

## $\pi$ -INVERSE ORDERED SEMIGROUPS

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### Abstract

This article deals with the generalization of  $\pi$ -inverse semigroups without order to ordered semigroups. Here we characterize  $\pi$ -inverse ordered semigroups by their ordered idempotents and bi-ideals.

**Keywords:** bi-ideals, ordered idempotent,  $\pi$ -regular,  $\pi$ -inverse, inverse.

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### 1. INTRODUCTION

A semigroup  $(S, \cdot)$  with an order relation  $\leq$  is called an ordered semigroup [2, 7] if for all  $a, b, x \in S$ ,  $a \leq b$  implies  $xa \leq xb$  and  $ax \leq bx$ . It is denoted by  $(S, \cdot, \leq)$ . Let  $(S, \cdot, \leq)$  be an ordered semigroup. For a subset  $A$  of  $S$ , let  $[A] = \{x \in S : x \leq a, \text{ for some } a \in A\}$ .

An element  $a$  of  $S$  is said to be regular (completely regular) [9] if there exists  $x \in S$  such that  $a \leq axa$  ( $a \leq a^2xa^2$ ).  $S$  is called a regular (completely regular) ordered semigroup if every element of  $S$  is regular (completely regular). Note that  $S$  is regular (completely regular) if and only if  $a \in (aSa]$  ( $a \in (a^2Sa^2]$ ) for all  $a \in S$ .

An element  $b \in S$  is called an inverse [5] of  $a$  if  $a \leq aba$  and  $b \leq bab$ . The set of all inverses of an element  $a \in S$  is denoted by  $V_{\leq}(a)$ .  $a', a''$  are the inverse of  $a$  unless otherwise stated.

An element  $e \in S$  is said to be an ordered idempotent if  $e \leq e^2$ . The set of all ordered idempotents of  $S$  is denoted by  $E_{\leq}(S)$ .

Bhuniya and Hansda [1] studied the ordered semigroups in which any two inverses of an element are  $\mathcal{H}$ -related. Class of these ordered semigroups are natural generalization of the class of all inverse semigroups. Hansda and Jamadar [5]

named these ordered semigroups as inverse ordered semigroups and studied their different aspects. In this paper, we further extend inverse ordered semigroups to  $\pi$ -inverse ordered semigroups.

A nonempty subset  $A$  of  $S$  is called a left (right) ideal [8] of  $S$ , if  $SA \subseteq A$  ( $AS \subseteq A$ ) and  $(A] = A$ . A nonempty subset  $A$  is called a (two-sided) ideal of  $S$  if it is both a left and a right ideal of  $S$ . Following Kehayopulu [9], a nonempty subset  $B$  of an ordered semigroup  $S$  is called a bi-ideal of  $S$  if  $BSB \subseteq B$  and  $(B] = B$ . Hansda [4] studied algebraic properties of bi-ideals in completely regular and Clifford ordered semigroups.

The principal [8] left ideal, right ideal, ideal and bi-ideal [9] generated by  $a \in S$  are denoted by  $L(a)$ ,  $R(a)$ ,  $I(a)$  and  $B(a)$  respectively. It is easy to show that

$$L(a) = (a \cup Sa], R(a) = (a \cup aS], I(a) = (a \cup Sa \cup aS \cup SaS] \text{ and } B(a) = (a \cup aSa].$$

Kehayopulu [8] defined Green's relations  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{J}$  and  $\mathcal{H}$  on an ordered semigroup  $S$  as follows

$$a\mathcal{L}b \text{ if } L(a) = L(b), a\mathcal{R}b \text{ if } R(a) = R(b), a\mathcal{J}b \text{ if } I(a) = I(b) \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on  $S$ .

An ordered semigroup  $S$  is called  $\pi$ -regular (resp. completely  $\pi$ -regular) [3] if for every  $a \in S$  there is  $m \in \mathbb{N}$  such that  $a^m \in (a^m Sa^m]$  (resp.  $a^m \in (a^{2m} Sa^{2m}]$ ). The set of all regular, completely regular, inverse and  $\pi$ -regular elements in an ordered semigroup  $S$  is denoted by  $Reg_{\leq}(S)$ ,  $Gr_{\leq}(S)$ ,  $V_{\leq}(S)$  and  $\pi Reg_{\leq}(S)$  respectively.

