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# ON THE STRUCTURE SPACE OF PRIME CONGRUENCES ON SEMIRINGS

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**Abstract**

In the present paper, we study some of the topological properties of the space of prime congruences on a semiring endowed with the hull kernel topology.

**Keywords:** semiring, congruence, prime congruence, hull kernel topology, structure space.

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## 1. Introduction

The notion of semirings was introduced by Vandiver [16] in 1934. Semirings constitute a fairly natural generalization of both rings and distributive lattices. Since unlike in rings, additive inverses are not required in semirings, there are considerable differences between the theories of the two structures viz. there is no bijection between the set of all ideals and congruences on semirings in general. As a result many theories like $k$-ideals have been developed to enrich this algebraic system and narrow the gap between rings and semirings. Also the structure space of semirings, formed by the class of prime ideals, prime $k$-ideals and maximal $k$-ideals etc. have been studied by many authors [2, 6, 13].

Moreover the properties, viz. separation axioms, compactness and connectedness in that space have been investigated as well. In [11], Lescot proved that the set of all prime $k$-ideals in a characteristic one semiring with Zariski topology is a spectral space. He also defined the prime congruences on the characteristic one semirings which are basically additively idempotent commutative semiring with zero and unity, to accomplish his study to provide a proper algebraic geometry in characteristic one semirings. Recently in [5], it has been shown that the space of all prime $k$-congruences (following the notion of prime congruence as of Lescot) is homeomorphic to the space of all prime $k$-ideals equipped with Zariski topology which is known as the $k$-prime spectrum in any commutative semiring with zero and unity and it has been found that both the spaces are spectral spaces as well.

To study the representation of arbitrary rings as rings of continuous functions on topological spaces, Jacobson [8] defined a topology for the set of primitive ideals of a ring : the hull kernel topology. In the context of semirings, many authors have used a similar construction to topologize sets of ideals and congruences (see, for instance, [7, 9, 14]). In the paper of Sen and Bandyopadhyay [14], they introduced the notion of hull kernel topology for the space of maximal regular congruences for the semi algebra and studied its topological properties. In 1993, Acharyya *et al.* [1] introduced the notion of prime congruence on a semiring which is different from that of Lescot. In 2014, Joo and Mincheva [10, 12] followed the same approach of twisted products of pair of elements to define prime congruence on an additively idempotent semiring as this class of congruences exhibit some analogous properties to the prime ideals of commutative rings. In order to establish a good notion of radical congruences they showed that the intersection of all prime congruences on a semiring can be characterized by certain twisted power formulas. To establish the structure space of the semiring $C_+(X)$ of all non-negative real-valued continuous functions on a topological space $X$ as another model of the Stone-$\check{C}$ech compactification, Acharyya *et al.* [1] considered the structure space of all maximal congruences which are prime on a semiring with zero equipped with the hull kernel topology. They proved that the space is $T_1$ and it is compact if the semiring is with unity and found a necessary and sufficient condition for $T_2$. Motivated by these works, in this paper we have studied the space of all prime congruences on any semiring with zero (not necessarily with unity) endowed with the hull kernel topology adopting the same definition of prime congruence on a general semiring structure as that of Acharyya *et al.* [1]. Here we intend to study the topological properties on the space of prime congruences on any semiring without any restrictions of assumptions like unity or commutativity, which is certainly a larger space than the space considered by Acharyya *et al.* We have found a base and characterized the closed sets of the space. Then we have investigated for the topological properties viz. separation axioms, compactness, connectedness, irreducibility etc. of that space and have found some necessary and / or sufficient conditions for $T_1$, $T_2$, compactness, connectedness etc. Also, we have identified the structure space of the semiring $Z_0^+$ of all non-negative integers and found that it is $T_1$, compact, connected but neither $T_2$ nor regular.

## 2. Preliminaries

**Definition 2.1.** Let $S$ be a non-empty set and '$+$' and '$\cdot$' be two binary operations on $S$, called addition and multiplication respectively. Then $(S,+,\cdot)$ is called a *semiring*[1] if (i) $(S,+)$ is a commutative semigroup, (ii) $(S,\cdot)$ is a semigroup and (iii) $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in S$. If there exists an element $0\in S$ such that $a+0=a$ for all $a\in S$ then 0 is called additive neutral element or the zero of $S$ and $S$ is called a *semiring with zero.* Moreover, if $a\cdot 0=0\cdot a=0$ for all $a\in S$ then $S$ is called a *semiring with absorbing zero.*

Throughout this paper we consider semiring $S$ with absorbing zero.

