula on triangle algebra as a classical logic structure. Moreover, we show that some set-theoretical solutions are preserved under some conditions when these mappings are written by using these new operators. Also, we state the relationship of transferring of set-theoretical solutions of Yang-Baxter equation among (MTL,BL, RL)-(state) triangle algebras and state (MTL,BL, RL)-(state) triangle algebras. Then, we give a scheme to explain clearly these relations.

The investigation of other such applications in different algebraic structures can be an interesting object for further work.

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