Let  $S$  be an ordered semigroup and  $\rho$  be an equivalence relation on  $S$ . Following Hansda and Jamadar [5], an element  $a \in S$  of type  $\tau$  is said to be a  $\rho$ -unique element in  $S$  if for every other element  $b \in S$  of type  $\tau$  we have  $a\rho b$ .

**Theorem 1** [5]. *The following conditions are equivalent on an ordered semigroup  $S$ .*

1.  $S$  is an inverse ordered semigroup;
2.  $S$  is regular and its idempotents are  $\mathcal{H}$ -commutative;
3. For every  $e, f \in E_{\leq}(S)$ ,  $e\mathcal{L}f(e\mathcal{R}f)$  implies  $e\mathcal{H}f$ .

## 2. $\pi$ -INVERSE ORDERED SEMIGROUP

This section deals with the characterization of the class of  $\pi$ -inverse ordered semigroups.

Let  $S$  be a  $\pi$ -regular ordered semigroup. Then for every  $a \in S$  there is  $m \in \mathbb{N}$  such that  $a^m \leq a^m x a^m \leq a^m (x a^m x) a^m$  and  $x a^m x \leq x a^m x (a^m) x a^m x$ . Thus for every  $a \in S$  there is  $m \in \mathbb{N}$  such that  $V_{\leq}(a^m) \neq \phi$ .

**Definition.** A  $\pi$ -regular ordered semigroup  $S$  is called  $\pi$ -inverse if for every  $a \in S$ , there is  $m \in \mathbb{N}$  such that any two inverses of  $a^m$  are  $\mathcal{H}$ -related.

For  $a \in S$ , there is  $m \in \mathbb{N}$  such that every principal left ideal and every principal right ideal generated by  $a^m$  in a  $\pi$ -inverse ordered semigroup have  $\mathcal{H}$ -unique ordered idempotent generator. This has been shown in the following theorem.

**Theorem 2.** A  $\pi$ -regular ordered semigroup  $S$  is  $\pi$ -inverse if and only if for every  $a \in S$  there is  $m \in \mathbb{N}$  such that  $(Sa^m]$  and  $(a^mS]$  are generated by an  $\mathcal{H}$ -unique ordered idempotent.

**Proof.** Suppose that  $S$  is  $\pi$ -inverse. Let  $a \in S$ . Since  $S$  is  $\pi$ -regular, there is  $m \in \mathbb{N}$  such that  $a^m \leq a^m z a^m$  for some  $z \in S$ . Let  $I = (Sa^m]$ . Then clearly  $I = (Sa^m z a^m] = (Se]$ , where  $e = z a^m \in E_{\leq}(S)$ . If possible let  $I = (Sf]$  for some  $f \in E_{\leq}(S)$ . Then  $e \mathcal{L} f$  and so  $e \leq x f$  and  $f \leq y e$  for some  $x, y \in S$ . Now  $e \leq e e \leq e e e \leq e x f e$ . Therefore  $e x f \leq e x f e x f$  so that  $e x f \in E_{\leq}(S)$ . Also  $e x f \leq e x f e x f \leq e x f (f e) e x f$  and  $f e \leq f e e e \leq f e x f e \leq f e (e x f) f e$ . Therefore  $f e \in V_{\leq}(e x f)$ . Also  $e x f \in V_{\leq}(e x f)$ . Since  $S$  is  $\pi$ -inverse for  $f e, e x f \in V_{\leq}(e x f)$  we have  $f e \mathcal{H} e x f$ . Then  $e \leq e e \leq e e e \leq e x f e \leq e x f f e \leq f e t t_1 e x f$  for some  $t, t_1 \in S$  and so  $e \leq f z_1$ , where  $z_1 = e t t_1 e x f$ . Similarly  $f \leq e z_2$  for some  $z_2 \in S$ . So  $e \mathcal{R} f$ . Hence  $e \mathcal{H} f$ . Likewise  $(a^m S]$  is generated by an  $\mathcal{H}$ -unique ordered idempotent.