**Definition 2.2.** [4] An equivalence relation $\rho$ on a semiring $S$ is called a *congruence* on $S$ if for any $a,b,c\in S$,

$$(a,b)\in\rho \text{ implies } (a+c,b+c)\in\rho,\ (ac,bc)\in\rho \text{ and } (ca,cb)\in\rho.$$

**Definition 2.3** [1]**.** A proper congruence $\rho$ on a semiring $S$ is called a *prime congruence* if it satisfies the following condition:

$$(ad+bc,ac+bd)\in\rho \text{ implies either } (a,b)\in\rho \text{ or } (c,d)\in\rho, \text{ where } a,b,c,d\in S.$$

**Examples 2.4.** (i) Let us consider the semiring $C_+(X)$ of all non-negative real-valued continuous functions on a topological space $X$. Then for any element $x\in X$, the relation $\rho_x$ defined by,

$$(f,g)\in\rho_x \text{ if and only if } f(x)=g(x), \text{ where } f,g\in C_+(X)$$

is a prime congruence on $C_+(X)$.
(ii) For any prime number $p$, the relation $\rho_p$ on the semiring $Z_0^+$ of all non-negative integers defined by,

$$\rho_p=\{(m,n)\in Z_0^+\times Z_0^+ : m-n \text{ is divisible by } p\}$$

is a prime congruence.

**Remark 2.5.** There is an order preserving bijection between set of all prime congruences on two isomorphic semirings.

[1] This semiring is same as a semiring in [1] and [3] without the additive identity and multiplicative identity.

**Definition 2.6** [15]**.** An operator that assigns to each subset $A$ of a topological space $X$ a subset $\bar{A}$ of $X$ is called a *Kuratowski closure operator* if following four axioms called *Kuratowski closure axioms* hold.

(i) $\bar{\emptyset} = \emptyset$

(ii) $A \subseteq \bar{A}$

(iii) $\bar{\bar{A}} = \bar{A}$

(iv) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ for all $A, B \subseteq X$.

From these axioms, we have, for $A, B \subseteq X$, $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$.

## 3. Main results

**Definition 3.1.** Let $S$ be a semiring and $\mathcal{A}_S$ be the collection of all prime congruences on $S$. For any subset $A$ of $\mathcal{A}_S$, we define

$$\overline{A} = \left\{ \rho \in \mathcal{A}_S : \bigcap_{\rho_i \in A} \rho_i \subseteq \rho \right\}.$$

Evidently, $\bar{\emptyset} = \emptyset$.

The following Lemma 3.2 and Theorem 3.3 can be considered as counterparts of [14, Lemma 4.1] and [14, Theorem 4.1], in the setting of prime congruences on a semiring $S$.

**Lemma 3.2.** *Let $S$ be a semiring and $\rho_1$ and $\rho_2$ be two congruences on $S$. If $\rho$ is a prime congruence on $S$, then*

$$\rho_1 \cap \rho_2 \subseteq \rho \textit{ implies either } \rho_1 \subseteq \rho \textit{ or } \rho_2 \subseteq \rho.$$

***Proof.*** Let $\rho_1 \cap \rho_2 \subseteq \rho$ and $\rho_2 \nsubseteq \rho$. Then there exists $(c, d) \in S \times S$ such that $(c, d) \in \rho_2 \setminus \rho$. Now, let $(a, b) \in \rho_1$.

Then $(ad + bc, ac + bd) \in \rho_1$. Also, $(ad + bc, ac + bd) \in \rho_2$. Therefore

$$(ad + bc, ac + bd) \in \rho_1 \cap \rho_2 \subseteq \rho \text{ implies } (a, b) \in \rho$$

whence it follows that $\rho$ is a prime congruence on $S$. Hence $\rho_1 \subseteq \rho$. ■

**Theorem 3.3.** *Let $S$ be a semiring and the set of all prime congruences on $S$ be $\mathcal{A}_S$. Then the mapping from $A \mapsto \bar{A}$ is a Kuratowski closure operator on $\mathcal{A}_S$.*

***Proof.*** Let $A, B \subseteq \mathcal{A}_S$.

(i) $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\alpha$ for each $\alpha$ and hence $A \subseteq \bar{A}$.

(ii) From (i) it is clear that, $\bar{A} \subseteq \bar{\bar{A}}$. Let $\rho_\beta \in \bar{\bar{A}}$. Then $\bigcap_{\rho_\alpha \in \bar{A}} \rho_\alpha \subseteq \rho_\beta$. Again, $\bigcap_{\rho_\gamma \in A} \rho_\gamma \subseteq \rho_\alpha$ for all $\alpha$. Then

$$\bigcap_{\rho_\gamma \in A} \rho_\gamma \subseteq \bigcap_{\rho_\alpha \in \bar{A}} \rho_\alpha \subseteq \rho_\beta \text{ implies } \rho_\beta \in \bar{A}.$$

Thus $\bar{\bar{A}} \subseteq \bar{A}$. Therefore $\bar{\bar{A}} = \bar{A}$.