Conversely assume that given condition holds in  $S$ . Then  $S$  is  $\pi$ -regular. Let  $a \in S$  and  $a', a'' \in V_{\leq}(a^m)$  for some  $m \in \mathbb{N}$ . Clearly  $(Sa^m] = (Sa' a^m] = (Sa'' a^m]$ . Since  $a' a^m, a'' a^m \in E_{\leq}(S)$  we have that  $a' a^m \mathcal{H} a'' a^m$ , by given condition. Then there are  $s, v \in S$  such that  $a' \leq a' a^m a' \leq a'' a^m s a'$  and  $a'' \leq a' a^m v a''$ . Thus  $a' \mathcal{R} a''$ . Likewise  $a' \mathcal{L} a''$ , that is  $a' \mathcal{H} a''$ . Hence  $S$  is a  $\pi$ -inverse ordered semigroup. ■

The following theorem shows some equivalent conditions for an ordered semigroup  $S$  to be  $\pi$ -inverse.

**Theorem 3.** The following conditions are equivalent on an ordered semigroup  $S$ .

1.  $S$  is a  $\pi$ -inverse ordered semigroup;
2.  $S$  is  $\pi$ -regular and for every  $e, f \in E_{\leq}(S)$ , there is  $m \in \mathbb{N}$  such that  $(e f)^m \in (f S e]$ ;
3.  $S$  is  $\pi$ -regular and for every  $e, f \in E_{\leq}(S)$ ,  $e \mathcal{L} f (e \mathcal{R} f)$  implies  $e \mathcal{H} f$ .

**Proof.** (1)  $\Rightarrow$  (2) First suppose  $S$  is  $\pi$ -inverse. Then  $S$  is  $\pi$ -regular. Let  $e, f \in E_{\leq}(S)$ . Since  $S$  is  $\pi$ -regular, for  $e f \in S$  there is  $x \in S$  such that  $x \in V_{\leq}(e f)^m$  for some  $m \in \mathbb{N}$ . We consider the following cases.

Case 1. If  $m = 1$  then  $e f \in (f S e]$  holds, by Theorem 1.

*Case 2.* If  $m > 1$  then  $x \leq x(ef)^m x$  implies that  $fxe \leq fxe(ef)^m fxe$ . Also  $(ef)^m \leq (ef)^m x(ef)^m$  implies that  $(ef)^m \leq (ef)^m (fxe)(ef)^m$ . Thus  $(ef)^m \in V_{\leq}(fxe)$ . Now  $x \leq x(ef)^m x = xe(fe)^{m-1}fx$  so that  $fxe \leq fxe(fe)^{m-1}fxe \leq fxe(fe)^{m-1}fxe(fe)^{m-1}fxe$  and  $(fe)^{m-1}fxe(fe)^{m-1} \leq (fe)^{m-1}fxe(fe)^{m-1}fxe(fe)^{m-1}$ . This gives  $(fe)^{m-1}fxe(fe)^{m-1} \in V_{\leq}(fxe)$ . Thus  $(ef)^m, (fe)^{m-1}fxe(fe)^{m-1} \in V_{\leq}(fxe)$ . Since  $S$  is  $\pi$ -inverse, we have that  $(fe)^{m-1}fxe(fe)^{m-1}\mathcal{H}(ef)^m$ . Then there are  $s_1, s_2 \in S$  such that  $(ef)^m \leq (fe)^{m-1}fxe(fe)^{m-1}s_1$  and  $(ef)^m \leq s_2(fe)^{m-1}fxe(fe)^{m-1}$ . Thus from the inequality  $(ef)^m \leq (ef)^m x(ef)^m$  we have that  $(ef)^m \leq (fe)^{m-1}fxe(fe)^{m-1}s_1xs_2(fe)^{m-1}fxe(fe)^{m-1} \leq f(fe)^{m-1}fxe(fe)^{m-1}s_1xs_2(fe)^{m-1}fxe(fe)^{m-1}e$ . Therefore  $(ef)^m \leq fye$ , where  $y = (fe)^{m-1}fxe(fe)^{m-1}s_1xs_2$ . Hence  $(ef)^m \in (fSe]$ .

(2)  $\Rightarrow$  (3) Let  $e, f \in E_{\leq}(S)$  be such that  $e\mathcal{L}f$ . Then  $e \leq xf$  and  $f \leq ye$  for some  $x, y \in S$ . Now  $e \leq xf$  implies  $e \leq exf$  and so  $e \leq ee \leq exfe$ , which implies that  $exf \leq exfexf$ . So  $exf \in E_{\leq}(S)$ . Similarly  $f \leq fye$  and  $fye \in E_{\leq}(S)$ . Now

$$(1) \quad e \leq exf \leq exff \leq (exf)(fye).$$

Since  $exf, fye \in E_{\leq}(S)$ , there exists  $m \in \mathbb{N}$  such that  $(exffye)^m \in ((fye)S(exf))$ , by condition (2). Then there exists  $z \in S$  such that  $(exffye)^m \leq (fye)z(exf)$ . Thus  $e \leq e^m$  together with (1) implies that  $e \leq (exffye)^m$  and therefore  $e \in ((fye)S(exf)) \subseteq (fS]$ . Likewise  $f \in (eS]$ , that is,  $e\mathcal{R}f$ . Hence  $e\mathcal{H}f$ .

For  $e\mathcal{R}f$ ,  $e\mathcal{H}f$  follows dually.

(3)  $\Rightarrow$  (1) Let  $a \in S$  and  $a', a'' \in V_{\leq}(a^m)$  for some  $m \in \mathbb{N}$ . Now  $a^m a' \leq a^m a'' a^m a'$  and  $a^m a'' \leq a^m a' a^m a''$  which gives  $a^m a' \mathcal{R} a^m a''$  so that  $a^m a' \mathcal{H} a^m a''$ , by the condition (3). Likewise  $a' a^m \mathcal{H} a'' a^m$ . Then  $a' \leq a' a^m a'$  gives that  $a' \leq a'' a^m x a^m$  for some  $x \in S$ . Therefore  $a' \leq a'' t$  where  $t = a^m x a^m$ . In a similar manner it is possible to get  $u, v, w \in S$  such that  $a' \leq u a''$ ,  $a'' \leq a' v$  and  $a'' \leq w a'$ . So  $a' \mathcal{H} a''$ . Hence  $S$  is a  $\pi$ -inverse ordered semigroup. ■

Let  $S$  be a  $\pi$ -regular ordered semigroup. Then for every  $a \in S$  there is  $m \in \mathbb{N}$  such that  $a^m \leq a^m x a^m$  for some  $x \in S$  which gives that  $a^m \leq a^m x (a^m) x a^m$ . Here  $a^m x, x a^m \in E_{\leq}(S)$  so that  $a^m \in (eSf]$ , for  $e = a^m x$  and  $f = x a^m$ .

Following this idea we find a condition for a  $\pi$ -regular ordered semigroup to be  $\pi$ -inverse.

**Theorem 4.** *A  $\pi$ -regular ordered semigroup  $S$  is  $\pi$ -inverse if and only if for every  $e, f \in E_{\leq}(S)$  and  $x \in S$  whenever  $x^m \in (eSf]$  for some  $m \in \mathbb{N}$ , then  $x' \in (fSe]$  for every  $x' \in V_{\leq}(x^m)$ .*