(iii) Clearly, $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. Now let $\rho_\beta \in \overline{A \cup B}$. Then $\bigcap_{\rho_\alpha \in A\cup B} \rho_\alpha \subseteq \rho_\beta$. It can be easily seen that,

$$\bigcap_{\rho_\alpha \in A\cup B} \rho_\alpha = \left(\bigcap_{\rho_\alpha \in A} \rho_\alpha\right) \cap \left(\bigcap_{\rho_\alpha \in B} \rho_\alpha\right).$$

Since $(\bigcap_{\rho_\alpha \in A} \rho_\alpha)$ and $(\bigcap_{\rho_\alpha \in B} \rho_\alpha)$ are congruences on the semiring $S$ and $\rho_\beta$ is a prime congruence on $S$ then by Lemma 3.2,

$$\left(\bigcap_{\rho_\alpha \in A} \rho_\alpha\right) \cap \left(\bigcap_{\rho_\alpha \in B} \rho_\alpha\right) \subseteq \rho_\beta \text{ implies either } \bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta \text{ or } \bigcap_{\rho_\alpha \in B} \rho_\alpha \subseteq \rho_\beta.$$

Hence $\rho_\beta \in \bar{A} \cup \bar{B}$. Consequently, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Therefore $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(iv) Let us suppose that $A \subseteq B$. Let $\rho_\beta \in \bar{A}$. Then $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta$. Since $A \subseteq B$, it follows that $\bigcap_{\rho_\alpha \in B} \rho_\alpha \subseteq \bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta$. This implies that $\rho_\beta \in \bar{B}$ and hence $\bar{A} \subseteq \bar{B}$. ∎

**Definition 3.4.** The topology $\tau_S$ induced by the Kuratowski closure operator on $\mathcal{A}_S$ is known as Hull Kernel topology. We consider this topological space to be the structure space of the semiring $S$.

Throughout this paper, $S$ is a semiring and the space of all prime congruences on $S$ with the hull kernel topology is denoted as $\mathcal{A}_S$.

**Notations 3.5.** Let $\rho$ be a congruence on a semiring $S$. We define,

$$\Delta(x,y) = \{\rho \in \mathcal{A}_S : (x,y) \in \rho\};\ C\Delta(x,y) = \{\rho \in \mathcal{A}_S : (x,y) \notin \rho\}$$

$$\Delta(\rho) = \{\rho' \in \mathcal{A}_S : \rho \subseteq \rho'\};\ C\Delta(\rho) = \{\rho' \in \mathcal{A}_S : \rho \nsubseteq \rho'\}.$$

**Proposition 3.6.** *Any closed set in $\mathcal{A}_S$ is of the form $\Delta(\rho)$, where $\rho$ is a congruence on $S$.*

***Proof.*** Let $\bar{A}$ be any closed set in $\mathcal{A}_S$, where $A \subseteq \mathcal{A}_S$. Let $A = \{\rho_\alpha : \alpha \in \Lambda\}$ and $\rho = \bigcap_{\rho_\alpha \in A} \rho_\alpha$. Then $\rho$ is a congruence on $S$. Let $\rho' \in \bar{A}$. Then $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho'$, i.e., $\rho \subseteq \rho'$. Consequently, $\rho' \in \Delta(\rho)$. So, $\bar{A} \subseteq \Delta(\rho)$. By the reverse implication, we obtain that, $\Delta(\rho) \subseteq \bar{A}$. Thus $\bar{A} = \Delta(\rho)$. ∎

**Corollary 3.7.** *$\Delta(\rho) = \bigcap\{\Delta(x,y) : (x,y) \in \rho\}$, where $\rho$ is a congruence on $S$.*

***Proof.*** Let $\rho' \in \Delta(\rho)$. Then $\rho \subseteq \rho'$ implies that $\rho' \in \Delta(x,y)$ for all $(x,y) \in \rho$. Therefore $\Delta(\rho) \subseteq \bigcap_{(x,y)\in\rho} \Delta(x,y)$. Conversely, let $\rho_1 \in \bigcap\{\Delta(x,y) : (x,y) \in \rho\}$. Then $(x,y) \in \rho_1$ for all $(x,y) \in \rho$. Therefore $\Delta(\rho) = \bigcap\{\Delta(x,y) : (x,y) \in \rho\}$. ∎

**Proposition 3.8.** *$\{C\Delta(a,b) : (a,b) \in S \times S\}$ is an open base for $\mathcal{A}_S$.*

***Proof.*** Let $U$ be an open set in $\mathcal{A}_S$. Then $A = \mathcal{A}_S \setminus U$ is a closed set in $\mathcal{A}_S$. By the Proposition 3.6, $A = \Delta(\rho)$ for some congruence $\rho$ on $S$. Then $\sigma \in U$ implies $\sigma \notin A$, i.e., $\rho \nsubseteq \sigma$. Then there exists $(a,b) \in \rho$ such that $(a,b) \notin \sigma$. Hence $\sigma \in C\Delta(a,b)$. Now, let $\sigma' \in C\Delta(a,b)$. Then $(a,b) \notin \sigma'$. This implies that $\rho \nsubseteq \sigma'$ whence it follows that $\sigma' \in U$. Hence $C\Delta(a,b) \subseteq U$. Consequently, $\sigma \in C\Delta(a,b) \subseteq U$. Thus $\{C\Delta(a,b) : (a,b) \in S \times S\}$ is an open base for $\mathcal{A}_S$. ∎

Now, we will study the separation properties of the space $\mathcal{A}_S$. Before going to the main result, we prove the following lemma.