**Proof.** First suppose that  $S$  is a  $\pi$ -inverse ordered semigroup. Then there is  $m \in \mathbb{N}$  such that  $V_{\leq}(x^m) \neq \phi$ . Let  $x' \in V_{\leq}(x^m)$ . Suppose  $x^m \in (eSf]$  for  $e, f \in E_{\leq}(S)$ . Then  $x^m \leq es_1 f$  for some  $s_1 \in S$ . Now  $x' \leq x' x^m x' \leq x' es_1 f x'$

and so  $es_1fx' \leq es_1fx'es_1fx'$ , that is  $es_1fx' \in E_{\leq}(S)$ . Similarly  $x'es_1f \in E_{\leq}(S)$ . Therefore  $x' \leq x'(es_1fx')^r$  and  $x' \leq (x'es_1f)^rx'$  for all  $r \in \mathbb{N}$ . Now since  $S$  is  $\pi$ -inverse, for  $f, x'es_1f \in E_{\leq}(S)$  there are  $s_2 \in S$  and  $n \in \mathbb{N}$  such that  $(x'es_1ff)^n \leq fs_2x'es_1f$ , by Theorem 3(2). Similarly for  $e, es_1fx' \in E_{\leq}(S)$  we have  $(ees_1fx')^k \leq es_1fx's_3e$ , for some  $s_3 \in S$  and  $k \in \mathbb{N}$ . Then  $x' \leq x'x^mx'$  implies that  $x' \leq (x'es_1ff)^nx'(ees_1fx')^k \leq fs_2x'es_1x'fes_1fx's_3e$ . Hence  $x' \in (fSe]$ .

Conversely, assume that the given condition holds in  $S$ . Let  $e, f \in E_{\leq}(S)$  be such that  $e\mathcal{L}f$ , this yields that  $e \leq ee \leq ezf$  for some  $z \in S$ . Therefore  $e^m \in (eSf]$ . Since  $e \in V_{\leq}(e^m)$  we have  $e \in (fSe]$ , by given condition. Likewise  $f \in (eSf]$ . This implies that  $e\mathcal{R}f$  and so  $e\mathcal{H}f$ . Thus by Theorem 3,  $S$  is a  $\pi$ -inverse ordered semigroup. ■

**Corollary 5.** *The following conditions are equivalent on a  $\pi$ -regular ordered semigroup  $S$ .*

1.  $S$  is a  $\pi$ -inverse ordered semigroup;
2. Let  $a \in S$ . Then there are  $m, n \in \mathbb{N}$  such that  $(a^ma'a^m)^n \in (a'Sa']$ , for every  $a' \in V_{\leq}(a^m)$ ;
3. Any two inverses of an ordered idempotent in  $S$  are  $\mathcal{H}$ -related;
4. All inverses of  $e$  are  $\mathcal{H}$ -commutative, for every  $e \in E_{\leq}(S)$ ;
5. For any  $e \in E_{\leq}(S)$  and  $e' \in V_{\leq}(e)$ ,  $ee'e'e \in (e'Se']$ .

**Proof.** (1)  $\Rightarrow$  (2), (2)  $\Rightarrow$  (3), (3)  $\Rightarrow$  (4) These are obvious.

(4)  $\Rightarrow$  (5) Let  $e \in E_{\leq}(S)$  and  $e' \in V_{\leq}(e)$ . Then  $ee'e'e \leq e's_1ees_2e'$  for some  $s_1, s_2 \in S$ . Hence  $ee'e'e \in (e'Se']$ .

(5)  $\Rightarrow$  (1) Let  $a \in S$  and  $a', a'' \in V_{\leq}(a^m)$  for some  $m \in \mathbb{N}$ . Then  $a' \leq a'a^ma' \leq a'a^ma''a^ma' \leq a''a^ms_4a'a^ma'$ , for some  $s_4 \in S$ . Therefore  $a' \leq a''t_1$  where  $t_1 = a^ms_4a'a^ma'$ . Similarly there exists  $t_2 \in S$  such that  $a' \leq t_2a''$ . Also there are  $t_3, t_4 \in S$  such that  $a'' \leq t_3a'$  and  $a'' \leq a't_4$ . Thus  $a'\mathcal{H}a''$ . Hence  $S$  is a  $\pi$ -inverse ordered semigroup. ■

**Corollary 6.** *Let  $S$  be a  $\pi$ -inverse ordered semigroup and  $a, b \in S$ . If  $m, n \in \mathbb{N}$  are such that  $V_{\leq}(a^m), V_{\leq}(b^n) \neq \emptyset$ , then the following statements hold in  $S$ .*

1.  $a^m\mathcal{L}b^n$  if and only if  $a'a^m\mathcal{H}b'b^n$  for every  $a' \in V_{\leq}(a^m)$  and  $b' \in V_{\leq}(b^n)$ ;
2.  $a^m\mathcal{R}b^n$  if and only if  $a^ma'\mathcal{H}b^n b'$  for every  $a' \in V_{\leq}(a^m)$  and  $b' \in V_{\leq}(b^n)$ ;
3.  $a^m\mathcal{H}b^n$  if and only if  $a'a^m\mathcal{H}b'b^n$  and  $a^ma'\mathcal{H}b^n b'$  for every  $a' \in V_{\leq}(a^m)$  and  $b' \in V_{\leq}(b^n)$ .

**Proof.** (1) Let  $a, b \in S$ . Since  $S$  is  $\pi$ -inverse, there are  $m, n \in \mathbb{N}$  such that  $V_{\leq}(a^m), V_{\leq}(b^n) \neq \emptyset$ . Let  $a' \in V_{\leq}(a^m)$ ,  $b' \in V_{\leq}(b^n)$ . Let  $a^m\mathcal{L}b^n$ . Since  $a^m \leq$

$a^m a' a^m$  and  $a' a^m \leq a' a^m a' a^m$ , we have  $a^m \mathcal{L} a' a^m$ , which implies that  $b^n \mathcal{L} a' a^m$ . Also  $b^n \mathcal{L} b' b^n$ . Hence  $a' a^m \mathcal{L} b' b^n$ . Since  $a' a^m, b' b^n \in E_{\leq}(S)$  and  $S$  is  $\pi$ -inverse we have  $a' a^m \mathcal{H} b' b^n$ , by Theorem 3(3).

Conversely suppose that given condition holds in  $S$ . Let  $a, b \in S$  with  $a' \in V_{\leq}(a^m)$  and  $b' \in V_{\leq}(b^n)$  for some  $m, n \in \mathbb{N}$ . Then by given condition  $a' a^m \mathcal{H} b' b^n$ . Also we have  $a^m \mathcal{L} a' a^m$  and  $b^n \mathcal{L} b' b^n$  so that  $a^m \mathcal{L} b^n$ .

(2) and (3) These follow dually. ■

### 3. BI-IDEALS IN $\pi$ -INVERSE ORDERED SEMIGROUPS

In this section we characterize a  $\pi$ -inverse ordered semigroup  $S$  by the principal bi-ideals of  $S$ .

**Theorem 7.** *Let  $S$  be a  $\pi$ -regular ordered semigroup. Then the following conditions are equivalent.*

1.  $S$  is a  $\pi$ -inverse ordered semigroup;
2. For any  $a \in S$ , there is  $m \in \mathbb{N}$  such that  $B(a') = B(a'')$  for every  $a', a'' \in V_{\leq}(a^m)$ ;
3. For any  $e, f \in E_{\leq}(S)$ ,  $B((ef)^m) \subseteq B(e) \cap B(f)$  for some  $m \in \mathbb{N}$ ;
4. For any  $e \in E_{\leq}(S)$  and  $x \in V_{\leq}(e)$ ,  $B(ex) = B(xe)$ .

**Proof.** (1)  $\Rightarrow$  (2) First suppose that  $S$  is a  $\pi$ -inverse ordered semigroup. Let  $a \in S$ . Then there is  $m \in \mathbb{N}$  such that  $a', a'' \in V_{\leq}(a^m)$ . Suppose  $x \in B(a')$ . Therefore  $x \leq a'$  or  $x \leq a' y a'$  for some  $y \in S$ . Since  $S$  is  $\pi$ -inverse,  $a' \mathcal{H} a''$ . If  $x \leq a'$  then  $x \leq a' a^m a' \leq a'' s_1 a^m s_2 a''$  for some  $s_1, s_2 \in S$ . Therefore  $x \leq a'' s a''$  where  $s = s_1 a^m s_2$ . If  $x \leq a' y a'$  then there is  $s_3 \in S$  such that  $x \leq a'' s_3 a''$ . Thus in either case  $x \in B(a'')$ . Also  $a' \in B(a'')$  implies that  $B(a') \subseteq B(a'')$ . Similarly  $B(a'') \subseteq B(a')$ . Hence  $B(a') = B(a'')$ .