**Lemma 3.9.** *Let $S$ be a semiring. For $a,b,c,d \in S$,*

$$\Delta(a,b) \cup \Delta(c,d) = \Delta(ac+bd, ad+bc).$$

***Proof.*** Let $\rho \in \Delta(a,b) \cup \Delta(c,d)$. Then either $\rho \in \Delta(a,b)$ or $\rho \in \Delta(c,d)$. If $\rho \in \Delta(a,b)$ then $(a,b) \in \rho$. Hence $(ac+bd, ad+bc) \in \rho$ which implies $\rho \in \Delta(ac+bd, ad+bc)$. Therefore

$$\Delta(a,b) \cup \Delta(c,d) \subseteq \Delta(ac+bd, ad+bc).$$

Again, let $\sigma \in \Delta(ac+bd, ad+bc)$. Then

$$(ac+bd, ad+bc) \in \sigma \text{ implies that either } (a,b) \in \sigma \text{ or } (c,d) \in \sigma$$

as $\sigma$ is a prime congruence. Therefore $\sigma \in \Delta(a,b) \cup \Delta(c,d)$ from which it follows that,

$$\Delta(ac+bd, ad+bc) \subseteq \Delta(a,b) \cup \Delta(c,d).$$

Hence for $a,b,c,d \in S$,

$$\Delta(a,b) \cup \Delta(c,d) = \Delta(ac+bd, ad+bc).$$

∎

**Corollary 3.10.** *Let $S$ be a semiring. For $a,b,c,d \in S$,*

$$C\Delta(a,b) \cap C\Delta(c,d) = C\Delta(ac+bd, ad+bc).$$

In the following theorem we proof the separation properties.

**Theorem 3.11.** (i) *The space $\mathcal{A}_S$ is $T_0$.*

(ii) *The space $\mathcal{A}_S$ is $T_1$ if and only if no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$.*

(iii) *The space $\mathcal{A}_S$ is $T_2$ if and only if for any two distinct elements $\rho_1, \rho_2$ of $\mathcal{A}_S$, there exists two pairs $(a,b),(c,d)$ of elements of $S \times S$ such that*

$$(a,b) \notin \rho_1,\ (c,d) \notin \rho_2 \ \ and \ \ (ac+bd, ad+bc) \in \rho \ \ for\ all \ \ \rho \in \mathcal{A}_S.$$

(iv) *The space $\mathcal{A}_S$ is a regular space if and only if for any $\rho \in \mathcal{A}_S$ and $(a,b) \notin \rho$, there exists a congruence $\sigma$ on $S$ and $(c,d) \in S \times S$ such that*

$$\rho \in C\Delta(c,d) \subseteq \Delta(\sigma) \subseteq C\Delta(a,b).$$

***Proof.*** (i) Let $\rho_1$ and $\rho_2$ be two distinct elements of $\mathcal{A}_S$. Then there is an element $(a,b)$ either in $\rho_1 \setminus \rho_2$ or in $\rho_2 \setminus \rho_1$. Let us suppose that $(a,b) \in \rho_1 \setminus \rho_2$. Then $C\Delta(a,b)$ is a neighbourhood of $\rho_2$ not containing $\rho_1$. Hence the space $\mathcal{A}_S$ is $T_0$.

(ii) Let the space $\mathcal{A}_S$ be $T_1$. Let us suppose that $\rho_1$ and $\rho_2$ be any two distinct elements of $\mathcal{A}_S$. Then each of $\rho_1$ and $\rho_2$ has a neighbourhood not containing the other. Since $\rho_1$ and $\rho_2$ are arbitrary elements of $\mathcal{A}_S$, it follows that no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$.

Conversely, let us suppose that no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$. Let $\rho_1$ and $\rho_2$ be any two distinct elements of $\mathcal{A}_S$. Then by hypothesis, $\rho_1 \nsubseteq \rho_2$ and $\rho_2 \nsubseteq \rho_1$. This implies that there exist $(a,b),(c,d) \in S \times S$ such that $(a,b) \in \rho_1$ but $(a,b) \notin \rho_2$ and $(c,d) \in \rho_2$ but $(c,d) \notin \rho_1$. Consequently, we have $\rho_1 \in C\Delta(c,d)$ but $\rho_1 \notin C\Delta(a,b)$ and $\rho_2 \in C\Delta(a,b)$ but $\rho_2 \notin C\Delta(c,d)$, i.e., each of $\rho_1$ and $\rho_2$ has a neighbourhood not containing the other. Hence the space $\mathcal{A}_S$ is $T_1$.

(iii) Let the space $\mathcal{A}_S$ be $T_2$. Then for any two distinct congruences $\rho_1, \rho_2$ of $\mathcal{A}_S$, there exist two open sets $C\Delta(a,b)$ and $C\Delta(c,d)$ such that $\rho_1 \in C\Delta(a,b)$ and $\rho_2 \in C\Delta(c,d)$ and $C\Delta(a,b) \cap C\Delta(c,d) = \emptyset$.