(2)  $\Rightarrow$  (3) Let  $e, f \in E_{\leq}(S)$  and  $x \in V_{\leq}(ef)^m$  for some  $m \in \mathbb{N}$ . Clearly  $(ef)^m, (fe)^{m-1} f x e (fe)^{m-1} \in V_{\leq}(f x e)$  and so by the condition (2) it follows that  $B((ef)^m) = B((fe)^{m-1} f x e (fe)^{m-1})$ . Now  $(ef)^m \in B((fe)^{m-1} f x e (fe)^{m-1})$  implies  $(ef)^m \leq (fe)^{m-1} f x e (fe)^{m-1}$  or  $(ef)^m \leq (fe)^{m-1} f x e (fe)^{m-1} h (fe)^{m-1} f x e (fe)^{m-1}$  for some  $h \in S$ . So in either case  $(ef)^m \leq h_1 (fe)^{m-1} f x e (fe)^{m-1}$  and  $(ef)^m \leq (fe)^{m-1} f x e (fe)^{m-1} h_2$  for some  $h_1, h_2 \in S$ . Likewise there are  $h_3, h_4 \in S$  such that  $(fe)^{m-1} f x e (fe)^{m-1} \leq h_3 (ef)^m$  and  $(fe)^{m-1} f x e (fe)^{m-1} \leq (ef)^m h_4$ . Hence  $(ef)^m \mathcal{H} (fe)^{m-1} f x e (fe)^{m-1}$ .

Let  $w \in B(ef)^m$ . Then either  $w \leq (ef)^m$  or  $w \leq (ef)^m s_1 (ef)^m$  for some  $s_1 \in S$ . If  $w \leq (ef)^m$  then  $w \leq (ef)^m \leq (ef)^m x (ef)^m \leq (ef)^m x s_2 (fe)^{m-1} f x e (fe)^{m-1}$  for some  $s_2 \in S$ .

Also  $w \leq (ef)^m s_1 (ef)^m$  gives  $w \leq ef s_1 s_3 (fe)^{m-1} fxe (fe)^{m-1}$  for some  $s_3 \in S$ . So in either case  $w \in B(e)$ . Likewise  $w \in B(f)$ . Therefore  $w \in B(e) \cap B(f)$  and hence  $B(ef)^m \subseteq B(e) \cap B(f)$ .

(3)  $\Rightarrow$  (4) Let  $e \in E_{\leq}(S)$  and  $x \in V_{\leq}(e)$ . Then  $e, xe, ex \in E_{\leq}(S)$ . Now by condition (3)  $B((exe)^m) \subseteq B(e) \cap B(xe)$  for some  $m \in \mathbb{N}$ . Let  $y \in B(e)$ . Then either  $y \leq e$  or  $y \leq es_3 e$  for some  $s_3 \in S$ . If  $y \leq e$  then  $y \leq exe \leq eexe \leq exeeexe \leq \dots \leq (exe)^m$ . So  $y \in B((exe)^m)$ . Likewise  $y \in B((exe)^m)$  for the case  $y \leq es_3 e$ . Therefore  $B(e) = B((exe)^m)$  and so  $B(e) \subseteq B(xe)$ . Also  $B((xee)^n) \subseteq B(e) \cap B(xe)$  for some  $n \in \mathbb{N}$ , then by a similar argument  $B(xe) \subseteq B(e)$ . Therefore  $B(e) = B(xe)$ . Likewise  $B(e) = B(ex)$ . Therefore  $B(xe) = B(ex)$ .

(4)  $\Rightarrow$  (1) By condition (4) we have  $ex\mathcal{H}xe$ . Also  $ex \in B(e)$  and  $ex \in B(x)$ . Then  $ex \leq e$  or  $ex \leq eb_1 e$  and  $ex \leq x$  or  $ex \leq xb_2 x$  for some  $b_1, b_2 \in S$ . Here following cases arise.