Therefore, $(a,b) \notin \rho_1$ and $(c,d) \notin \rho_2$. Let if possible, there exists $\rho$ in $\mathcal{A}_S$ such that $(ac+bd, ad+bc) \notin \rho$. That means, by Corollary 3.10,

$$\rho \in C\Delta(ac+bd, ad+bc) = C\Delta(a,b) \cap C\Delta(c,d) = \emptyset,$$

which is a contradiction. Hence the given condition holds in $\mathcal{A}_S$. Conversely, let the condition hold. Let $\rho_1, \rho_2$ be two distinct elements of $\mathcal{A}_S$. Then there exists two pairs $(a,b),(c,d)$ of elements of $S \times S$ such that

$$(a,b) \notin \rho_1 \text{ and } (c,d) \notin \rho_2 \text{ and } (ac+bd, ad+bc) \in \rho \text{ for all } \rho \in \mathcal{A}_S.$$

Therefore

$$\rho_1 \in C\Delta(a,b) \text{ and } \rho_2 \in C\Delta(c,d) \text{ and } \rho \in \Delta(ac+bd, ad+bc) \text{ for all } \rho \in \mathcal{A}_S,$$

i.e., $C\Delta(ac + bd, ad + bc) = \emptyset$. Thus there exist two open sets $C\Delta(a, b)$ and $C\Delta(c, d)$ containing $\rho_1$ and $\rho_2$ respectively such that

$$C\Delta(a, b) \cap C\Delta(c, d) = C\Delta(ac + bd, ad + bc) = \emptyset.$$

Therefore the space is $T_2$.

(iv) Let the space $\mathcal{A}_S$ be regular. Let $\rho \in \mathcal{A}_S$ and $(a, b) \notin \rho$. Then $\rho \in C\Delta(a, b)$ and $\mathcal{A}_S \setminus C\Delta(a, b)$ is a closed set not containing $\rho$. Since $\mathcal{A}_S$ is a regular space, there exists two disjoint open sets $U$ and $V$ such that $\rho \in U$ and $\mathcal{A}_S \setminus C\Delta(a, b) \subseteq V$, i.e., $\mathcal{A}_S \setminus V \subseteq C\Delta(a, b)$. $\mathcal{A}_S \setminus V$ is a closed set, which means $\mathcal{A}_S \setminus V = \Delta(\sigma) \subseteq C\Delta(a, b)$ for some congruence $\sigma$ on $S$ (*cf.* Proposition 3.6). ..... (1)

Since $U \cap V = \emptyset$, $V \subseteq \mathcal{A}_S \setminus U$ and $\mathcal{A}_S \setminus U$ being a closed set, is of the form $\mathcal{A}_S \setminus U = \Delta(\sigma')$ for some congruence $\sigma'$ on $S$. Since $\rho \in U$ then $\rho \notin \mathcal{A}_S \setminus U = \Delta(\sigma')$ which implies $\sigma' \nsubseteq \rho$. Therefore there exists $(c, d) \in \sigma'$ such that $(c, d) \notin \rho$ whence it follows that $\rho \in C\Delta(c, d)$. ..... (2)

Now, we are to show that, $V \subseteq \Delta(c, d)$. Let $\rho_1 \in V$. Then $V \subseteq \Delta(\sigma')$ implies $\sigma' \subseteq \rho_1$. Since $(c, d) \in \sigma'$, $(c, d) \in \rho_1$ and hence $\rho_1 \in \Delta(c, d)$. Consequently, $C\Delta(c, d) \subseteq \mathcal{A}_S \setminus V = \Delta(\sigma)$. ..... (3)

Thus combining (1), (2), (3), we find that,

$$\rho \in C\Delta(c, d) \subseteq \Delta(\sigma) \subseteq C\Delta(a, b).$$

Conversely, let the given condition hold and let $\rho \in \mathcal{A}_S$ and $A$ be a closed set not containing $\rho$. Then $A = \Delta(\sigma')$ for some congruence $\sigma'$ on $S$. Since $\rho \notin \Delta(\sigma')$, we have $\sigma' \nsubseteq \rho$. This implies that there exists $(a, b) \in \sigma'$ such that $(a, b) \notin \rho$. Now, by the given condition, there exists a congruence $\sigma$ on $S$ and $(c, d) \in S \times S$ such that

$$\rho \in C\Delta(c, d) \subseteq \Delta(\sigma) \subseteq C\Delta(a, b).$$

Since $(a, b) \in \sigma'$, $C\Delta(a, b) \cap \Delta(\sigma') = \emptyset$. Indeed, if $C\Delta(a, b) \cap \Delta(\sigma') \neq \emptyset$, then $\rho' \in C\Delta(a, b) \cap \Delta(\sigma')$ would imply that $(a, b) \notin \rho'$ and $\sigma' \subseteq \rho'$ which is a contradiction to the fact that $(a, b) \in \sigma'$. Hence

$$\Delta(\sigma') \subseteq \mathcal{A}_S \setminus C\Delta(a, b) \subseteq \mathcal{A}_S \setminus \Delta(\sigma).$$