*Case 1.* If  $ex \leq e$  and  $ex \leq x$ , then  $ex \leq exex \leq xe \leq xexe = xae$  where  $a = ex$ .

*Case 2.* If  $ex \leq e$  and  $ex \leq xb_2 x$ , then  $ex \leq exex \leq xb_2 xe = xbe$  where  $b = b_2 x$ .

*Case 3.* If  $ex \leq eb_1 e$  and  $ex \leq x$ , then  $ex \leq exex \leq xeb_1 e = xce$  where  $c = eb_1$ .

*Case 4.* If  $ex \leq eb_1 e$  and  $ex \leq xb_2 x$ , then  $ex \leq exex \leq xb_2 xeb_1 e = xde$  where  $d = b_2 xeb_1$ . Therefore in either case  $ex \leq xse$  for some  $s \in S$ . Similarly  $xe \leq etx$  for some  $t \in S$ . Thus  $e, x$  are  $\mathcal{H}$ -commutative. Hence by Corollary 5,  $S$  is a  $\pi$ -inverse ordered semigroup. ■

**Corollary 8.** A  $\pi$ -regular ordered semigroup  $S$  is  $\pi$ -inverse if and only if for any  $e \in E_{\leq}(S)$  and  $x \in V_{\leq}(e)$ ,  $B(ex) = B(e) \cap B(x) = B(xe) = B(e) = B(x)$ .

**Proof.** This follows from Theorem 7. ■

**Corollary 9.** A  $\pi$ -regular ordered semigroup  $S$  is  $\pi$ -inverse if and only if for any  $e, f \in E_{\leq}(S)$ ,  $e\mathcal{L}f(e\mathcal{R}f)$  implies  $B(e) = B(f)$ .

**Proof.** Let  $S$  be a  $\pi$ -inverse ordered semigroup. Since  $S$  is  $\pi$ -inverse  $e\mathcal{L}f(e\mathcal{R}f)$  implies  $e\mathcal{H}f$  by Theorem 3. So it is easy to check that  $B(e) = B(f)$ .

Conversely suppose that the condition holds in  $S$ . Now  $B(e) = B(f)$  gives that  $e \in B(f)$  and  $f \in B(e)$ . Therefore  $e \leq f$  or  $e \leq fxf$  and  $f \leq e$  or  $f \leq eye$  for some  $x, y \in S$ . In either case  $e\mathcal{R}f$ . So  $e\mathcal{L}f$  implies  $e\mathcal{H}f$ . Hence  $S$  is a  $\pi$ -inverse ordered semigroup, by Theorem 3. ■

**Corollary 10.** *Let  $S$  be a  $\pi$ -inverse ordered semigroup and  $a, b \in S$ . If  $a' \in V_{\leq}(a^m)$ ,  $b' \in V_{\leq}(b^n)$ , for some  $m, n \in \mathbb{N}$ , then the following conditions hold on  $S$ .*

1.  $a^m \mathcal{L} b^n$  if and only if  $B(a'a^m) = B(b'b^n)$ .
2.  $a^m \mathcal{R} b^n$  if and only if  $B(a^m a') = B(b^n b')$ .

**Proof.** (1) Let  $S$  be a  $\pi$ -inverse ordered semigroup and  $a, b \in S$ . Also let  $a' \in V_{\leq}(a^m)$ ,  $b' \in V_{\leq}(b^n)$  for some  $m, n \in \mathbb{N}$ , such that  $a^m \mathcal{L} b^n$ . So by Corollary 6  $a'a^m \mathcal{H} b'b^n$ . Let  $x \in B(a'a^m)$ . Therefore  $x \leq a'a^m$  or  $x \leq a'a^m s_1 a'a^m$  for some  $s_1 \in S$ . So it is easy to verify that  $x \in B(b'b^n)$ . Also  $a'a^m \in B(b'b^n)$ . So  $B(a'a^m) \subseteq B(b'b^n)$ . Similarly  $B(b'b^n) \subseteq B(a'a^m)$ . So  $B(a'a^m) = B(b'b^n)$ .

Converse follows easily.

- (2) This is similar to (1). ■

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