Therefore $\mathcal{A}_S \setminus \Delta(\sigma)$ is an open set containing $\Delta(\sigma')$. It is clear that, $C\Delta(c, d) \cap (\mathcal{A}_S \setminus \Delta(\sigma)) = \emptyset$. So, we find that, $C\Delta(c, d)$ and $\mathcal{A}_S \setminus \Delta(\sigma)$ are two disjoint open sets containing $\rho$ and $\Delta(\sigma')$ respectively. Hence the space $\mathcal{A}_S$ is a regular space. ∎

**Theorem 3.12.** *Let $S$ be a semiring. Then the structure space $\mathcal{A}_S$ is compact if and only if for any collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in $S$, there exists a finite subcollection $\{(a_i, b_i) : i = 1, 2, \ldots, n\}$ in $S \times S$ such that for any $\rho \in \mathcal{A}_S$, there exists some $(a_i, b_i)$ such that $(a_i, b_i) \notin \rho$.*

***Proof.*** Let $\mathcal{A}_S$ be compact. Then the open cover $\{C\Delta(a_\alpha, b_\alpha) : (a_\alpha, b_\alpha) \in S \times S\}$ of $\mathcal{A}_S$ has a finite subcover $\{C\Delta(a_i, b_i) : i = 1, 2, \dots, n\}$. Then for any $\rho \in \mathcal{A}_S$, $\rho \in C\Delta(a_i, b_i)$ for some $(a_i, b_i) \in S \times S$. This implies that $(a_i, b_i) \notin \rho$. Hence $\{(a_i, b_i) : i = 1, 2, \dots, n\}$ is the required finite subcollection of elements of $S \times S$ such that for any $\rho \in \mathcal{A}_S$, there exists some $(a_i, b_i)$ for $i = 1, 2, \dots, n$ such that $(a_i, b_i) \notin \rho$. Conversely, let us suppose that the given condition holds. Let $\{C\Delta(a_i, b_i) : (a_i, b_i) \in S \times S\}$ be an open cover of $\mathcal{A}_S$. Suppose to the contrary that no finite subcollection of $\{C\Delta(a_i, b_i) : (a_i, b_i) \in S \times S\}$ covers $\mathcal{A}_S$. This means that for any finite set $\{(a_i, b_i) : i = 1, 2, \dots, n\}$ of elements of $S \times S$, $\bigcup_{i=1}^{n} C\Delta(a_i, b_i) \neq \mathcal{A}_S$ whence $\bigcap_{i=1}^{n} \Delta(a_i, b_i) \neq \emptyset$. Then there exists $\rho \in \mathcal{A}_S$ such that $\rho \in \bigcap_{i=1}^{n} \Delta(a_i, b_i)$ which implies $(a_i, b_i) \in \rho$ for $i = 1, 2, \dots, n$ and this leads to a contradiction. So the open cover $\{C\Delta(a_i, b_i) : (a_i, b_i) \in S \times S\}$ has a finite subcover and hence $\mathcal{A}_S$ is compact. ∎

Recall that, if a topological space $X$ is connected then only subsets which are both open and closed (clopen sets) are $X$ and the empty set. Otherwise the space is disconnected.

**Theorem 3.13.** *Let $S$ be a semiring. Then the structure space $\mathcal{A}_S$ is disconnected if and only if there exists a congruence $\rho$ on $S$ and a collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in $S$ not belonging to $\rho$ such that if $\rho' \in \mathcal{A}_S$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$ then $\rho \setminus \rho' \neq \emptyset$.*

***Proof.*** Let $\mathcal{A}_S$ be disconnected. Then there exists a nontrivial clopen subset of $\mathcal{A}_S$. Let $\rho$ be a congruence on $S$ for which $\Delta(\rho)$ is closed as well as open. Then $\Delta(\rho) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha)$, where $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ is a collection of pairs of elements in $S$. Now, since $C\Delta(a_\alpha, b_\alpha) \subseteq \Delta(\rho)$ for all $\alpha \in \Lambda$, for any $\rho_\alpha \in C\Delta(a_\alpha, b_\alpha)$, we have $\rho \subseteq \rho_\alpha$. Therefore $(a_\alpha, b_\alpha) \notin \rho$ as $(a_\alpha, b_\alpha) \notin \rho_\alpha$ for all $\alpha \in \Lambda$. Now, if $\rho' \in \mathcal{A}_S$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$, we have $\rho' \notin \Delta(\rho)$. So, $\rho \nsubseteq \rho'$, i.e., $\rho \setminus \rho' \neq \emptyset$. Conversely, let the given condition hold. Then $\Delta(\rho) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha)$ is clopen subset of $\mathcal{A}_S$ and hence $\mathcal{A}_S$ is disconnected. ∎

**Remark 3.14.** Intersection of two prime congruences is not a prime congruence on a semiring. Therefore the set of all prime congruences on a semiring does not form a lattice.

**Theorem 3.15.** *Let $S$ be a semiring and $\{\rho_i : i \in \Lambda\}$ be the collection of prime congruences on $S$ such that $\{\rho_i : i \in \Lambda\}$ forms a chain of congruences. Then $\bigcap_{i \in \Lambda} \rho_i$ is a prime congruence on $S$.*

***Proof.*** Let $(a, b) \notin \bigcap_{i \in \Lambda} \rho_i$ and $(c, d) \notin \bigcap_{i \in \Lambda} \rho_i$, where $a, b, c, d \in S$. That means there exists $\alpha, \beta$ such that $(a, b) \notin \rho_\alpha$ and $(c, d) \notin \rho_\beta$. As $\{\rho_i : i \in \Lambda\}$ forms a chain of prime congruences, let us suppose that $\rho_\beta \subseteq \rho_\alpha$. Then

$$(a,b) \notin \rho_\beta \text{ implies } (ac+bd, ad+bc) \notin \rho_\beta.$$

Therefore $(ac+bd, ad+bc) \notin \bigcap_{i\in\Lambda} \rho_i$. So, $\bigcap_{i\in\Lambda} \rho_i$ is a prime congruence on $S$. ■

**Definition 3.16.** Let $S$ be a semiring. The structure space $\mathcal{A}_S$ is called *irreducible* if for any decomposition $\mathcal{A}_S = A_1 \cup A_2$, where $A_1, A_2$ are closed subsets of $\mathcal{A}_S$, we have either $\mathcal{A}_S = A_1$ or $\mathcal{A}_S = A_2$.

**Theorem 3.17.** *Let $S$ be a semiring and $A$ be a closed subset of $\mathcal{A}_S$. Then $A$ is irreducible if and only if $\bigcap_{\rho_i\in A} \rho_i$ is a prime congruence on $S$.*

***Proof.*** Let $A$ be a closed subset of $\mathcal{A}_S$ which is irreducible. Let $(ac+bd, ad+bc) \in \bigcap_{\rho_i\in A} \rho_i$. Since $\rho_i$ is a prime congruence on $S$ for each $i$ then either $(a,b) \in \rho_i$ or $(c,d) \in \rho_i$ for each $i$. Hence either $\rho_i \in \Delta(a,b)$ or $\rho_i \in \Delta(c,d)$ for each $i$. Thus $A = (A \cap \Delta(a,b)) \cup (A \cap \Delta(c,d))$. Since $A$ is irreducible, it implies that $A \subseteq \Delta(a,b)$ or $A \subseteq \Delta(c,d)$. Therefore it implies that

$$(a,b) \in \bigcap_{\rho_i\in A} \rho_i \quad \text{or} \quad (c,d) \in \bigcap_{\rho_i\in A} \rho_i.$$

Conversely, let us suppose that, $\bigcap_{\rho_i\in A} \rho_i$ is a prime congruence on $S$. Let $A = B \cup C$, where $B, C$ are closed subsets of $A$. So,

$$\bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in B} \rho_i \quad \text{and} \quad \bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in C} \rho_i.$$

Therefore by the fact that, a prime congruence can not be obtained as the intersection of two strictly larger congruences (*cf.* Proposition 2.5 of [10]), it follows that,

$$\text{either} \quad \bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i \quad \text{or} \quad \bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in C} \rho_i.$$

If we assume that $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i$ then for any $\rho_k \in A$, $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i \subseteq \rho_k$. Since $B$ is a closed subset of $A$, it follows that $\rho_k \in \overline{B} = B$ and hence $A = B$. Similarly, if we assume that $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in C} \rho_i$ then this implies that $A = C$. Hence $A$ is irreducible. ■

**Theorem 3.18.** *Let $S$ be a semiring and $A$ be a subset of $\mathcal{A}_S$. $A$ is dense in $\mathcal{A}_S$ if and only if $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in \mathcal{A}_S} \rho_i$.*

***Proof.*** Let $A$ be a subset of $\mathcal{A}_S$ which is dense in $\mathcal{A}_S$. Obviously, $\bigcap_{\rho_i\in \mathcal{A}_S} \rho_i \subseteq \bigcap_{\rho_i\in A} \rho_i$. Since $A$ is dense in $\mathcal{A}_S$, $\bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in \mathcal{A}_S} \rho_i$. Therefore $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in \mathcal{A}_S} \rho_i$. To prove the converse, let us assume that $\mathcal{A}_S \setminus \bar{A} \neq \emptyset$. Then there exists a prime congruence $\rho$ on $S$ such that $\rho \in \mathcal{A}_S \setminus \bar{A}$. Therefore there exists an open neighbourhood $U$ of $\rho$ in $\mathcal{A}_S$ such that $U \cap A = \emptyset$, i.e., $C\Delta(x,y) \cap A = \emptyset$

for some $(x,y) \in S \times S$. Then $A \subseteq \Delta(x,y)$ implies $(x,y) \in \bigcap_{\rho_i \in A} \rho_i$. Now, if possible, let $(x,y) \in \bigcap_{\rho_i \in \mathcal{A}_S} \rho_i$. Then $\rho_i \in \Delta(x,y)$ for each $\rho_i \in \mathcal{A}_S$. It implies that $\mathcal{A}_S = \Delta(x,y)$, i.e., $C\Delta(x,y) = \emptyset$ which is a contradiction to the fact that it is an open neighbourhood of $\rho$. That means $(x,y) \notin \bigcap_{\rho_i \in \mathcal{A}_S} \rho_i$. Therefore $\bigcap_{\rho_i \in \mathcal{A}_S} \rho_i \subsetneqq \bigcap_{\rho_i \in A} \rho_i$. Hence $\bigcap_{\rho_i \in A} \rho_i = \bigcap_{\rho_i \in \mathcal{A}_S} \rho_i$ implies that $A$ is dense in $\mathcal{A}_S$. ■

In the following result we characterize the prime congruences on the semiring $Z_0^+$ of all non-negative integers.

**Theorem 3.19.** *The prime congruences on the semiring $Z_0^+$ of all non-negative integers are precisely of the form*

$$\rho_p = \{(m,n) \in Z_0^+ \times Z_0^+ : m - n \textit{ is divisible by } p\} \textit{ for some prime number } p.$$

***Proof.*** For any prime number $p$,

$$\rho_p = \big\{(m,n) \in Z_0^+ \times Z_0^+ : m - n \text{ is divisible by } p\big\}$$

is a prime congruence (*cf.* Examples 2.4(ii)) on $Z_0^+$. Let $\sigma$ be any prime congruence on $Z_0^+$. We are to prove that, $\sigma$ is of the form $\rho_p$ for some prime number $p$. The zero-class of $\sigma$ denoted by $\sigma(0)$ is a $k$-ideal of $Z_0^+$, by [4, Lemma 3.1]. Since any $k$-ideal is the form $aZ_0^+$ for $a \in Z_0^+$ (*cf.* [3, Example 6.6]), $\sigma(0) = kZ_0^+$ for some $k \in Z_0^+$. Therefore it can be easily shown that the congruence

$$\rho_k = \{(m,n) \in Z_0^+ \times Z_0^+ : m - n \text{ is divisible by } k\}$$

is contained in $\sigma$, i.e., $\rho_k \subseteq \sigma$. Also, $k$ must be a prime number $p$ (say). In fact, if $k$ is not a prime then $k = n_1 n_2$ (say). Since $\sigma$ is a prime congruence on $Z_0^+$, $(k,0) \in \rho_k \subseteq \sigma$ implies either $n_1 \in kZ_0^+$ or $n_2 \in kZ_0^+$ which is absurd. Again, $\rho_p$ is a maximal congruence on $Z_0^+$ which implies that $\rho_p = \sigma$. Therefore the prime congruences on $Z_0^+$ are precisely of the form $\rho_p$ for any prime number $p$. ■

Next, we give one example of a semiring $S$ whose structure space $\mathcal{A}_S$ is $T_0$, $T_1$, compact, connected but neither $T_2$ nor regular.

**Example 3.20.** Let us consider the semiring $S = Z_0^+$ of all non-negative integers and let $\mathcal{A}_S = \{\rho_p : p \text{ is a prime}\}$ be the space of all prime congruences on $Z_0^+$ with the hull kernel topology defined in it (we have already proved in Theorem 3.19 that the prime congruences on $Z_0^+$ are $\rho_p$ for any prime number $p$). Then we have the following properties of the structure space $\mathcal{A}_S$ of the semiring $S$:

(i) $\mathcal{A}_S$ is a $T_0$-space by (i) of Theorem 3.11.

(ii) $\mathcal{A}_S$ is a $T_1$-space by (ii) of Theorem 3.11.

(iii) $\mathcal{A}_S$ is a compact space by Theorem 3.12.

Now, let $\rho_{p_1}$ and $\rho_{p_2}$ be two distinct elements of $\mathcal{A}_S$ and let $(a,b),(c,d)$ be two pairs of elements of $S$ such that $(a,b) \notin \rho_{p_1}$ and $(c,d) \notin \rho_{p_2}$ which means $p_1$ does not divide $(a-b)$ and $p_2$ does not divide $(c-d)$. Then there always exists a prime number $p$ such that $p$ does not divide $(a-b)$ and $p$ does not divide $(c-d)$, i.e., $p$ does not divide $(a-b)(c-d) = (ac+bd)-(ad+bc)$. This implies that $(ad+bc, ac+bd) \notin \rho_p$. Again, $C\Delta(a,b)$ for $(a,b) \in S \times S$ is infinite and its complement $\Delta(a,b)$ is finite which is also a closed set. Hence it follows that any two non-trivial open sets intersect. Therefore we have the following:

(iv) $\mathcal{A}_S$ is not a $T_2$-space by (iii) of Theorem 3.11.

(v) $\mathcal{A}_S$ is not a regular space.

(vi) $\mathcal{A}_S$ is a connected space.

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